K-THEORY FOR RING C*-ALGEBRAS – THE CASE OF
NUMBER FIELDS WITH HIGHER ROOTS OF UNITY

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Abstract. We compute K-theory for ring C*-algebras in the case of higher roots of unity and thereby completely determine the K-theory for ring C*-algebras attached to rings of integers in arbitrary number fields.

1. Introduction

Recently, a new type of constructions was introduced in the theory of operator algebras, so-called ring C*-algebras. The construction goes as follows: Given a ring \( R \), take the Hilbert space \( \ell^2(R) \) where \( R \) is viewed as a discrete set. Consider the C*-algebra generated by all addition and multiplication operators induced by ring elements. This is the reduced ring C*-algebra of \( R \). It is denoted by \( \mathfrak{A}_r[R] \). Such an algebra was first introduced and studied by J. Cuntz in [Cun] in the special case \( R = \mathbb{Z} \). As a next step, J. Cuntz and the first named author considered the case of integral domains satisfying a certain finiteness condition in [Cu-Li1]. Motivating examples for such rings are given by rings of integers from algebraic number theory. It turns out that the associated ring C*-algebras carry very interesting structures and admit surprising alternative descriptions (see [Cu-Li1] and [Cu-Li2]). Finally, the most general case of rings without left zero-divisors was treated in [Li].

Of course, whenever new constructions of C*-algebras appear, one of the first problems is to compute their topological K-theory. Usually, this helps a lot in understanding the inner structure of the C*-algebras. In our situation, it even turns out that the ring C*-algebras attached to rings of integers are Kirchberg algebras satisfying the UCT (see [Cu-Li1], § 3 and [Li], § 5). For such C*-algebras, topological K-theory is a complete invariant. This is why computing K-theory is of particular interest and importance. The first K-theoretic computations were carried out in [Cu-Li2] for ring C*-algebras attached to rings of integers, but only in the special case where the roots of unity in the number field are given by \(+1\) and \(-1\). The reason why the general case could not be treated was that a K-theoretic computation for a certain group C*-algebra was missing.

In the present paper, we treat the remaining case of higher roots of unity. The missing ingredient is provided by [La-Lű], where for each number field, the K-theory of the group C*-algebra attached to the semidirect product of the additive group of the ring of integers by the multiplicative group of roots of unity in the number field has been computed. This computation serves as a starting point for our present paper and allows us to follow the strategy from [Cu-Li2] to completely

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determine K-theory for ring C*-algebras associated with rings of integers in number fields.

Our present K-theoretic computations heavily rely on the so-called duality theorem from [Cu-Li2]. In the present paper, we show that the imprimitivity bimodules constructed in the proof of the duality theorem from [Cu-Li2], Section 4, are equivariant with respect to the canonical semigroup actions. Apart from being of independent interest, our observations lead to a unified approach to duality theorems as in [Cu-Li2] and [Lar-Li], and they simplify our K-theoretic computations as well.

Let us now formulate our main results. Let $K$ be a number field, i.e., a finite field extension of $\mathbb{Q}$. The ring of integers in $K$, i.e., the integral closure of $\mathbb{Z}$ in $K$, is denoted by $R$. Let $\mathfrak{A}_r[R]$ be the ring C*-algebra of $R$ defined at the beginning of the introduction (see also § 2.1). Moreover, the multiplicative group $K^\times$ always admits a decomposition of the form $K^\times = \mu \times \Gamma$ where $\mu$ is the group of roots of unity in $K$ and $\Gamma$ is a free abelian subgroup of $K^\times$. Here is our result treating the case of higher roots of unity:

**Theorem 1.1.** Assume that our number field contains higher roots of unity, i.e., $|\mu| > 2$. Then

$$K_*(\mathfrak{A}_r[R]) \cong K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma).$$

The isomorphism above is meant as an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. Here $K_*(\mathfrak{A}_r[R])$ is the $\mathbb{Z}/2\mathbb{Z}$-graded abelian group $K_0(\mathfrak{A}_r[R]) \oplus K_1(\mathfrak{A}_r[R])$. The exterior $\mathbb{Z}$-algebra $\Lambda^*(\Gamma)$ over $\Gamma$ is endowed with its canonical grading, the group $K_0(C^*(\mu))$ is trivially graded, and we take graded tensor products.

This theorem is the main result of this paper. In combination with the results from [Cu-Li2], it gives the following complete description of the K-theory of $\mathfrak{A}_r[R]$ without restrictions on $\mu$:

**Theorem 1.2.** With the same notations as in the previous theorem, the K-theory of the ring C*-algebra attached to the ring of integers $R$ in a number field $K$ is given by

$$K_*(\mathfrak{A}_r[R]) \cong \begin{cases} K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) & \text{if } \# \{v_R\} = 0, \\ \Lambda^*(\Gamma) & \text{if } \# \{v_R\} \text{ is odd,} \\ \Lambda^*(\Gamma) \oplus ((\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)) & \text{if } \# \{v_R\} \text{ is even and at least 2} \end{cases}$$

again as $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups.

Here $\# \{v_R\}$ denotes the number of real places of $K$.

Note that since we are just identifying $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups, our K-theoretic formulas could be further simplified. But we present the formulas in this way because this description of K-theory naturally comes out of our computations.

Using the classification result for UCT Kirchberg algebras due to E. Kirchberg and C. Phillips, we obtain from our K-theoretic computations

**Corollary 1.3.** Given two rings of integers $R_1$ and $R_2$ in number fields, the corresponding ring C*-algebras $\mathfrak{A}_r[R_1]$ and $\mathfrak{A}_r[R_2]$ are isomorphic if and only if either...
both of the number fields corresponding to \( R_1 \) and \( R_2 \) satisfy the condition that the numbers of their real places are even and at least 2 or none of the number fields satisfies this condition.

This result should be contrasted with the observation due to J. Cuntz and C. Deninger that the \( C^* \)-dynamical system \( (\mathfrak{A}[R], R, \sigma) \) (the higher dimensional analogue of the system introduced in [Cun], it is also the analogue for ring \( C^* \)-algebras of the system introduced in [C-D-L]) determines the number field in the case of Galois extensions of \( \mathbb{Q} \).

This paper is structured as follows: We start with a review on ring \( C^* \)-algebras attached to rings of integers. Then, we revisit the duality theorem from [Cu-Li2] in Section 3. Along the way, we introduce the notion of semigroup equivariant Morita equivalence. After these preparations, we compute the K-theory for ring \( C^* \)-algebras following the same strategy as in [Cu-Li2]. This is the main part of the paper. Here we use results from [La-Lü] on the K-theory of certain group \( C^* \)-algebras (see Section 4). In the last section, we show how the formalism of the Baum-Connes conjecture can be used to deduce injectivity for certain homomorphisms on the level of K-theory. To do so, we apply tools from algebraic topology. This injectivity result is used in Section 5, but its proof is given in Section 6 because it is somewhat independent from the main part.

2. Review

From now on, let \( K \) be a global field, i.e., a finite field extension of the rational numbers \( \mathbb{Q} \) or of \( \mathbb{F}_p(T) \), the quotient field of the polynomial ring \( \mathbb{F}_p[T] \) over the finite field \( \mathbb{F}_p \). Let \( R \) be the ring of integers of \( K \). This means that \( R \) is the integral closure of \( \mathbb{Z} \) if \( K \) is a field extension of \( \mathbb{Q} \) and of \( \mathbb{F}_p[T] \) if \( K \) is a field extension of \( \mathbb{F}_p(T) \).

2.1. Constructions. First of all, let us recall the construction of ring \( C^* \)-algebras: The reduced ring \( C^* \)-algebra is simply given as the \( C^* \)-algebra generated by the left regular representation of the ring. More precisely, we consider the Hilbert space \( \ell^2(R) \) with its canonical orthonormal basis \( \{ \varepsilon_r : r \in R \} \). Then we use addition and multiplication in \( R \) to define unitaries \( U_b \) via \( \varepsilon_r \mapsto \varepsilon_{b+r} \) and isometries \( S_a \) via \( \varepsilon_r \mapsto \varepsilon_{ar} \) for all \( b \) in \( R \), \( a \) in \( R^\times = R \setminus \{0\} \). The reduced ring \( C^* \)-algebra is then given by \( \mathfrak{A}_R := C^*(\{U_b\}, \{S_a\}) \), the smallest involutive norm-closed algebra of bounded operators on \( \ell^2(R) \) containing the families \( \{U_b\} \) and \( \{S_a\} \). Note that we write \( \{U_b\} \) for \( \{U^b : b \in R\} \) and \( \{S_a\} \) for \( \{S_a : a \in R^\times\} \). We use analogous notations for other families of generators as well. There is also a full version denoted by \( \mathfrak{A}[R] \). It is defined as the universal \( C^* \)-algebra generated by unitaries \( \{u^b : b \in R\} \) and isometries \( \{s_a : a \in R^\times\} \) satisfying

I. \( u^b s_a u^d s_c = u^{b+ad} s_{ac} \)

II. \( \sum u^b s_a s_a^* u^{-b} = 1 \)

where we sum over \( R/aR = \{b + aR : b \in R\} \) in II. More precisely, Relation I implies that each of the summands \( u^b s_a s_a^* u^{-b} \) only depends on the coset \( b + aR \), not on the particular representative of the coset. Thus we obtain one summand for each coset.
in \( R/aR \) and we sum up these elements in Relation II. Here \( aR \) is the principal ideal of \( R \) generated by \( a \). We write \( e_a \) for the range projection \( s_a s_a^* \) of \( s_a \).

Here we use the notion of universal C*-algebras as explained in [Bla2], § II.8.3. Since all our generators have norm less or equal to 1, the universal C*-algebra \( A[R] \) exists. Moreover, we know that \( A[R] \) is not the zero algebra because by universal property, there exists a non-zero homomorphism \( A[R] \to A_f[R] \) sending \( u^b \) to \( U^b \) and \( s_a \) to \( S_a \) (it is easy to check relations I and II for \( U^b \) and \( S_a \) in place of \( u^b \) and \( s_a \)). Actually, it is proven in [Cu-Li1] that \( A[R] \) is purely infinite and simple (see [Cu-Li1], Theorem 3.6) so that the canonical homomorphism \( A[R] \to A_f[R] \) is an isomorphism. Thus, we can (and will) always identify \( A[R] \) and \( A_f[R] \), i.e., there is no need to distinguish between full and reduced versions.

2.2. Crossed product descriptions. The universal property of \( A[R] \) implies that \( A[R] \cong C(\bar{R}) \rtimes (R \rtimes R^\times) \). The reader may consult [Cu-Li1] for more details. Here \( \rtimes \) stands for “semigroup crossed product by endomorphisms”. This notion is defined in [Laca], § 1.3 or [Li], Appendix A.1. The \( R \rtimes R^\times \)-action we consider is given by affine transformations as follows: \( R \) sits as a subring in its profinite completion \( \bar{R} \) and thus acts additively and multiplicatively. More precisely, \( (R, +) \) acts additively and \( (R \times, \cdot) \) acts multiplicatively, and these two actions give rise to an action of \( R \rtimes R^\times \) on \( \bar{R} \), thus on \( C(\bar{R}) \).

Moreover, dilation theory as presented in [Laca] (we also explain the basic ideas in Section 3.3) tells us that \( C(\bar{R}) \rtimes (R \rtimes R^\times) \sim_M C_0(\mathcal{A}_f) \rtimes K \rtimes K^\times \). Here \( K \rtimes K^\times \) acts via affine transformations on \( C_0(\mathcal{A}_f) \), again using that \( K \) sits as a subring in its finite adele space \( \mathcal{A}_f \). The idea is to formally invert the \( R \rtimes R^\times \)-action, so that we obtain a bigger C*-algebra with a \( K \rtimes K^\times \)-action (by automorphisms) which is still very close to the original semigroup action.

2.3. The duality theorem. At this point, the duality theorem enters the game. It says that \( C_0(\mathcal{A}_\infty) \rtimes K \rtimes K^\times \) is Morita equivalent to \( C_0(\mathcal{A}_f) \rtimes K \rtimes K^\times \). For the first crossed product, we let \( K \rtimes K^\times \) act on \( C_0(\mathcal{A}_\infty) \) via affine transformations where \( \mathcal{A}_\infty \) is the infinite adele space of \( K \). So on the whole, we obtain \( \mathfrak{A}_f[R] \sim_M C_0(\mathcal{A}_\infty) \rtimes K \rtimes K^\times \).

Actually, the most general version of the duality theorem says that for every (multiplicative) subgroup \( \Gamma \) of \( K^\times \), the crossed products \( C_0(\mathcal{A}_\infty) \rtimes K \rtimes \Gamma \) and \( C_0(\Gamma \cdot \bar{R}) \rtimes (\Gamma \cdot R) \rtimes \Gamma \) are Morita equivalent (see [Cu-Li2], Theorem 4.1). Here \( \Gamma \cdot \bar{R} \) is the subring of \( \mathcal{A}_f \) generated by \( \Gamma \) and \( \bar{R} \), and \( \Gamma \cdot R \) is the subring of \( K \) generated by \( \Gamma \) and \( R \).

3. The duality theorem revisited

Again, let \( K \) be a global field and let \( R \) be the ring of integers in \( K \).
3.1. Motivation and introductory comments. As we have explained in Section 2.3, the duality theorem (Theorem 4.1 in [Cu-Li2]) was proven for each multiplicative subgroup $\Gamma$ of $K^\times$ separately in [Cu-Li2]. It is clear that these individual statements are closely connected because their proofs are more or less identical (see [Cu-Li2], §4), but the close connection was not made precise in [Cu-Li2].

Our aim in this section is to fill this gap and thereby improve our understanding of the duality theorem.

More precisely, we will show that $C_0(A_\infty) \rtimes K$ and $C(R) \rtimes R$ are $R^\times$-equivariantly Morita equivalent, so that the single $R^\times$-equivariant Morita equivalence $C_0(A_\infty) \rtimes K \sim_M C(R) \rtimes R$ induces Morita equivalences

$$C_0(A_\infty) \rtimes K \rtimes \langle S \rangle \sim_M C(R) \rtimes R \rtimes S$$

for every subsemigroup $S$ of $R^\times$. Here $\langle S \rangle$ is the subgroup of $K^\times$ generated by $S$, and $\rtimes$ stands for semigroup crossed products by endomorphisms which will be defined in the next paragraph. In particular, taking $S = R^\times$, we obtain

$$C_0(A_\infty) \rtimes K \rtimes K^\times \sim_M C(R) \rtimes R \rtimes R^\times.$$ 

Actually, it turns out that already the $C_0(A_\infty) \rtimes K \rtimes C(R) \rtimes R$-imprimitivity bimodule constructed in [Cu-Li2], §4 is $R^\times$-equivariant. The reason why $R^\times$-equivariance was not observed before is that in order to prove or even state these equivariance results, we first have to introduce the notion of equivariance for Morita equivalences or imprimitivity bimodules with respect to semigroup actions. To the authors’ knowledge, this notion has not appeared before in the literature. The idea is to extend the definitions and results from [C-M-W] and [Com] from the case of group actions to the case of semigroup actions.

As a direct consequence of our observation concerning equivariance, we obtain that the $R^\times$-equivariant Morita equivalence $C_0(A_\infty) \rtimes K \rtimes C(R) \rtimes R$ induces on the level of K-theory an isomorphism $K_* C_0(A_\infty) \rtimes K \rtimes \mu) \cong K_* C(R) \rtimes R \rtimes \mu)$ which is compatible with the canonical $R^\times$-actions. This result allows us to simplify parts of the K-theoretic computations in [Cu-Li2], and it also proves to be useful in our present K-theoretic computations.

Moreover, we observe that the imprimitivity bimodules we construct are compatible with the canonical actions of the canonical automorphism group of our global field (Aut $(K/Q)$ or Aut $(K/F_p(T))$ depending on whether $K$ is a number or function field). In addition, we will also establish a couple of new Morita equivalences which can be viewed as variations of the duality theorem from [Cu-Li2].

3.2. Semigroup crossed products by endomorphisms. To formulate our results about equivariance of Morita equivalences with respect to semigroup actions, we first have to recall the notion of semigroup crossed products by endomorphisms. We will have to deal with not necessarily unital C*-algebras. Our exposition here follows [Laca], Appendix A.1 in [Li] and [Lar].

Definition 3.1. A $C^*$-dynamical semisystem $(A, P, \alpha)$ consists of a $C^*$-algebra $A$, a semigroup $P$ and a semigroup homomorphism $\alpha : P \rightarrow \text{End}(A), p \mapsto \alpha_p$. 
Here, by a semigroup, we mean a set $P$ with an associative binary operation $P \times P \to P$, $(p, q) \mapsto pq$. We will only look at discrete semigroups. Moreover, we write $\text{End}(A)$ for the semigroup of endomorphisms of $A$. We restrict our discussion to semigroups with unit elements and require the map $\alpha : P \to \text{End}(A)$ to preserve the unit elements, i.e., if $e$ is the unit element of $P$, then we want $\alpha e = \text{id}_A$.

We now introduce the crossed product associated with such a semisystem. The basic idea is to “enlarge” the C*-algebra $A$ so that the endomorphisms $\alpha_p$ are implemented by isometries, in the sense that for every $\alpha_p$, there exists an isometry such that conjugation by this isometry gives $\alpha_p$. However, the definition making this idea precise is a bit technical.

**Definition 3.2.** Let $(A, P, \alpha)$ be a C*-dynamical semisystem. The crossed product associated to this semisystem is a triple $(A \rtimes_\alpha P, i_A, i_P)$ consisting of a C*-algebra $A \rtimes_\alpha P$, a *-homomorphism $i_A : A \to A \rtimes_\alpha P$ and a semigroup homomorphism $i_P : P \to \text{Isom}(M(A \rtimes_\alpha P))$ with the properties

1. $i_A(A) \cdot (A \rtimes_\alpha P) = A \rtimes_\alpha P$
2. $i_P(p) i_A(a) i_P(p)^* = i_A(\alpha_p(a))$ for all $p \in P$, $a \in A$
3. $A \rtimes_\alpha P$ is generated as a C*-algebra by $\{i_A(a) i_P(p) : a \in A, p \in P\}$

and satisfying the following universal property:

4. Given a non-degenerate representation $\pi$ of $A$ on a Hilbert space $H$ and a semigroup homomorphism $\pi : P \to \text{Isom}(H)$, $p \mapsto V_p$ such that $V_p \pi(a) V_p^* = \pi(\alpha_p(a))$ for all $p \in P$ and $a \in A$, there exists a (necessarily non-degenerate) representation

   $\pi \rtimes V$ of $A \rtimes_\alpha P$ on $H$ such that $(\pi \rtimes V) \circ i_A = \pi$ and $M(\pi \rtimes V) \circ i_P = V$.

Here are a few explanations: $\text{Isom}(M(A \rtimes_\alpha P))$ denotes the semigroup of isometries in the unital C*-algebra $M(A \rtimes_\alpha P)$, where $M$ stands for multiplier algebra, and $\text{Isom}(H)$ is the semigroup of isometries on $H$. The map $M(\pi \rtimes V) : M(A \rtimes_\alpha P) \to \mathcal{L}(H)$ is the extension of $\pi \rtimes V$ to the multiplier algebra of $A \rtimes_\alpha P$. This extension exists as $\pi \rtimes V$ is non-degenerate.

**Remark 3.3.** If such a triple $(A \rtimes_\alpha P, i_A, i_P)$ exists, it is unique up to canonical isomorphism. Existence of such triples is for example guaranteed in the case where $P$ acts via extendible endomorphisms. An endomorphism of a C*-algebra is called extendible if there exists a strictly continuous extension of the endomorphism to the multiplier algebra. We say that $P$ acts via extendible endomorphisms if $\alpha : P \to \text{End}(A)$ has the property that for every $p \in P$, the endomorphism $\alpha_p$ is extendible. In the cases of interest for us, this extendibility condition is satisfied. For example, if our semigroup $P$ happens to be a group, then extendibility is guaranteed because automorphisms always extend to multiplier algebras. In the group case, our construction yields the ordinary full crossed product by the group. Moreover, if our C*-algebra $A$ is unital, all the multiplier algebras disappear. In that case, we ask for a unital C*-algebra $A \rtimes_\alpha P$, and we may replace condition (1) by the condition

   $i_A : A \to A \rtimes_\alpha P$ preserves the units.
Furthermore, there is no need for condition (3). In other words, we are precisely in the situation of [Laca] (with all circle-valued multipliers set to be 1) or [Li], Appendix A.1. Actually, these two cases, the group case and the unital case, are the cases of interest for the applications we have in mind.

3.3. Dilation theory. Our goal is to develop a concrete model for the semigroup crossed products by endomorphisms we have just introduced. For this, we need two extra assumptions on our C*-dynamical semisystems.

First, the semigroup $P$ is called an left Ore semigroup if $P$ is cancellative (which means that $pq_1 = pq_2 \Rightarrow q_1 = q_2$ and $qp = q_2p \Rightarrow q_1 = q_2$ for all $p, q_1, q_2$ in $P$) and if for all $p, q$ in $P$, the intersection $Pp \cap Pq$ is non-trivial. Here $Pp := \{xp: x \in P\}$ and $Pq := \{xq: x \in P\}$ are the left ideals of $P$ generated by $p$ and $q$, respectively. For our purposes, it is convenient to observe that a cancellative semigroup $P$ is an left Ore semigroup if and only if $P$ is directed with respect to the pre-order $p \leq q \iff$ there exists $x \in P$ such that $xp = q$.

“Directed” means that for all $p$ and $q$ there exists $r \in P$ with $p \leq r$ and $q \leq r$. Moreover, we have the following

**Theorem 3.4.** A semigroup $P$ can be embedded into a group $G$ with $P^{-1}P = G$ if and only if $P$ is an left Ore semigroup. The group $G$ is determined up to canonical isomorphism by the universal property that every semigroup homomorphism from $P$ to a group $H$ extends uniquely to a group homomorphism from $G$ to $H$.

The reader is referred to [Laca] or [Cl-Pr] for the proof of this theorem and further details on left Ore semigroups.

Secondly, we want the endomorphisms $\alpha_p : A \to A$ to be injective and extendible for all $p \in P$. The reason why we ask for extendibility is partly explained in Remark 3.3 and will partly become clear later on. Injectivity of the endomorphisms $\alpha_p$ is only assumed for the sake of simplicity and also because in the cases of interest for us, the endomorphisms will be injective.

Let us now assume that the C*-dynamical semisystem $(A, P, \alpha)$ satisfies the conditions that $P$ is an left Ore semigroup and that for every $p \in P$, the homomorphism $\alpha_p$ is injective and extendible. Under these two conditions, we now construct an ordinary C*-dynamical system $(A_\infty, G, \alpha^{(\infty)})$ which dilates $(A, P, \alpha)$ (in a sense we will make precise) such that the C*-algebra $A \rtimes_\alpha P$ can be identified with a full corner in the ordinary crossed product $A_\infty \rtimes_{\alpha^{(\infty)}} G$. Here $G$ is the enveloping group of $P$ from Theorem 3.4. The construction and the proof follow more or less exactly the proof of Theorems 2.1.1 and 2.2.1 in [Laca] as it is explained in [P-R-Y]. For the reader’s convenience and also to set up notation, we present the construction and its basic properties. The basic idea is to formally invert our endomorphisms with the help of an inductive limit construction.

To begin with, consider the inductive system over the directed set $(P, \leq)$ with $A_p := \{x \in P\}$ for all $p \in P$ and connecting homomorphism from $A_p = A$ to $A = A_q$ given by $\alpha_x$ whenever $xp = q$ for $x \in P$. We will write $qp^{-1}$ for such an $x$. This notation makes perfectly sense once we go over to the enveloping group of $P$. Let $A_\infty$ be the inductive limit of this system. Moreover, we denote by $\iota_q$ the canonical
embeddings $A = A_q \xrightarrow{\iota_q} A_\infty$ so that $A_\infty = \bigcup_{q \in P} t_q(A)$. These $\iota_q$ are injective because we assume that the $\alpha_p$ are injective. The embedding of the C*-algebra labelled by the unit element $e$ of $P$ plays a special role and is therefore denoted by $\iota$, i.e., we set $\iota := \iota_e$.

The next step is to define a $G$-action on $A_\infty$. According to Theorem 3.4, we only have to construct a semigroup homomorphism $P \to \text{Aut}(A_\infty)$. Such a semigroup homomorphism can be defined as follows: For $p, q \in P$ choose $r \in P \cap P_q$. Then set for every $a \in A$ $\alpha_p(\infty)(\iota_q(a)) := \iota_{pq^{-1}}(\iota_{q^{-1}}(a))$. It is then easy to check that $\alpha_p(\infty)$ is well-defined on $\bigcup_{q \in P} t_q(A)$ and that it extends to an automorphism of $A_\infty = \bigcup_{q \in P} t_q(A)$ which we still denote by $\alpha_p(\infty)$. It follows from our construction that we obtain a semigroup homomorphism $\alpha(\infty) : P \to \text{Aut}(A_\infty), p \mapsto \alpha_p(\infty)$ which extends to a group homomorphism $\alpha(\infty) : G \to \text{Aut}(A_\infty), g \mapsto \alpha_g(\infty)$. The following observation is a direct consequence of the construction:

**Proposition 3.5.** The C*-dynamical system $(A_\infty, G, \alpha(\infty))$ and the embedding $\iota : A \to A_\infty$ constructed above satisfy $\alpha_p(\infty) \circ \iota = \iota \circ \alpha_p$ for all $p \in P$ and is uniquely determined by the following universal property:

Whenever given a C*-dynamical system $(\overline{A}, G, \overline{\alpha})$ with an embedding $\overline{\iota} : A \to \overline{A}$ satisfying $\overline{\iota} \circ \iota = \iota \circ \alpha_p$ for all $p \in P$, there exists a unique $G$-equivariant homomorphism $\Phi : A_\infty \to \overline{A}$ with $\Phi \circ \iota = \overline{\iota}$.

The reader may compare this with [Laca], Theorem 2.1.1 and Definition 2.1.2. Property 2. in [Laca], Theorem 2.1.1 is equivalent to uniqueness of $\Phi$.

The crucial observation at this point is

**Lemma 3.6.** $\iota : A \to A_\infty$ is extendible, i.e., there exists a strictly continuous extension $M(\iota) : M(A) \to M(A_\infty)$ to multiplier algebras.

**Proof.** By [Jen-Thom], Proposition 1.1.13, we have to prove that if $(u_\lambda)_\lambda$ is an approximate unit of $A$, then $\iota(u_\lambda)_\lambda$ converges strictly to a projection in $M(A_\infty)$. Let $(u_\lambda)_\lambda$ be an approximate unit of $A$. By our assumption that every $\alpha_q$ is extendible, there exist for every $p \in P$ a projection $f_p \in M(A)$ such that $(\alpha_p(u_\lambda))_\lambda$ converges strictly to $f_p$. For $p \in P$ and $a \in A$, we then have

$$\iota(u_\lambda)(\alpha_q(\infty))^{-1}(\iota(a)) = (\alpha_q(\infty))^{-1}(\iota(\alpha_p(u_\lambda)))(\iota(a)) = (\alpha_q(\infty))^{-1}(\iota(\alpha_q(u_\lambda)a)) \xrightarrow{\lambda} (\alpha_q(\infty))^{-1}(\iota(f_p a))$$

strictly.

It is then straightforward to check that the assignment

$$(\alpha_q(\infty))^{-1}(\iota(a)) \mapsto (\alpha_q(\infty))^{-1}(\iota(f_p a))$$

is well-defined on $\bigcup_{q \in P} t_q(A)$ and that it extends to a left multiplier $L : A_\infty \to A_\infty$. The corresponding right multiplier $R$ is obtained in an analogous manner if we start with the observation

$$(\alpha_q(\infty))^{-1}(\iota(a))\alpha_p(u_\lambda) = (\alpha_q(\infty))^{-1}(\iota(a)\alpha_q(\infty)(\iota(u_\lambda))) = (\alpha_q(\infty))^{-1}(\iota(\alpha_q(u_\lambda)))$$

$\xrightarrow{\lambda} (\alpha_q(\infty))^{-1}(\iota(a)f_p)$ strictly.

Thus we obtain a projection $f := (L, R) \in M(A_\infty)$ with $\lim_\lambda \iota(u_\lambda) = f$ strictly. □
We keep the notation \( f = \lim_\lambda \iota(u_\lambda) \in \text{Proj}(M(A_\infty)) \) so that \( \iota(A) \subseteq f A_\infty f \subseteq f (A_\infty \rtimes _\alpha(\infty) G) f \). Moreover, let \( u_g \) be the canonical unitaries in \( M(A_\infty \rtimes _\alpha(\infty) G) \) implementing \( \alpha(\infty) \). Our goal now is to prove

**Theorem 3.7.** The C*-algebra \( f (A_\infty \rtimes _\alpha(\infty) G) f \), the embedding \( A \overset{\iota}{\to} f (A_\infty \rtimes _\alpha(\infty) G) f \) and the semigroup homomorphism \( P \to \text{Isom}(M(f(A_\infty \rtimes _\alpha(\infty) G)f)), p \mapsto u_p f \) satisfy (1), (2), (3) and (4) from Definition 3.2.

**Proof.** We have \( \lim_\lambda u_\lambda = f \) strictly. This settles (1). Condition (2) is also immediate. Condition (3) follows from \( A_\infty \rtimes _\alpha(\infty) G = \varprojlim \{ a_\infty u_g : a_\infty \in A_\infty, g \in G \} \) and our assumption that \( P \) is a left Ore semigroup. It therefore remains to prove (4). Suppose we are given a covariant representation \( (\pi, V) \) of \( (A, P, \alpha) \) as in (4) from Definition 3.2. We now dilate \( (\pi, V) \) to a covariant representation of \( (A_\infty, G, \alpha(\infty)) \). The procedure is exactly the same as in [Laca]. But we use the language of inductive limits of Hilbert spaces to make clear that we are applying exactly the same dilation process as in the construction of \( (A_\infty, G, \alpha(\infty)) \).

First, consider the inductive system of Hilbert spaces \( \{(H_p, V_p, q) : p, q \in P, p \leq q \} \) with \( H_p := H \) for all \( p \in P \) and \( V_p : V_{q^{-1}} : H_p = H \to H = H_q \) whenever \( p \leq q \). We let \( H_\infty := \lim_\lambda (H_p, V_{p,q}) \) be the corresponding inductive limit. The reader may also consult [Ka-Ri], 11.5.26. The inductive limit \( H_\infty \) comes with isometric embeddings \( i^H_q : H = H_q \to H_\infty \) such that \( H_\infty = \bigcup_{q \in P} i^H_q(H) \). Again, the isometric embedding of \( H_\infty \) plays a special role and is denoted by \( i^H \), i.e., we set \( i^H := i^H_\infty \). Let us proceed and construct a \( G \)-action \( V^\infty(\infty) \) on \( H_\infty \) out of \( V \) in the same way as we have constructed \( \alpha(\infty) \) out of \( \alpha \). For \( p, q \in P \) choose \( r \in Pp \cap Pq \) and set for \( \xi \in H : V^\infty(\infty)(i^H_q(\xi)) := i^H_r(V^\infty(\infty)(r^{-1}(r \eta^{-1}) \xi) \). This formula gives rise to a unitary \( V^\infty(\infty) \) on \( H_\infty \). Moreover, we obtain a semigroup homomorphism \( P \to U(H_\infty), p \mapsto V^\infty(\infty) \) which extends by Theorem 3.4 to a group homomorphism \( G \to U(H_\infty), g \mapsto V^\infty(\infty) \). We have the analogue of Proposition 3.5:

**Proposition 3.8.** \( (H_\infty, G, i^H) \) satisfies \( V^\infty(\infty) \circ i^H = i^H \circ V_p \) for all \( p \in P \) and is determined by the following universal property:

Whenever given a Hilbert space \( \overline{H} \) together with a group homomorphism \( \overline{V} : G \to U(\overline{H}) \) and an isometry \( \overline{i}^H : H \to \overline{H} \) satisfying \( \overline{V_p} \circ \overline{i}^H = \overline{i}^H \circ V_p \) for all \( p \in P \), there exists a unique \( G \)-equivariant isometry \( W : H_\infty \to \overline{H} \) such that \( U \circ \overline{i}^H = \overline{W} \).

Now, the point is that we can further define a representation \( \pi_\infty \) of \( A_\infty \) on \( H_\infty \). As before, choose for \( p, q \in P \) some \( r \in Pp \cap Pq \) and define for all \( a \in A \) and \( \xi \in H \)

\[
\pi_\infty((\alpha_p^\infty)^{-1}(\iota(a)))(V^\infty_q)(\xi) = (V^\infty_q)^*i^H(V^\infty_{r^{-1}}(\pi(a)V^\infty_{r^{-1}}\xi).
\]

It is easy to check that this formula indeed defines a representation of \( A_\infty \) on \( H_\infty \). Moreover, we have \( V^\infty_q \pi_\infty(a_\infty)(V^\infty_q)^* = \pi_\infty(\alpha_p^\infty(a_\infty)) \) for all \( a_\infty \in A_\infty \) and also that \( \pi_\infty \) is non-degenerate. Because of these observations, we obtain by the universal property of the crossed product associated with \( (A_\infty, G, \alpha(\infty)) \) a homomorphism \( \pi_\infty : V^\infty(\infty) : A_\infty \rtimes _\alpha(\infty) G \to \mathcal{L}(H_\infty) \). Moreover, by our construction, for some (hence every) approximate unit \( (u_\lambda)_\lambda \) of \( A \), the net \( (\pi_\infty(\iota(a_\lambda)))_\lambda \) converges strongly to the orthogonal projection onto \( i^H(H) \subseteq H_\infty \). Therefore, we can define the homomorphism

\[
\pi \circ V : f (A_\infty \rtimes _\alpha(\infty) G)f \to \mathcal{L}(H), x \mapsto (i^H)^*(\pi_\infty \circ V^\infty(\infty))(\xi) \circ i^H.
\]
It follows from these constructions that \((\pi \ltimes V)(\iota(a)) = \pi(a)\) for all \(a \in A\) and \(M(\pi \ltimes V)(u_p f) = V_p f\) for all \(p \in P\). We have therefore completed the proof of the theorem. \(\square\)

In addition, note that the projection \(f \in \text{Proj}(M(A_{\infty} \rtimes_{\alpha(\infty)} G))\) is full. The reason is that \(\lim_p (\alpha_p^{(\infty)})^{-1}(f) = 1\) strictly in \(M(A_{\infty})\) as \(\bigcup_{p \in P} (\alpha_p^{(\infty)})^{-1}(\iota(A))\) is dense in \(A_{\infty}\). Hence it follows that we also have \(\lim_p (\alpha_p^{(\infty)})^{-1}(f) = 1\) strictly in \(M(A_{\infty} \rtimes_{\alpha(\infty)} G)\). We obtain from this observation and the previous theorem

**Corollary 3.9.** \(A \rtimes_{\alpha} P \cong f(A_{\infty} \rtimes_{\alpha(\infty)} G) f \sim_M A_{\infty} \rtimes_{\alpha(\infty)} G\).

### 3.4. Equivariant Morita equivalence.

**Definition 3.10.** Let \((A, P, \alpha)\) and \((B, P, \beta)\) be two \(C^*\)-dynamical semisystems. An imprimitivity bimodule between these two semisystems is an \(A\)-\(B\)-imprimitivity bimodule \(X\) together with a semigroup homomorphism \(\tau : P \to \text{End}(X), p \mapsto \tau_p\) such that

\[
\begin{align*}
(5) & \quad A \langle \tau_p(x), \tau_p(y) \rangle = \alpha_p(A \langle x, y \rangle) \\
(6) & \quad \langle \tau_p(x), \tau_p(y) \rangle_B = \beta_p(\langle x, y \rangle_B)
\end{align*}
\]

for all \(p \in P\) and \(x, y \in X\).

We write \((X, P, \tau)\) for such an \((A, P, \alpha)\)–\((B, P, \beta)\)-imprimitivity bimodule. Moreover, we say that \(A\) and \(B\) are \(P\)-equivariantly Morita equivalent if there exists an \((A, P, \alpha)\)–\((B, P, \beta)\)-imprimitivity bimodule (here the actions on \(A\) and \(B\) are to be understood and do not appear in the notation). In (5) and (6), \(A \langle \cdot, \cdot \rangle\) and \(\langle \cdot, \cdot \rangle_B\) denote the \(A\) and \(B\)-valued inner products on \(X\). Furthermore, \(\text{End}(X)\) simply denotes the semigroup of \(C\)-linear maps from \(X\) to \(X\). The reason why we only ask for \(C\)-linearity is that once (5) is satisfied, all the \(\tau_p\) will automatically be norm decreasing and satisfy \(\tau_p(ax) = \alpha_p(a)\tau_p(x)\) for all \(a \in A\) and \(x \in X\). Analogously, (6) implies \(\tau_p(xb) = \tau_p(x)\beta_p(b)\) for all \(x \in X\) and \(b \in B\). The reader may also compare [E-K-Q-R], Chapter 1, § 2.

**Remark 3.11.** If \(P\) happens to be a group, then our notion of \(P\)-equivariant Morita equivalence coincides with the notions introduced in [Com] and [C-M-W].

**Remark 3.12.** By [Bla2], Theorem II.7.6.9, \(A\) and \(B\) are Morita equivalent if and only if there exists a \(C^*\)-algebra \(C\) such that \(A\) and \(B\) are isomorphic to complementary full corners in \(C\). Given this alternative characterisation of Morita equivalence, it is straightforward to see that \(A\) and \(B\) are \(P\)-equivariantly Morita equivalent if and only if there exists a \(C^*\)-dynamical semisystem \((C, P, \gamma)\) such that \(A\) and \(B\) are \(P\)-equivariantly isomorphic to complementary full corners in \(C\).

**Theorem 3.13.** Let \((A, P, \alpha)\) and \((B, P, \beta)\) be \(C^*\)-dynamical semisystems. Assume that \(P\) is an left Ore semigroup and that all the \(\alpha_p\) and \(\beta_p\) are injective and extendible. If \(A\) and \(B\) are \(P\)-equivariantly Morita equivalent, then \(A \rtimes_{\alpha} P\) and \(B \rtimes_{\beta} P\) are Morita equivalent.

Let us remark that since \(\alpha_p\) and \(\beta_p\) are isometric, the endomorphisms \(\tau_p\) of the \((A, P, \alpha)\)–\((B, P, \beta)\)-imprimitivity bimodule must be isometric as well.
The first step in the proof of this theorem is the following: Assume that \((X, P, \tau)\) is an \((A, P, \alpha) - (B, P, \beta)\)-imprimitivity bimodule. Let us now dilate \((X, P, \tau)\) to an \((A(\infty), G, \alpha(\infty)) - (B(\infty), G, \beta(\infty))\)-imprimitivity bimodule \((X(\infty), G, \tau(\infty))\). Here \(G\) is the enveloping group of \(P\) and \((A(\infty), G, \alpha(\infty)), \ (B(\infty), G, \beta(\infty))\) are the dilations of \((A, P, \alpha)\) and \((B, P, \beta)\) constructed in the previous paragraph. An \((A(\infty), G, \alpha(\infty)) - (B(\infty), G, \beta(\infty))\)-imprimitivity bimodule in our sense (Definition 3.10) is the same as an imprimitivity system in the sense of Definition 1 in [Com], §3 (the same notion is also introduced in [C-M-W]). Again, this is the same dilation process as in §3.3.

**Lemma 3.14.** Let \(A^{\text{alg}}_\infty := \text{alg-}\lim_{\infty}(A_p, \alpha_{p,q})\) and \(B^{\text{alg}}_\infty := \text{alg-}\lim_{\infty}(B_p, \beta_{p,q})\) be the algebraic inductive limits of the inductive systems \(\{(A_p, \alpha_{p,q}), \ (B_p, \beta_{p,q})\} : p, q \in P, p \leq q\) from the previous paragraph. Then the algebraic inductive limit \(X^{\text{alg}}_\infty\) of the inductive system \(\{(X_p, \tau_{p,q}) : p, q \in P, p \leq q\}\) (we set \(\tau_{p,q} := \tau_{q^{-1}}\) for \(p \leq q\)) carries the following structure of an \(A^{\text{alg}}_\infty - B^{\text{alg}}_\infty\)-pre-imprimitivity bimodule:

The left \(A^{\text{alg}}_\infty\)-action is given by \(\iota^A_q(a)\iota^X_q(x) = \iota^X(\alpha_q(x))\), for the \(A^{\text{alg}}_\infty\)-valued inner product we take \(A^{\text{alg}}_\infty\)-valued inner product \(\langle \iota^X_q(x), \iota^X_q(y) \rangle = \iota^A_q(\langle x, y \rangle)\), the right \(B^{\text{alg}}_\infty\)-action is given by \(\iota^B_q(B(b)) = \iota^X(\beta_q(x))\) and we set \(\langle \iota^B_q(B(b)), \iota^X_q(x) \rangle_{B^{\text{alg}}_\infty} = \iota^B_q(B(b), x)\) for the \(B^{\text{alg}}_\infty\)-valued inner product.

It suffices to define all the structures on \(\iota^A_q(A), \ i^B_q(B)\) and \(\iota^X_q(X)\) because \(A^{\text{alg}}_\infty = \bigcup_{q \in P} \iota^A_q(A), \ B^{\text{alg}}_\infty = \bigcup_{q \in P} \iota^B_q(B)\) and \(X^{\text{alg}}_\infty = \bigcup_{q \in P} \iota^X_q(X)\). Moreover, in this lemma, we view \(A^{\text{alg}}_\infty\) and \(B^{\text{alg}}_\infty\) as dense *-algebras of \(A_\infty\) and \(B_\infty\), respectively.

**Proof.** Just check that the formulas above indeed yield well-defined actions and inner products which satisfy the axioms of a pre-imprimitivity bimodule.

Let us now define a \(G\)-action on \(X^{\text{alg}}_\infty\).

**Lemma 3.15.** For \(p, q \in P\) choose \(r \in Pp \cap Pq\) and set for all \(x \in X^{(\infty)}_p(\iota^X_q(x)) = \iota^X_{\tau_{q^{-1}}^{-1}}(\tau_{q^{-1}}^{-1}(x))\). This gives a well-defined linear automorphism \(\tau^{(\infty)}_p\) of \(X^{\text{alg}}_\infty\). Moreover, the map \(P \ni p \mapsto \tau^{(\infty)}_p \in \text{Aut}(X^{\text{alg}}_\infty)\) is a semigroup homomorphism, and we have

\[
A^{\text{alg}}_\infty \tau^{(\infty)}_p(x_\infty, \tau^{(\infty)}_p(y_\infty) = \alpha^{(\infty)}_p = \langle x_\infty, y_\infty \rangle \\
B^{\text{alg}}_\infty \tau^{(\infty)}_p(x_\infty, \tau^{(\infty)}_p(y_\infty) = \beta^{(\infty)}_p = \langle x_\infty, y_\infty \rangle_{B^{\text{alg}}_\infty}
\]

for all \(x_\infty, y_\infty \in X^{\text{alg}}_\infty\).

**Proof.** Again, these assertions can be checked via straightforward computations.

**Proof of the theorem.** Since \(X^{\text{alg}}_\infty\) is an \(A^{\text{alg}}_\infty - B^{\text{alg}}_\infty\)-pre-imprimitivity bimodule, the closure \(X_\infty := \overline{X^{\text{alg}}_\infty}\) is an \(A_\infty - B_\infty\)-imprimitivity bimodule (with the extended actions and inner products from \(X^{\text{alg}}_\infty\)). Moreover, for all \(p \in P\), \(\tau^{(\infty)}_p\) extends to an automorphism of \(X_\infty\) which we still denote by \(\tau^{(\infty)}_p\). We therefore obtain a
A Remark 3.17. C form the linking algebra of this pre-imprimitivity bimodule carries a canonical structure of a P of (5) and (6). As explained in the proof of the previous theorem, the closure of (9) is straightforward to check that

\[ \tau_g(x_{\infty}) = \alpha_g(x_{\infty}) = \beta_g(x_{\infty}) \]

for all \( x_{\infty}, y_{\infty} \in X_{\infty} \). Thus by [Com] or [C-M-W], we deduce \( A_{\infty} \times_{\alpha(\infty)} G \sim_M B_{\infty} \times_{\beta(\infty)} G \). Therefore we conclude

\[ A \times_{\alpha} P \sim_M A_{\infty} \times_{\alpha(\infty)} G \sim_M B_{\infty} \times_{\beta(\infty)} G \sim_M B \times_{\beta} P. \]

This completes the proof of the theorem.

Here are a couple of remarks concerning this result.

**Remark 3.16.** In our theorem, we start with a \( P \)-equivariant \( A \rightarrow B \)-imprimitivity bimodule. Actually, it suffices to start with an \( A \rightarrow B \)-pre-imprimitivity bimodule (for dense \(*\)-algebras \( A \) and \( B \) of \( A \) and \( B \), respectively) satisfying the analogues of (5) and (6). As explained in the proof of the previous theorem, the closure of this pre-imprimitivity bimodule carries a canonical structure of a \( P \)-equivariant \( A \rightarrow B \)-imprimitivity bimodule.

**Remark 3.17.** Assume that we start with an \( A \rightarrow B \)-pre-imprimitivity bimodule \( X \) as in the previous remark. Going through [Com], § 6, it is possible to describe explicitly, in terms of \( X \), the \( A_{\infty} \times_{\alpha(\infty)} G - B_{\infty} \times_{\beta(\infty)} G \)-imprimitivity bimodule and the actions and inner products on it constructed in the proof of the previous theorem.

**Remark 3.18.** Let us discuss an alternative way of proving Theorem 3.13. Namely, the proofs in [C-M-W] and [Com] in the group case suggest the following procedure: Given an \( (A, P, \alpha) \rightarrow (B, P, \beta) \)-imprimitivity bimodule \( X \) as in Theorem 3.13, we first form the linking algebra \( C := (X^e_B) \) as in [Bla2], II.7.6.9. The algebra \( C \) carries a canonical \( P \)-action, namely \( P \ni p \mapsto \gamma_p = (\tau_p^\alpha, \tau_p^\beta) \) with \( \tau_p^\alpha(x^e) = (\tau_p(x))^\alpha \). We then construct the semigroup crossed product \( C^e \times P \). It is then clear that once we know that the canonical homomorphisms \( A \times_{\alpha} P \rightarrow C \times_{e} P \) and \( C \times_{\beta} P \rightarrow B \times_{\beta} P \) are injective, these homomorphisms can be used to identify \( A \times_{\alpha} P \) and \( B \times_{\beta} P \) with full corners in \( C \times_{\beta} P \). Finally, we could conclude that \( A \times_{\alpha} P \sim_M C \times_{e} P \sim_M B \times_{\beta} P \). This would then be the exact analogue of the proofs presented in [C-M-W] and [Com]. However, to make this work, we need

**Proposition 3.19.** The canonical homomorphisms \( A \times_{\alpha} P \rightarrow C \times_{e} P \) and \( C \times_{\beta} P \rightarrow B \times_{\beta} P \) are injective.

**Proof.** Set \( C_{\infty} := \lim\limits_{\rightarrow} (C_p, \gamma_{p,q}) \) where \( \gamma_{p,q} := \gamma_{q p^{-1}}(p \leq q) \). As before, we can define a \( P \)-action \( \gamma(\infty) \) on \( C_{\infty} \) which dilates \( \gamma \). The action \( \gamma(\infty) \) is given by the by now familiar formula \( \gamma_p(\alpha(c)) = \alpha_{r p^{-1}}(\gamma_{r p^{-1}}(c)) \) for some \( r \in P_p \cap P_q \). It is then straightforward to check that \( C_{\infty} \) can be canonically identified with the linking algebra \( (A_{\infty} X_{\infty} \beta_{\infty}) \) so that \( \gamma(\infty) \) is transformed into the action

\[ \tau_p(\infty) = (\alpha_{p}(\infty), \beta_{p}(\infty)) \in \text{Aut} \left( A_{\infty} X_{\infty} B_{\infty} \right). \]
Hence we obtain a canonical isomorphism \((A \times_{\alpha} X_{\infty} B_{\infty}) \rtimes G) = C_{\infty} \rtimes_{G'} G\) where the first crossed product is taken with respect to the \(G\)-action which extends (9). Moreover, chasing through our constructions, it is immediate that the following diagrams commute:

\[
\begin{array}{c}
\xymatrix{ A \rtimes_{\alpha} P \ar[r] & C \rtimes_{\gamma} P \\
(A \times_{\alpha} X_{\infty} B_{\infty}) \rtimes G \ar[u] & C_{\infty} \rtimes_{G'} G \ar[u]}
\end{array}
\]

All the arrows we consider here are given by the corresponding canonical homomorphisms. Moreover, we know that the vertical arrows are injective by Theorem 3.7. And by [Com], § 5, we know that the upper horizontal arrows are injective. Thus we deduce that the canonical homomorphisms \(A \rtimes_{\alpha} P \rightarrow C \rtimes_{\gamma} P\) and \(C \rtimes_{\gamma} P \leftarrow B \rtimes_{\beta} P\) are injective, as desired.

From commutativity of the diagrams (10) and (11), we immediately obtain

**Corollary 3.20.** The imprimitivity bimodule constructed in the previous remark coincides with the one from the original proof of Theorem 3.13.

Here is an observation which will be useful later on: In the same situation as in Theorem 3.13, assume that \(P\) is abelian. Let \(Q\) be a subsemigroup of \(P\). By Remark 3.18, we can identify \(A \rtimes_{\alpha} Q\) and \(B \rtimes_{\beta} Q\) via the canonical embeddings with complementary full corners in \(C \rtimes_{\gamma} Q\). Here we restrict the actions to \(Q\). As \(P\) is assumed to be abelian, it acts on \(A \rtimes_{\alpha} Q\), \(B \rtimes_{\beta} Q\) and \(C \rtimes_{\gamma} Q\) so that the canonical embeddings \(A \rtimes_{\alpha} Q \rightarrow C \rtimes_{\gamma} Q\) and \(C \rtimes_{\gamma} Q \leftarrow B \rtimes_{\beta} Q\) are \(P\)-equivariant. Thus, by Remark 3.12, we deduce

**Corollary 3.21.** \(A \rtimes_{\alpha} Q\) and \(B \rtimes_{\beta} Q\) are \(P\)-equivariantly Morita equivalent.

With the notations introduced above, we can make another very useful observation: Assume that we are in the situation of Theorem 3.13. Let us take \(p \in P\) and denote by \((\alpha_p)_*\) and \((\beta_p)_*\) the endomorphisms of \(K_*(A)\) and \(K_*(B)\) induced by \(\alpha_p\) and \(\beta_p\) on K-theory. Moreover, the \(A\)-\(B\)-imprimitivity bimodule \(X\) gives rise to an element in \(KK(A,B)\) which induces a homomorphism \(K_*(A) \rightarrow K_*(B)\). Let us call this homomorphism \([X]\).

**Lemma 3.22.** Under the assumptions of Theorem 3.13, we have \([X] \circ (\alpha_p)_* = (\beta_p)_* \circ [X]\) for all \(p \in P\).

**Proof.** Let \(\kappa_A : A \rightarrow C, a \mapsto (0 0)\) and \(\kappa_B : B \rightarrow C, b \mapsto (0 0)\) be the canonical inclusions into the linking algebra \(C\). It is clear that \([X] = ((\kappa_B)_*)^{-1}(\kappa_A)_*.\)
Moreover, we have $\kappa_A \circ \alpha_p = \gamma_p \circ \kappa_A$ and $\gamma_p \circ \kappa_B = \kappa_B \circ \gamma_p$, where $\gamma$ is the $P$-action on $C$ from above. Thus we conclude $[X] \circ (\alpha_p)_* = ((\kappa_B)_*^{-1})(\kappa_A)_* (\alpha_p)_* = ((\kappa_B)_*^{-1})(\kappa_A)_* = (\beta_p)_* [X]$. \hfill $\Box$

Moreover, as a last comment, we note that Lemma 3.22 can also be proven by directly computing Kasparov products in $KK(A, B)$ without using the linking algebra.

3.5. Duality theorems as applications. Let us now apply the results from the previous paragraphs to C*-dynamical systems given by affine transformations of adele spaces. Recall that in [Cu-Li2], it was proven that for every global field $K$, we have

$$C_0(\mathbb{A}_\infty) \rtimes K \cong C(\mathbb{A}/K) \rtimes \mathbb{A}_\infty$$

and the Morita equivalence

$$C(\mathbb{A}/K) \rtimes \mathbb{A}_\infty \sim_M C(\mathcal{R}) \rtimes R.$$  

In (13), $\mathbb{A}$ stands for the full adele space of $K$, i.e., $\mathbb{A}$ is the direct product of the infinite adele space $\mathbb{A}_\infty$ and the finite adele space $\mathbb{A}_f$ associated with $K$, and $\mathbb{A}_\infty$ acts on $\mathbb{A}/K$ (hence on $C(\mathbb{A}/K)$) by additively shifting the first component of $\mathbb{A} = \mathbb{A}_\infty \times \mathbb{A}_f$ (see [Cu-Li2]).

Our aim is to show that the concrete imprimitivity bimodule from [Cu-Li2], Section 4, which implements the Morita equivalence (12), is $R^\times$-equivariant in the sense of Definition 3.10. For the $R^\times$-actions, we take the canonical multiplicative actions on $C_0(\mathbb{A}_\infty) \rtimes K$ and $C(\mathcal{R}) \rtimes R$. Actually, for our purposes, it turns out to be most convenient to consider the inverse of the canonical $R^\times$-action on $C_0(\mathbb{A}_\infty) \rtimes K$.

This action is transported by the concrete identification (13) in [Cu-Li2], Section 4 to the $R^\times$-action on $C(\mathbb{A}/K) \rtimes \mathbb{A}_\infty$ given by

$$a \cdot ([t \mapsto f_t]) = [t \mapsto ([N(a)]_\infty^{-1} f_{a^{-1}}(a^{-1}t))].$$

This is explained in [Cu-Li2]. The $R^\times$-action on $C(\mathcal{R}) \rtimes R$ is given by

$$a \cdot h(z, d) = \begin{cases} h(a^{-1} z, a^{-1} d) & \text{if } z \in a\mathcal{R} \text{ and } d \in aR, \\ 0 & \text{else} \end{cases}$$

for $h \in C_*(\mathcal{R} \rtimes R) \subseteq C(\mathcal{R}) \rtimes R$. Note that (15) gives an action of $R^\times$ via automorphisms, whereas (16) yields an $R^\times$-action by endomorphisms.

So our task is to show that the imprimitivity bimodule constructed in [Cu-Li2], Section 4 implementing (14) is $R^\times$-equivariant with respect to the $R^\times$-actions described in (15) and (16). In other words, we need to find an $R^\times$-action on our imprimitivity bimodule. The point is that such an $R^\times$-action is actually canonically given by construction, as we will see.
Let us first recall how this imprimitivity bimodule was constructed. An important role is played by the transformation groupoid $\mathcal{G}$ associated with the (right) action of $\mathbb{A}_\infty$ on $\mathbb{A}/K$ given by $[r, x] \cdot s = [r + s, x]$ for $r, s \in \mathbb{A}_\infty$ and $x \in \mathbb{A}_f$. Here $[r, x]$ denotes the equivalence class of $(r, x) \in \mathbb{A}$ in $\mathbb{A}/K$. It is shown in [Cu-Li2], Section 4 that for $N := \{(0, z) : z \in \mathbb{R}\} \subseteq \mathbb{A}/K$, the groupoids $\mathcal{G}$ and $G_N^N$ are equivalent. Theorem 2.8 in [M-R-W] then yields a concrete $C_c(G)-C_c(G_N^N)$-pre-imprimitivity bimodule. Here $C_c(G)$ is viewed as a dense sub-$*-$algebra of $C^*(G)$, and $C_c(G_N^N)$ is viewed as a dense sub-$*-$algebra of $C^*(G_N^N)$. Furthermore, it is known that the identity map on $C_c((\mathbb{A}/K) \times \mathbb{A}_\infty)$ extends to a canonical isomorphism $C(\mathbb{A}/K) \times \mathbb{A}_\infty \cong C^*(G)$ (see Remark (ii) after Definition 1.12 in Chapter II of [Ren]). Moreover, it is shown in [Cu-Li2] that $G_N^N$ is isomorphic, as a topological groupoid, to the transformation groupoid associated with the (right) action of $R$ on $\mathbb{R}$ given by $z \cdot d = z - d$. Thus we can identify $C(\mathbb{R}) \times R$ with $C^*(G_N^N)$ by sending $h \in C_c(\mathbb{R} \times R) \subseteq C(\mathbb{R}) \times R$ to $[(0, z), d] \mapsto h(z, d)] \in C_c((G_N^N) \subseteq C^*(G_N^N)$. Under this identification, the $R^\times$-action (16) on $C(\mathbb{R}) \times R$ corresponds to the $R^\times$-action on $C^*(G_N^N)$ determined by

$$
(17) \quad (a \cdot h)([0, z], d) = \begin{cases} h([0, a^{-1}z], a^{-1}d) & \text{if } z \in a\mathbb{R} \text{ and } d \in aR, \\ 0 & \text{else.} \end{cases}
$$

Now let $X$ denote the concrete $C_c(G)-C_c(G_N^N)$-pre-imprimitivity bimodule from the proof of Theorem 2.8 in [M-R-W]. In the following, we give explicit formulas for the inner products of $X$: The underlying $C$-vector space is $C_c(G_N^N)$ where $G_N = \{(r, x), t \in \mathcal{G} : (r + t, x) \in N\}$. The $C_c(G)$-valued inner product is given by

$$
(18) \quad C_c(G) \langle \phi_1, \phi_2 \rangle ([r, x], t) = \sum_{d \in K\cap(r+x+\mathbb{R})} \phi_1([r, x], -r + d) \overline{\phi_2([r + t, x], -(r + t) + d)}. \tag{18}
$$

The $C_c(G_N^N)$-valued inner product is given by

$$
(19) \quad \langle \phi_1, \phi_2 \rangle_{C_c(G_N^N)}((0, z), b) = \int_{\mathbb{A}_\infty} \overline{\phi_1([t, z], -t)\phi_2([t, z], -t + b)} dt. \tag{19}
$$

Moreover, we define an $R^\times$-action on $X$ as follows: For $a \in R^\times$ and $\phi \in X$ set

$$
(20) \quad a \cdot \phi([r, x], t) = \begin{cases} ([\mathbb{N}(a)\mathbb{A}_\infty]^{-\frac{1}{2}} \phi([a^{-1}r, a^{-1}x], a^{-1}t) & \text{if } x - (r + t) \in a\mathbb{R}, \\ 0 & \text{else.} \end{cases} \tag{20}
$$

**Lemma 3.23.** The $C_c(G)-C_c(G_N^N)$-pre-imprimitivity bimodule $X$ together with the $R^\times$-action given by (20) is an $R^\times$-equivariant $C_c(G)-C_c(G_N^N)$-pre-imprimitivity bimodule in the sense of Remark 3.16 with respect to the $R^\times$-actions on $C_c(G)$ and $C_c(G_N^N)$ given by (15) and (17).

**Proof.** We just have to establish (5) and (6) for $X$ and the $R^\times$-actions given by (15), (17) and (20). Let us start with the $C_c(G)$-valued inner product given by (18).

$$
(21) \quad C_c(G) \langle a \cdot \phi_1, a \cdot \phi_2 \rangle ([r, x], t) = \sum_d (a \cdot \phi_1)([r, x], -r + d)(a \cdot \overline{\phi_2})([r + t, x], -(r + t) + d) = |\mathbb{N}(a)|^{-\frac{1}{2}} \sum_d \left\{ \phi_1([a^{-1}r, a^{-1}x], -a^{-1}r + a^{-1}d) \overline{\phi_2([a^{-1}r + a^{-1}t, a^{-1}x], -(r + t) + a^{-1}d)} \right\}. \tag{21}
$$
Moreover,

\[ \langle a \cdot c_\phi (\phi_1, \phi_2) (\langle x, r \rangle, t) = |N(a)|^{-1} \sum_{d} \int_{A_\infty} \overline{\phi_2}(a^{-1}r + a^{-1}t, a^{-1}x), -a^{-1}(r + t) + d') \]. \]

\[ = |N(a)|^{-1} \sum_{d} \{ \phi_1([a^{-1}r, a^{-1}x], -a^{-1}r + d') \}. \]

By (18), the sum in (21) is taken over all \( d \in \mathbb{K} \cap \langle x + \mathbb{R} \rangle \). However, because of (20), only those \( d \in \mathbb{K} \) satisfying \( x - d \in a\mathbb{R} \Leftrightarrow d \in x + a\mathbb{R} \) will yield non-zero summands. Thus the sum in (21) is really taken over all \( d \in \mathbb{K} \cap \langle x + \mathbb{R} \rangle \). The sum in (22) is taken over all \( d' \in \mathbb{K} \cap \langle x^{-1}x + \mathbb{R} \rangle \). Now set \( d' = a^{-1}d \) to see that (21) and (22) coincide.

It remains to study the \( C_\infty(G^N_X) \)-valued inner product given by (19). We compute

\[ \langle a \cdot \phi_1, a \cdot \phi_2 \rangle_{C_\infty(G^N_X)} ([0, z], b) = \int_{A_\infty} \overline{\phi_1}([a^{-1}t, a^{-1}z], -a^{-1}t + a^{-1}b)dt \]

\[ = |N(a)|^{-1} \sum_{d} \{ \phi_1([a^{-1}r, a^{-1}x], -a^{-1}r + d') \}. \]

\[ = \int_{A_\infty} \overline{\phi_1}([a^{-1}t, a^{-1}z], -a^{-1}t + a^{-1}b)dt \]

if \( z \in a\mathbb{R} \wedge z - b \in a\mathbb{R} \Leftrightarrow z \in a\mathbb{R} \wedge b \in a\mathbb{R} \cap R = aR \). Otherwise \( \langle a \cdot \phi_1, a \cdot \phi_2 \rangle_{C_\infty(G^N_X)} ([0, z], b) = 0 \). Comparing this with

\[ a \cdot \langle \phi_1, \phi_2 \rangle_{C_\infty(G^N_X)} ([0, z], b) = \begin{cases} \langle \phi_1, \phi_2 \rangle_{C_\infty(G^N_X)} ([0, a^{-1}z], a^{-1}b) & \text{if } z \in a\mathbb{R} \wedge b \in a\mathbb{R} \\ 0 & \text{else} \end{cases} \]

\[ = \begin{cases} \int_{A_\infty} \overline{\phi_1}([t, a^{-1}z], -t) \phi_2([t, a^{-1}z], -t + a^{-1}b)dt & \text{if } z \in a\mathbb{R} \wedge b \in a\mathbb{R}, \\ 0 & \text{else,} \end{cases} \]

we see that \( \langle a \cdot \phi_1, a \cdot \phi_2 \rangle_{C_\infty(G^N_X)} = a \cdot \langle \phi_1, \phi_2 \rangle_{C_\infty(G^N_X)}. \)

Let \( X \) be the \( C(\mathbb{A}/\mathbb{K}) \times \mathbb{A}_\infty \)-\( C(\mathbb{R}) \times R \)-imprimitivity bimodule obtained from \( X \) by completing.

**Corollary 3.24.** The \( R^* \)-action on \( X \) extends to an \( R^* \)-action on \( X \) so that \( X \) and this extended action form an \( R^* \)-equivariant \( C(\mathbb{A}/\mathbb{K}) \times \mathbb{A}_\infty \)-\( C(\mathbb{R}) \times R \)-imprimitivity bimodule.

**Proof.** See Remark 3.16. \( \square \)

**Corollary 3.25.** For every (multiplicative) subsemigroup \( S \subseteq R^* \), let \( \langle S \rangle \) be the subgroup of \( K^* \) generated by \( S \). We then have

\[ C_0(\mathbb{A}_\infty) \rtimes K \rtimes \langle S \rangle \sim_M C(\mathbb{R}) \rtimes R \rtimes S. \]
Proof. Since the imprimitivity bimodule $X$ from above is $R^\times$-equivariant, it is also $S$-equivariant for every subsemigroup $S$ of $R^\times$ (just restrict the action). By Remark 3.3, the extendibility conditions in Theorem 3.13 are satisfied in our situation. Thus Theorem 3.13 implies that $C_0(A_\infty) \rtimes K \cong S \cong_M C(X) \rtimes R \cong S$. As $S$ acts on $C_0(A_\infty) \rtimes K$ by automorphisms, we can canonically identify $C_0(A_\infty) \rtimes K \cong S$ with $C_0(A_\infty) \rtimes K \cong \langle S \rangle$. This proves our claim.

If we apply the construction from Paragraph 3.3 to the $C^*$-dynamical system given by the multiplicative $S$-action on $C(X) \rtimes R$, we will obtain the dilated system (the system $(A_\infty, G, \alpha^{(\infty)})$ in the notation from Paragraph 3.3) given by the multiplicative action of $\langle S \rangle$ on $C_0(\langle S \rangle \cdot X) \rtimes \langle (S) \cdot R \rangle$. Thus Corollary 3.9 yields

$$C(X) \rtimes R \cong S \cong_M C_0(\langle S \rangle \cdot X) \rtimes \langle (S) \cdot R \rangle \rtimes \langle S \rangle.$$ 

Combining this with the last corollary, we get

$$C_0(A_\infty) \rtimes K \rtimes \langle S \rangle \cong_M C_0(\langle S \rangle \cdot X) \rtimes \langle (S) \cdot R \rangle \rtimes \langle S \rangle.$$

As a special case, namely for $S = R^\times$, we obtain

$$C_0(A_\infty) \rtimes K \rtimes K^\times \cong_M C(X) \rtimes R \rtimes R^\times \cong_M C(A_\infty) \rtimes K \rtimes K^\times.$$

Remark 3.26. So we have reproven Theorem 4.1 in [Cu-Li2] for $\Gamma = \langle S \rangle$. Actually, it turns out that the “new” imprimitivity bimodules we obtained in our new proof can be canonically identified with the “old” ones from [Cu-Li2]. We just have to compare our new actions and inner products (compare Remark 3.17) with the corresponding ones from [M-R-W] used in the proof of Theorem 4.1 in [Cu-Li2].

By Corollary 3.24, we know that $C(A_\infty / K) \rtimes A_\infty$ and $C(X) \rtimes R$ are $R^\times$-equivariantly Morita equivalent. Now apply Corollary 3.21 to $A = C(A_\infty / K) \rtimes A_\infty$, $B = C(X) \rtimes R$, $Q = \mu$ and $P = R^\times$ to deduce

Corollary 3.27. $C(A_\infty / K) \rtimes A_\infty \rtimes \mu$ and $C(X) \rtimes R \rtimes \mu$ are $R^\times$-equivariantly Morita equivalent.

We deduce the following useful consequence

Corollary 3.28. The isomorphism from Section 4 in [Cu-Li2] implementing (13) and the particular imprimitivity bimodule $X$ implementing (14) give rise to an $R^\times$-equivariant K-theoretic isomorphism $[X] : K_*(C_0(A_\infty) \rtimes K \rtimes \mu) \cong K_*(C(X) \rtimes R \rtimes \mu)$, in the sense that for every $a \in R^\times$, $[X] \circ (\beta_a)_* = (\beta^{(fin)}_a)_* \circ [X]$. Here $\beta$ and $\beta^{(fin)}$ denote the multiplicative $R^\times$-actions on $C_0(A_\infty) \rtimes K \rtimes \mu$ and $C(X) \rtimes R \rtimes \mu$, respectively.

Proof. This follows from equivariance of the particular isomorphism from Section 4 in [Cu-Li2] implementing (13) and the last corollary together with Lemma 3.22.

3.6. Variations of the duality theorem. For the sake of completeness, let us briefly discuss variations of the original duality theorem. These variations are not used in the sequel. Moreover, we point out that as in the case of the original duality theorem, the variations can be proven directly as in [Cu-Li2] or with the help of the notion of semigroup equivariant Morita equivalences developed in this paper.
Proposition 3.29. Let $K$ be a global field. Then for all natural numbers $n$ and all subgroups $\Gamma$ of $GL_n(K)$,
\[ C_0(M_n(\mathbb{A}_K)) \otimes M_n(K) \times \Gamma \sim M_n(\mathbb{R}) \times (\Gamma \cdot M_n(R)) \times \Gamma. \]
Here $\Gamma$ acts on $C_0(M_n(\mathbb{A}_K)) \otimes M_n(K)$ multiplicatively from the left, whereas $\Gamma$ acts on $C_0(\Gamma \cdot M_n(R)) \times (\Gamma \cdot M_n(R))$ via right multiplications.

Proof. We can carry over the proof of the original duality theorem in Section 4 of [Cu-Li2]. We just have to replace the pairing from Theorem 2.4 in [Cu-Li2] by
\[ M_n(\mathbb{A}_K) \times M_n(\mathbb{A}_K) \to \mathbb{T}, (A, B) \mapsto \chi(\text{Tr}(AB)). \]
Here $\text{Tr}$ denotes the canonical (unnormalized) trace on $n \times n$-matrices, and $\chi$ is a non-trivial character on $\mathbb{A}$ which is trivial on $K$ as in Theorem 2.4 from [Cu-Li2]. The rest is analogous to the proof of Theorem 4.1 in [Cu-Li2]. \qed

Note that since $\Gamma$ is in general not commutative, it is necessary to distinguish between left and right actions.

Here is another variation of the duality theorem. Let $R$ be the ring of integers in a number field $K$, and let $P$ be a subsemigroup of $R^\times$ and set $S := \{ v \text{ finite place of } K: P_v \cap P \neq \emptyset \}$ where $P_v = \{ x \in K: |x|_v < 1 \}$.

Proposition 3.30. Assume that there exists a character $\chi$ on $\mathbb{A}_\infty \times \left( \prod_{v \in S} K_v \right)$ such that the pairing $(\mathbb{A}_\infty \times \left( \prod_{v \in S} K_v \right)) \times (\mathbb{A}_\infty \times \left( \prod_{v \in S} K_v \right)) \to \mathbb{T}, (x, y) \mapsto \chi(xy)$ induces the identifications $(\mathbb{A}_\infty \times \left( \prod_{v \in S} K_v \right))/P^{-1} R \cong \hat{P}^{-1} \hat{R}$ and $\mathbb{A}_\infty \cong \hat{\mathbb{A}}_\infty$. Then
\[ C_0(\mathbb{A}_\infty) \times (P^{-1} R) \times (P) \sim M C_0(\prod_{v \in S} K_v) \times (P^{-1} R) \times (P). \]

Here $\prod'$ stands for restricted direct product, i.e., for all but finitely many finite places $v \in S$, the $v$-th coordinate has to lie in the maximal compact subring (also called the valuation ring) of $K_v$.

Proof. Just proceed as in the proof of the original duality theorem (see [Cu-Li2], § 4). \qed

It is not clear to the authors whether there always exists a character which satisfies the conditions of Proposition 3.30. But if we can write Dedekind’s complementary module $\mathfrak{c} = \mathfrak{C}_{R\mathbb{A}}$ (in the sense of Definition (2.1) in [Neu], Chapter III, § 2) as $(aR) \cdot Q$ for $a \in K^\times$ and a fractional ideal $Q$ of the form $Q = \prod_{v \in S} P_v^{m_v}$, then we can construct a character with the desired properties following Tate’s thesis (compare [Tate] or [Lang], Chapter XIV).

For example, taking $K = \mathbb{Q}$ and $P = [2]$ as in [Lar-Li], we have in this case $\mathbb{A}_\infty = \mathbb{R}$, $R = \mathbb{Z}$, $P^{-1} R = \mathbb{Z}[1/2]$ and $\prod_{v \in S} K_v = \mathbb{Q}_2$. A character with the desired properties is obtained as follows: The inclusion $\mathbb{Z}[1/2] \hookrightarrow \mathbb{Q}_2$ induces an isomorphism $\mathbb{Z}[1/2]/\mathbb{Z} \cong \mathbb{Q}_2/\mathbb{Z}_2$. Composing the inverse of this isomorphism and the canonical projection $\mathbb{Q}_2 \twoheadrightarrow \mathbb{Q}_2/\mathbb{Z}_2$, we obtain $p: \mathbb{Q}_2 \twoheadrightarrow \mathbb{Q}_2/\mathbb{Z}_2 \cong \mathbb{Z}[1/2]/\mathbb{Z}$. Then set $\chi(v, x) = \exp(2\pi iv) \cdot \exp(-2\pi ip(x))$. 

\[
\chi(\nu, x) = \exp(2\pi i\nu) \cdot \exp(-2\pi ip(x)).
\]
Compare also [Lang], Chapter XIV (our $p$ is called $\lambda_0$ in [Lang], Chapter XIV, § 1). So the last proposition yields

$$C_0(R) \times \mathbb{Z}[\frac{1}{2}] \times \langle 2 \rangle \sim_M C_0(\mathbb{Q}_2) \times \mathbb{Z}[\frac{1}{2}] \times \langle 2 \rangle.$$  

This is Theorem 7.5 in [Lar-Li].

As S. Neshveyev pointed out to the first named author, it is also possible to deduce

$$C(\mathfrak{A}/K) \times C(\mathfrak{R}) \times \mathbb{R}$$

by applying Green’s imprimitivity theorem (see [Wil], Corollary 4.11) to the commuting (additive) actions of $\mathfrak{A}_\infty$ and $\mathbb{R}$ on $\mathfrak{A}_\infty \times \mathbb{R}$ (using $\mathfrak{A}/K \cong (\mathfrak{A}_\infty \times \mathbb{R})/R$). It is then possible to deduce the last variation:

**Proposition 3.31.** Given any set of places $S$ of our global field $K$ and any subgroup $\Gamma$ of $K^\times$, we have

$$C_0(\prod_{v \in S} K_v) \times K \rtimes \Gamma \sim_M C_0(\prod_{v \notin S} K_v) \times K \rtimes \Gamma.$$  

**Proof.** As in the original proof, we need two ingredients. The first is the identification

$$C_0(\prod_{v \in S} K_v) \times K \rtimes \Gamma \cong C(\mathfrak{A}/K) \times (\prod_{v \in S} K_v) \times \Gamma$$

which can be proven as in [Cu-Li2] using Lemma 4.3 in [Cu-Li2]. The second ingredient is to deduce

$$C(\mathfrak{A}/K) \times (\prod_{v \in S} K_v) \rtimes \Gamma \sim_M C_0(\prod_{v \notin S} K_v) \times \Gamma \rtimes \Gamma.$$  

This can be done by first applying Green’s imprimitivity theorem to the commuting (additive) actions of $\prod_{v \in S} K_v$ and $K$ on $\mathfrak{A} = (\prod_{v \in S} K_v) \times (\prod_{v \notin S} K_v)$ and then observing that the resulting imprimitivity bimodule is $\Gamma$-equivariant. Alternatively, we could also simply apply Green’s imprimitivity theorem to the commuting actions of $(\prod_{v \in S} K_v) \rtimes \Gamma$ and $K \rtimes \Gamma$ on $\mathfrak{A} \rtimes \Gamma$ in order to directly deduce $C(\mathfrak{A}/K) \times (\prod_{v \in S} K_v) \rtimes \Gamma \sim_M C_0(\prod_{v \notin S} K_v) \times K \rtimes \Gamma$. Putting together these two ingredients, we obtain our result. □

**Remark 3.32.** Note that all these Morita equivalences are equivariant with respect to the canonical $\text{Aut}(K/\mathbb{Q})$- or $\text{Aut}(K/\mathbb{F}_p(T))$-actions in case $K$ is a number or function field. The reason is that each of the imprimitivity bimodules we have constructed carries a canonical $\text{Aut}(K/\mathbb{Q})$- or $\text{Aut}(K/\mathbb{F}_p(T))$-action which is compatible with the actions on the $C^*$-algebras (since $\text{Aut}(K/\mathbb{Q})$ or $\text{Aut}(K/\mathbb{F}_p(T))$ acts on $\mathfrak{A}_\infty$ and $\mathfrak{A}_f$).

4. **K-theory for certain group C*-algebras**

We now turn to the case of a number field $K$ and present the proof of Theorem 1.1. Let $R$ be the ring of integers in $K$. Moreover, let $\mu$ be the group of roots of unity in $K$. This group is always a finite cyclic group generated by a root of unity, say $\zeta$.

The starting point for our K-theoretic computations is the work of M. Langer and the second named author on the K-theory of certain group C*-algebras. More precisely, in [La-Lui], the K-theory of the group C*-algebra of $R \rtimes \mu$ has been computed. Here $R \rtimes \mu$ is the semidirect product obtained from the multiplicative action of $\mu$ on the additive group $(R, +)$. The corresponding group C*-algebra is denoted by $C^*(R \rtimes \mu)$. It is very useful for our purposes that it is even possible to give an almost complete list of generators for the corresponding K-groups. Let us now summarize the results from [La-Lui].

To do so, we first need to introduce some notation. The additive group $(R, +)$ of our ring $R$ sits as a subgroup in $R \rtimes \mu$. Let $\iota: R \to R \rtimes \mu$ be the canonical inclusion,
and denote by \( \iota_* \) the homomorphism \( \K_0(C^*(R)) \to \K_0(C^*(R \rtimes \mu)) \) induced by \( \iota \) on \( \K_0 \). Moreover, given a finite subgroup \( M \) of \( R \rtimes \mu \), consider the canonical projection \( M \to \{ e \} \) from \( M \) onto the trivial group. This projection induces a homomorphism \( C^*(M) \to C \) of the group \( C^* \)-algebras, hence a homomorphism on \( \K_0: \K_0(C^*(M)) \to \K_0(C) \). Let us denote the kernel of this homomorphism by \( \bar{R}_C(M) \). The canonical inclusion \( M \to R \rtimes \mu \) induces a homomorphism \( \iota_M : C^*(M) \to C^*(R \rtimes \mu) \), hence a homomorphism \( \K_0(C^*(M)) \to \K_0(C^*(R \rtimes \mu)) \).

Restricting this homomorphism to \( \bar{R}_C(M) \), we obtain \( (\iota_M)_* : \bar{R}_C(M) \to \K_0(C^*(R \rtimes \mu)) \). Here are the main results from [La-Lü]:

**Theorem 4.1** (Langer-Lück). With the notations from above, we have

\[ (*) \quad \K_0(C^*(R \rtimes \mu)) \text{ is finitely generated and torsion-free.} \]

\[ (**) \quad \text{Let } \mathcal{M} \text{ be the set of conjugacy classes of maximal finite subgroups of } R \rtimes \mu. \text{ Then } \sum_{(M) \in \mathcal{M}} (\iota_M)_* : \Theta_{(M) \in \mathcal{M}} \bar{R}_C(M) \to \K_0(C^*(R \rtimes \mu)) \text{ is injective, i.e., for every } (M) \in \mathcal{M}, \text{ the map } (\iota_M)_* \text{ is injective and for every } (M_1), (M_2) \in \mathcal{M} \text{ with } (M_1) \neq (M_2), \text{ we have } \im((\iota_{M_1})_*) \cap \im((\iota_{M_2})_*) = \{0\}. \text{ Moreover, } \im((\iota_*)) \cap \left( \sum_{(M) \in \mathcal{M}} \im((\iota_M)_*) \right) = \{0\} \text{ and } \im((\iota_*)) + \left( \sum_{(M) \in \mathcal{M}} \im((\iota_M)_*) \right) \text{ is of finite index in } \K_0(C^*(R \rtimes \mu)). \]

\[ (***) \quad \K_1(C^*(R \rtimes \mu)) \text{ vanishes.} \]

**Proof.** \( (*) \) is Theorem 0.1, (iii) in [La-Lü]. \( (**) \) is Theorem 0.1, (ii) in [La-Lü]. Note that the maps \( \iota_M \) in our notation are denoted by \( \iota_M \) in [La-Lü], and that \( \iota \) in our notation is denoted by \( k \) in [La-Lü]. The group \( \bar{R}_C(M) \) coincides with the corresponding one in [La-Lü] upon the canonical identification of the representation ring \( \bar{R}_C(M) \) of \( M \) with \( \K_0(C^*(M)) \) as abelian groups. Furthermore, \( (***) \) is Theorem 0.1, (iv) in [La-Lü].

Let us now describe \( \K_0(C^*(R \rtimes \mu)) \) in a way which is most convenient for our K-theoretic computations. The idea is to use \( (*) \) and \( (**) \) from Theorem 4.1 to decompose \( \K_0(C^*(R \rtimes \mu)) \) into direct summands. However, we cannot simply use the subgroups \( \im((\iota_*)) \) and \( \im((\iota_M)_*) \) for \( (M) \in \mathcal{M} \) which appear in \( (**) \) because these subgroups might not be direct summands. To solve this problem, we proceed as follows: First of all, we set

\[ (25) \quad K_{\text{nf}} := \{ x \in \K_0(C^*(R \rtimes \mu)) : \exists \ N \in \mathbb{Z}_{>0} \text{ such that } N x \in \im((\iota_*)) \}. \]

Now take a finite subgroup \( M \) of \( R \rtimes \mu \). It has to be a cyclic group. Let \( (b, \zeta^i) \) in \( R \rtimes \mu \) be a generator of \( M \). Note that \( i = m/|M| \) (up to multiples of \( m \)), where \( m = |\mu| \). Let \( \chi \) be a character of \( \mathbb{Z}/|M|\mathbb{Z} \), and denote by \( p_\chi(u^b s^j \zeta^i) \) the spectral projection \( \frac{1}{|M|} \sum_{j=0}^{M-1} \chi(j + |M|\mathbb{Z})(u^b s^j \zeta^i) \). Then \( \im((\iota_M)_*) \) is generated by \( \{ p_\chi(u^b s^i \zeta^i) : 1 \neq \chi \in \mathbb{Z}/|M|\mathbb{Z} \} \). Here \( [\cdot] \) denotes the \( K_0 \)-class of the projection in question and \( 1 \in \mathbb{Z}/|M|\mathbb{Z} \) is the trivial character.

It is then clear that the \( K_0 \)-classes \( \{ p_\chi(u^b s^i \zeta^i) \} \) for \( (\mu) \neq (M) \in \mathcal{M}, M = \langle (b, \zeta^i) \rangle \) and \( 1 \neq \chi \in \mathbb{Z}/|M|\mathbb{Z} \) form a \( \mathbb{Z} \)-basis of \( \sum_{(\mu) \neq (M) \in \mathcal{M}} \im((\iota_M)_*) \). Let us enumerate these \( K_0 \)-classes \( \{ p_\chi(u^b s^i \zeta^i) \} \) by \( y_1, y_2, y_3, \ldots, y_{f_{\text{nf}}} \), where \( f_{\text{nf}} \) is the rank of
\[ \sum_{(\mu) \neq (M) \in M} \text{im} \left( (\iota_{M})_{*} \right). \] As \( K_0(C^*(R \times \mu)) \) is free abelian, we can recursively find \( \mathcal{g}_1, \mathcal{g}_2, \mathcal{g}_3, \ldots, \mathcal{g}_{r_{k_{f_{\text{fin}}}}} \) in \( K_0(C^*(R \times \mu)) \) such that for every \( 1 \leq j \leq r_{k_{f_{\text{fin}}}}, \)

\[
K_{\text{fin}} + \text{span} \left\{ \mathcal{g}_1, \mathcal{g}_2, \mathcal{g}_3, \ldots, \mathcal{g}_{r_{k_{f_{\text{fin}}}}} \right\} = \left\{ x \in K_0(C^*(R \times \mu)) : \exists N \in \mathbb{Z}_{>0} \text{ such that } Nx \in K_{\text{fin}} + \left\langle y_1, \ldots, y_j \right\rangle \right\}.
\]

By construction, these elements \( \mathcal{g}_1, \mathcal{g}_2, \mathcal{g}_3, \ldots, \mathcal{g}_{r_{k_{f_{\text{fin}}}}} \) are linearly independent. We set \( K_{\text{fin}}^{\mathcal{g}} = \left\langle \mathcal{g}_1, \mathcal{g}_2, \mathcal{g}_3, \ldots, \mathcal{g}_{r_{k_{f_{\text{fin}}}}} \right\rangle \). By construction, \( K_{\text{fin}} \cap K_{\text{fin}}^{\mathcal{g}} = \{0\} \). Finally, \( \{[p_\chi(s_\zeta)] : 1 \neq \chi \in \mathbb{Z}/m\mathbb{Z} \} \) is a \( \mathbb{Z} \)-basis of \( \text{im}((\iota_{\mu})_*) \). Enumerate the elements \( [p_\chi(s_\zeta)] \), \( 1 \neq \chi \in \mathbb{Z}/m\mathbb{Z} \), by \( z_1, \ldots, z_{m-1} \). Again, there exist \( \tau_1, \ldots, \tau_{m-1} \) in \( K_0(C^*(R \times \mu)) \) with the property that for every \( 1 \leq l \leq m-1, \)

\[
K_{\text{fin}} + \mathcal{g}_{r_{k_{f_{\text{fin}}}}} + \langle \tau_1, \ldots, \tau_l \rangle = \left\{ x \in K_0(C^*(R \times \mu)) : \exists N \in \mathbb{Z}_{>0} \text{ s.t. } Nx \in K_{\text{fin}} + \mathcal{g}_{r_{k_{f_{\text{fin}}}}} + \langle z_1, \ldots, z_l \rangle \right\}.
\]

It is again clear that \( \tau_1, \ldots, \tau_{m-1} \) are linearly independent. We set \( K_{f_{\text{fin}}} = \langle \tau_1, \ldots, \tau_{m-1} \rangle \). By construction, we have \( (K_{\text{fin}} + \mathcal{g}_{r_{k_{f_{\text{fin}}}}}) \cap K_{f_{\text{fin}}}^{\mathcal{g}} = \{0\} \). Thus \( K_0(C^*(R \times \mu)) = K_{\text{fin}} \oplus K_{f_{\text{fin}}}^{\mathcal{g}} \oplus K_{f_{\text{fin}}}^{\mathcal{g}} \) as \( \text{im}((\iota_{\mu})_*) \) is of finite index in \( K_0(C^*(R \times \mu)) \) by (***) from Theorem 4.1.

5. K-theory for ring \( C^* \)-algebras

Our goal is to prove Theorem 1.1. As explained in Paragraph 2.1, we will not distinguish between \( \mathfrak{A}_n[R] \) and \( \mathfrak{A}[R] \).

5.1. The strategy. Let us recall the strategy of the previous K-theoretic computations from [Cu-Li2] for number field without higher roots of unity. We will use the same strategy to treat the case of higher roots of unity.

The first step is to compute K-theory for the sub-C*-algebra \( C^*(\{u^b\}, s_\zeta, \{e_\alpha\}) \) of \( \mathfrak{A}[R] \). Recall that \( e_\alpha \) is the range projection of \( s_\alpha \), i.e., \( e_\alpha = s_\alpha s_\alpha^* \). This sub-C*-algebra can be identified with the inductive limit of the system given by the algebras \( C^*(\{u^b\}, s_\zeta, e_\alpha) \) for \( a \in R^\times \). Moreover, we can prove that for fixed \( a \in R^\times \), the algebra \( C^*(\{u^b\}, s_\zeta, e_\alpha) \) is isomorphic to a matrix algebra over \( C^*(\{u^b\}, s_\zeta) \cong C^*(R \times \mu) \). In this situation, Theorem 4.1 allows us to compute K-theory for \( C^*(\{u^b\}, s_\zeta, \{e_\alpha\}) \) using its inductive limit structure.

The next step is to use the duality theorem from [Cu-Li2] and Section 3 to pass over to the infinite adele space. The main point is to prove that the additive action of \( K \) is negligible for K-theory, i.e., that the canonical homomorphism

\[
C_0(\mathbb{A}_\infty) \times K^\times \to C_0(\mathbb{A}_\infty) \times K \times K^\times
\]

induces an isomorphism on K-theory, at least rationally. This is good enough once we can show that all the K-groups are torsion-free. At this point, we need to know that the canonical homomorphism \( C_0(\mathbb{A}_\infty) \times \mu \to C_0(\mathbb{A}_\infty) \times K \times \mu \) is injective on K-theory. The proof of this statement is postponed to Section 6.
The last step is to compute K-theory for $C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times$ using our results from the previous steps. The idea is to compare its K-theory with the K-theory of $C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times$ using homotopy arguments and the Pimsner-Voiculescu exact sequence. As we know that $\mathbb{A}[R]$ is Morita equivalent to $C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times$, we finally obtain the K-theory for the ring C*-algebra $\mathbb{A}[R]$.

5.2. Identifying inductive limits. The first step is to compute the K-theory of $C^*(\{u^b\}, s_\zeta, \{e_a\})$ using its inductive limit structure. Here $C^*(\{u^b\}, s_\zeta, \{e_a\})$ is the sub-C*-algebra of $\mathbb{A}[R]$ generated by $\{u^b: b \in R\}, s_\zeta$ and $\{e_a: a \in R^\times\}$. First of all, note that for all $a$ and $c$ in $R^\times$, we have $e_a = \sum_{b+c \in R/R} u^b e_a u^{-ab}$. Just conjugate Relation II in Paragraph 2.1 (for $c$ in place of $a$), $1 = \sum u^b e_c u^{-b}$, by $s_a$. Therefore, the C*-algebras $C^*(\{u^b\}, s_\zeta, e_a)$ for $a$ in $R^\times$ and the inclusion maps

$$\iota_{a,ac}: C^*(\{u^b\}, s_\zeta, e_a) \to C^*(\{u^b\}, s_\zeta, e_{ac})$$

form an inductive system. Here $R^\times$ is ordered by divisibility. It is clear that the inductive limit of this system can be identified with $C^*(\{u^b\}, s_\zeta, \{e_a\})$. Thus our goal is to compute the K-theory of $C^*(\{u^b\}, s_\zeta, \{e_a\})$ and to determine the structure maps $\iota_{a,ac}$ on K-theory. Note that $C^*(\{u^b\}, s_\zeta, e_a)$ is obtained from $C^*(\{u^b\}, s_\zeta)$ by adding one single projection $e_a$ and not the whole set of projections $\{e_a\}$.

Let $a$ and $c$ be arbitrary elements in $R^\times$. Choose a minimal system $\mathcal{R}_a$ of representatives for $R/aR$ in $R$. “Minimal” means that for arbitrary elements $b_1$ and $b_2$ in $\mathcal{R}_a$, the difference $b_1 - b_2$ lies in $aR$ (if and) only if $b_1 = b_2$. We always choose $\mathcal{R}_a$ in such a way that 0 is in $\mathcal{R}_a$.

Using the decomposition $R = \bigcup_{b \in \mathcal{R}_a} (b + aR)$ and the inverse of the isomorphism $\ell^2(R) \cong \ell^2(b + aR); \varepsilon_r \mapsto \varepsilon_{b+ar}$, we can construct the unitary

$$\ell^2(R) = \bigoplus_{b \in \mathcal{R}_a} \ell^2(b + aR) \cong \bigoplus_{\mathcal{R}_a} \ell^2(R).$$

Conjugation with this unitary gives rise to an isomorphism

$$\mathcal{L}(\ell^2(R)) \cong \mathcal{L}(\ell^2(R/aR)) \otimes \mathcal{L}(\ell^2(R)), T \mapsto \sum_{b, b' \in \mathcal{R}_a} e_{b, b'} \otimes (s_a^u u^{-b} T u^{b'} s_a).$$

Here $e_{b, b'}$ is the canonical rank 1 operator in $\mathcal{L}(\ell^2(R/aR))$ corresponding to $b + aR$ and $b' + aR$ sending a vector $\xi$ in $\ell^2(R/aR)$ to $\{\xi, \varepsilon_{b'+aR}\} \delta_{b+aR}$. In this formula, $\{\varepsilon_{b+aR}: b \in \mathcal{R}_a\}$ is the canonical orthonormal basis of $\ell^2(R/aR)$.

Let us denote the restriction of this isomorphism to $C^*(\{u^b\}, s_\zeta, e_{ac})$ by $\vartheta_{ac, c}$.

Lemma 5.1. For every $a$ and $c$ in $R^\times$, the image of $\vartheta_{ac, c}$ is $\mathcal{L}(\ell^2(R/aR)) \otimes (C^*(\{u^b\}, s_\zeta, e_c))$. Thus $\vartheta_{ac, c}$ induces an isomorphism

$$C^*(\{u^b\}, s_\zeta, e_{ac}) \cong \mathcal{L}(\ell^2(R/aR)) \otimes (C^*(\{u^b\}, s_\zeta, e_c)).$$

Since $R/aR$ is always finite, we know that $\mathcal{L}(\ell^2(R/aR))$ is just a matrix algebra. So it does not matter which tensor product we choose.
Proof. A direct computation yields
\[
\vartheta_{ac,c}(u^b e_a u^{-b'}) = e_{b,b'} \otimes 1 \text{ for all } b, b' \in R_a;
\]
\[
\vartheta_{ac,c}(u^{ab}) = 1 \otimes u^b \text{ for all } b \in R;
\]
\[
\vartheta_{ac,c}(\sum_{b \in R_a} u^b e_a u^{-cb} s_c) = 1 \otimes s_c;
\]
\[
\vartheta_{ac,c}(\sum_{b \in R_a} u^b e_{ac} u^{-b}) = 1 \otimes e_c.
\]

Our claim follows from the observation that \(C^*(\{u^b\}, s_c, e_{ac})\) is generated by
\[
u^b e_a u^{-b'} (b, b' \in R_a); u^{ab} (b \in R); \sum_{b \in R_a} u^b e_a u^{-cb} s_c \text{ and } \sum_{b \in R_a} u^b e_{ac} u^{-b}.
\]

\[\square\]

Let us now fix minimal systems of representatives \(R_a\) for every \(a\) in \(R^\times\) as explained before the previous lemma (we will always choose \(0 \in R_a\)). As \(R/aR\) is finite for every \(a\) in \(R^\times\), we know that \(\mathcal{L}(\ell^2(R/aR))\) is simply a matrix algebra of finite dimension. Thus we can use the previous lemma to identify \(K_0(C^*(\{u^b\}, s_c, e_{ac}))\) and \(K_0(C^*(\{u^b\}, s_c))\) via \((\rho_{1,a})_*^{-1}(\vartheta_{1,c})_*\). Here \(\rho_{c,a}\) (for \(a \in R \times c \in R^\times\)) is the canonical homomorphism
\[
C^*(\{u^b\}, s_c, e_{ac}) \to \mathcal{L}(\ell^2(R/aR)) \otimes (C^*(\{u^b\}, s_c, e_{ac})); x \mapsto e_{0,0} \otimes x.
\]

Lemma 5.2. We have
\[
(\rho_{1,a})_*^{-1}(\vartheta_{ac,c})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* = (\rho_{1,a})_*^{-1}(\vartheta_{ac,c})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_* (\vartheta_{ac,c}^{-1})_*.
\]

In other words, under the K-theoretic identifications above, the map \((\vartheta_{ac,c})_*\) corresponds to \((\vartheta_{ac,c})_*\). This observation is helpful because it says that we only have to determine the homomorphisms \(\vartheta_{1,c}\) on K-theory.

Proof. It is immediate that
\[
(\rho_{c,a}) = \vartheta_{ac,c} \circ \text{Ad} (s_a)
\]

as homomorphisms \(C^*(\{u^b\}, s_c, e_{ac}) \to \mathcal{L}(\ell^2(R/aR)) \otimes (C^*(\{u^b\}, s_c, e_{ac})).\) Here we mean by \(\text{Ad} (s_a)\) the homomorphism \(C^*(\{u^b\}, s_c, e_{ac}) \to C^*(\{u^b\}, s_c, e_{ac}); x \mapsto s_a x s_a^\ast.\) It would be more precise to write \(\text{Ad} (s_a)|_{C^*(\{u^b\}, s_c, e_{ac})}\), but it will become clear from the context on which domain \(\text{Ad} (s_a)\) is defined.

We know by Relation I in Paragraph 2.1 that
\[
\text{Ad} (s_a) \circ \text{Ad} (s_c) = \text{Ad} (s_{ac}).
\]

So, using (26), we can deduce from (27) that
\[
\vartheta_{ac,c}^{-1} \circ \rho_{c,a} \circ \vartheta_{ac,c}^{-1} \circ \rho_{1,c} \circ \vartheta_{ac,c}^{-1} \circ \rho_{1,a} \circ \vartheta_{ac,c}^{-1} \circ \rho_{1,a} \circ \vartheta_{ac,c}^{-1} \circ \rho_{1,a}.
\]

Moreover, \(\text{Ad} (s_a) \circ \vartheta_{1,c} = \vartheta_{ac,c} \circ \text{Ad} (s_a)\) and (26) imply
\[
\vartheta_{ac,c}^{-1} \circ \rho_{c,a} \circ \vartheta_{ac,c}^{-1} \circ \rho_{1,c} \circ \vartheta_{ac,c}^{-1} \circ \rho_{1,a} \circ \vartheta_{ac,c}^{-1} \circ \rho_{1,a}.
\]
Finally, we compute
\[
(\rho_{1,ac})^{-1}_*(\theta_{ac,1})_*((\tau_{a,ac})_*(\theta^{-1}_{a,1})_*(\rho_{1,a})_*) \\
\cong (28) (\rho_{1,c})^{-1}_*(\theta_{c,1})_*((\rho_{c,a})_*(\theta_{ac,c})_*((\tau_{a,ac})_*(\theta^{-1}_{a,1})_*(\rho_{1,a})_*) \\
\cong (29) (\rho_{1,c})^{-1}_*(\theta_{c,1})_*((\rho_{c,a})_*(\theta_{ac,c})_*((\theta^{-1}_{ac,c})_*(\rho_{c,a})_*((1,c)_* \\
= (\rho_{1,c})^{-1}_*(\theta_{c,1})_*((1,c)_*)
\]

Therefore it remains to determine
\[
(\rho_{1,c})^{-1}_*(\theta_{c,1})_*((1,c)_*) : K_0(C^*([u^b],s_\zeta)) \to K_0(C^*([u^b],s_\zeta))
\]

Let us denote this map by \( \eta_c \), i.e., \( \eta_c = (\rho_{1,c})^{-1}_*(\theta_{c,1})_*((1,c)_*) \). In conclusion, we have identified the K-theory of \( C^*([u^b],s_\zeta) \) with \( C^*(\{u^b\},s_\zeta) \) with the particular homomorphism \( \eta_c \).

5.3. The structure maps. First of all, we can canonically identify \( C^*(\{u^b\},s_\zeta) \) with \( C^*(R \rtimes \mu) \) because \( R \rtimes \mu \) is amenable. Therefore, all the results from Section 4 carry over to \( C^*(\{u^b\},s_\zeta) \). In the sequel, we use the same notations as in Section 4, but everything should be understood modulo this canonical isomorphism \( C^*(\{u^b\},s_\zeta) \cong C^*(R \rtimes \mu) \).

To determine \( \eta_c \), we use the decomposition \( K_0(C^*(\{u^b\},s_\zeta)) = K_{inf} \oplus K_{fin}^c \oplus K_{fin}^\mu \) with the particular \( \mathbb{Z} \)-basis \( \{ y_1, \ldots, y_{rk_{fin}}^c \} \) and \( \{ \tau_1, \ldots, \tau_{m-1} \} \) of \( K_{fin}^c \) and \( K_{fin}^\mu \), respectively (see § K(group-C)). Moreover, let \( [1] \in K_0(C^*(\{u^b\},s_\zeta)) \) be the \( K_0 \)-class of the unit in \( C^*(\{u^b\},s_\zeta) \) and denote by \( \langle [1] \rangle \) the subgroup of \( K_0(C^*(\{u^b\},s_\zeta)) \) generated by \( [1] \). As the canonical inclusion \( C \cdot 1 \hookrightarrow C^*(\{u^b\},s_\zeta) \) splits (a split is given by \( C^*(\{u^b\},s_\zeta) \cong C^*(R \rtimes \mu) \to C^*(\{e\}) \cong C \cdot 1 \), it is clear that \( \langle [1] \rangle \) is a direct summand of \( K_0(C^*(\{u^b\},s_\zeta)) \), hence of \( K_{inf} \).

Furthermore, note that it suffices to determine the structure maps \( \eta_c \) for \( c \in \mathbb{Z}_{>1} \) with the property that \( \prod_{i|m, 1 \leq i \leq m} (1 - \zeta^i) \) divides \( c \) because these elements form a cofinal set in \( R^\times \) with respect to divisibility.

Our goal is to prove

**Proposition 5.3.** There exists a subgroup \( K_{inf}^c \) of \( K_{inf} \) together with a \( \mathbb{Z} \)-basis \( \{ y_1, \ldots, y_{rk_{inf}}^c \} \) of \( K_{inf}^c \) such that \( K_{inf} = \langle [1] \rangle \oplus K_{inf}^c \) and that with respect to the \( \mathbb{Z} \)-basis \( \{ [1], y_1, \ldots, y_{rk_{inf}}^c, \bar{y}_1, \ldots, \bar{y}_{rk_{fin}}^c, \tau_1, \ldots, \tau_{m-1} \} \) of \( K_0(C^*(\{u^b\},s_\zeta)) \),
for every $c$ in $\mathbb{Z}_{>1}$ with the property that $\prod_{i|m,1\leq i<m}(1-\zeta^i)$ divides $c$, the homo-

morphism $\eta_c$ is of the form

$$
\begin{pmatrix}
    c^n & * & * \\
    0 & \ddots & * \\
    0 & 0 & c^2 \\
    0 & 0 & 1 \\
    0 & 0 & \ddots \\
    0 & 0 & 0 & 1
\end{pmatrix}.
$$

This matrix is subdivided according to the decomposition $K_0(C^*(\{u^b\}, s_\zeta)) = \langle[1]\rangle \oplus K_0^\mathcal{C} \oplus K_0^\mathcal{C} \oplus K_0^\mathcal{C}$. Moreover, the diagonal of the box describing the $K_0^\mathcal{C}$-part of this matrix consists of powers of $c$ with decreasing exponents. The least exponent $\gamma$ can be $0$ only if $n$ is even, and in that case, the $0$-th power $c^0$ can appear only once on the diagonal.

The proof of this proposition consists of two parts which are treated in the following two paragraphs.

5.3.1. The infinite part.

**Lemma 5.4.** For $c \in \mathbb{Z}_{>1}$, we have $\eta_c(K_{inf}) \subseteq K_{inf}$. Moreover, there is a subgroup $K_{inf}^\mathcal{C}$ of $K_{inf}$ and a $\mathbb{Z}$-basis $\{x_1, \ldots, x_{rk}^\mathcal{C}\}$ of $K_{inf}^\mathcal{C}$ such that $K_{inf} = \langle[1]\rangle \oplus K_{inf}^\mathcal{C}$ and, for every $c \in \mathbb{Z}_{>1}$, $\eta_c|_{K_{inf}}$, as a map $K_{inf} \rightarrow K_{inf}$, is of the form

$$
\begin{pmatrix}
    c^n & * \\
    0 & \ddots \\
    0 & 0 & c^2 \\
    0 & 0 & \ldots \\
    0 & 0 & 0 & 1
\end{pmatrix}
$$

with respect to the decomposition $K_{inf} = \langle[1]\rangle \oplus K_{inf}^\mathcal{C}$ and the chosen $\mathbb{Z}$-basis of $K_{inf}^\mathcal{C}$. Here, as in the proposition, $\gamma$ can be $0$ only if $n$ is even, and in that case, the $0$-th power $c^0$ can only appear at most once on the diagonal.

**Proof.** Let us choose a suitable $\mathbb{Z}$-basis for $K_{inf}$ and determine $\eta_c|_{K_{inf}}$. First of all, under the canonical identification $C^*(\{u^b\}, s_\zeta) \cong C^*(R \rtimes \mu)$, the sub-$C^*$-algebra $C^*(\{u^b\})$ corresponds to $C^*(R)$. So the inclusion map $\iota : C^*(R) \hookrightarrow C^*(R \rtimes \mu)$ corresponds to the canonical inclusion $C^*(\{u^b\}) \hookrightarrow C^*(\{u^b\}, s_\zeta)$ which we denote by $\iota$ as well. Let $\omega_1, \ldots, \omega_n$ be a $\mathbb{Z}$-basis for $R$ and let $u(i) := u^{\omega_i}$. Since $C^*(\{u^b\})$ is isomorphic to $C^*(R) \cong C^*(\mathbb{Z}_n)$ ($R$ is viewed as an additive group), a $\mathbb{Z}$-basis for $K_0(C^*(\{u^b\}))$ is given by

$\{[u(i_1)] \times \cdots \times [u(i_k)] : i_1 < \cdots < i_k, k \text{ even}\}$. 

Let \( \nu_c \) be the endomorphism on \( C^*\{\{u^b\}\} \) defined by \( \nu_c(u^b) = u^{cb} \). We have
\[
(\vartheta_{e,1} \circ \iota_{1,e} \circ \iota \circ \nu_e)(u^b) = \vartheta_{e,1}(u^{cb}) = 1 \otimes u^b
\]
(30) for all \( b \) in \( R \). Thus \( \vartheta_{e,1} \circ \iota_{1,e} \circ \iota \circ \nu_e = (1 \otimes \text{id}) \circ \iota \).

Proceeding inductively, we obtain that for every even number \( \eta \),
\[
(33) \eta_c(\iota_{s}([u(i_1)]_1 \times \cdots \times [u(i_k)]_1)) = c^{-k}(p_{1,c})^{-1}(\vartheta_{e,1})_{s}((\iota_{1,e})_{s}([u(i_1)]_1 \times \cdots \times [u(i_k)]_1))
\]
(34) \( = c^{-k}(p_{1,c})^{-1}(\vartheta_{e,1})_{s}((\iota_{1,e})_{s}([u(i_1)]_1 \times \cdots \times [u(i_k)]_1))
\]
(30) \( = c^{-k}(p_{1,c})^{-1}(1 \otimes [i_1])_{s}([u(i_1)]_1 \times \cdots \times [u(i_k)]_1)
\]
(32) \( = c^{-n}h_{s}([u(i_1)]_1 \times \cdots \times [u(i_k)]_1) \)

Now, let \( H_k \) be the subgroup of \( K_0(C^*\{\{u^b\}\}) \) generated by the \( K_0 \)-classes \([u(i_1)]_1 \times \cdots \times [u(i_k)]_1 \) for \( i_1 < \cdots < i_k \) where \( k \) is fixed. We have
\[
K_0(C^*\{\{u^b\}\}) = \bigoplus_{k \geq 0 \text{ even}} H_k.
\]

We claim that \( \ker(\iota_{s}) \) is compatible with this decomposition, i.e.,
\[
\ker(\iota_{s}) = \bigoplus_{k \geq 0 \text{ even}} (H_k \cap \ker(\iota_{s})).
\]

Proof of the claim:

Let \( h \) be in \( \ker(\iota_{s}) \). We can write
\[
h = \sum_{k \geq 0 \text{ even}} h_k
\]
(32) with \( h_k \in H_k \). We have to show that for every \( k \), the summand \( h_k \) lies in \( \ker(\iota_{s}) \).

Let us assume that there are at least two non-zero summands in (32), because otherwise, there is nothing to show. Now, equation (31) tells us that
\[
\eta_c(\iota_{s}([u(i_1)]_1 \times \cdots \times [u(i_k)]_1)) = c^{-k}(p_{1,c})^{-1}(\vartheta_{e,1})_{s}((\iota_{1,e})_{s}([u(i_1)]_1 \times \cdots \times [u(i_k)]_1))
\]
(33) \( = c^{-k}(p_{1,c})^{-1}(1 \otimes [i_1])_{s}([u(i_1)]_1 \times \cdots \times [u(i_k)]_1)
\]
(30) \( = c^{-n}h_{s}([u(i_1)]_1 \times \cdots \times [u(i_k)]_1) \)

on \( K_0(C^*\{\{u^b\}\}) \) = \( \bigoplus_k H_k \). Thus \( \bigoplus_k (c^{-n} \cdot \text{id}_{H_k}) \)(h) = \( \sum_k c^{-n} \cdot h_k \) lies in \( \ker(\iota_{s}) \) as well. This implies
\[
\ker(\iota_{s}) \ni c^n h - \bigoplus_k (c^{-n} \cdot \text{id}_{H_k})(h) = \sum_{k \geq 2 \text{ even}} (c^n - c^{-n}) \cdot h_k.
\]

Proceeding inductively, we obtain that for every even number \( j \geq 2 \),
\[
(34) \sum_{k \geq j \text{ even}} (c^{n-j} - c^{-n-j})(c^{n-2} - c^{-n}) \cdots (c^{n-j+2} - c^{-n}) \cdot h_k
\]
lies in \( \ker(\iota_{s}) \). Taking \( j \) to be the highest index for which the summand \( h_j \) in (32) is not zero, the term in (34) will be a non-zero multiple of the highest term in (32).

As both \( K_0(C^*\{\{u^b\}\}) \) and \( K_0(C^*\{\{u^b\} \cup s_j\}) \) are free abelian, we conclude that the highest term itself must lie in \( \ker(\iota_{s}) \). Working backwards, we obtain that for every \( k \), the summand \( h_k \) lies in \( \ker(\iota_{s}) \). This proves our claim.

Now, for every \( k \), \( H_k \cap \ker(\iota_{s}) = \ker(\iota_{s} \mid H_k) \) is a direct summand of \( H_k \) because \( K_0(C^*\{\{u^b\} \cup s_j\}) \) is free abelian. Thus we can choose subgroups \( I_k \) of \( H_k \) so that
\[
H_k = I_k \oplus (H_k \cap \ker(\iota_{s})).
\]
As \( \ker (\iota_\ast) = \bigoplus_k (H_k \cap \ker (\iota_\ast)) \), we have \( K_0(C^*(\{u^b\})) = (\bigoplus_k I_k) \oplus \ker (\iota_\ast) \). We can choose a \( \mathbb{Z} \)-basis for \( \bigoplus_k I_k \) in \( \bigcup_k H_k \). As \( H_0 \cap \ker (\iota_\ast) = \{0\} \), we have \( I_0 = H_0 = \langle [1] \rangle \) so that we can let \([1]\) be a basis element. Moreover, \( H_n \) is non-trivial only if \( n \) is even, and in that case \( \text{rk} (H_n) = 1 \) so that there is at most one basis element in \( H_n \).

Now \( \iota_\ast \) maps \( \bigoplus_k I_k \) isomorphically into \( \text{im} (\iota_\ast) \subseteq K_0(C^*(\{u^b\}, s_\zeta)) \), so that the \( \mathbb{Z} \)-basis of \( \bigoplus_k I_k \) chosen above is mapped to a \( \mathbb{Z} \)-basis of \( \text{im} (\iota_\ast) \). By (33), we know that if we order this \( \mathbb{Z} \)-basis in the right way (corresponding to the index \( k \)), we obtain that \( \eta_c|_{\text{im} (\iota_\ast)} \) as an endomorphism of \( \text{im} (\iota_\ast) \) is given by

\[
\begin{pmatrix}
c^n & 0 \\
\vdots & \ddots & 0 \\
0 & \cdots & c^n
\end{pmatrix}
\]

where the exponents of \( c \) on the diagonal are monotonously decreasing. The entry \( c^n \) corresponds to the basis element \([1]\), and \( c^0 \) can only appear at most once on the diagonal (if it appears, it has to be in the lower right corner according to our ordering). We enumerate this \( \mathbb{Z} \)-basis of \( \text{im} (\iota_\ast) \) by \( x_0, \ldots, x_{\text{rk}_{\text{fin}}} \) according to our ordering, so \( x_0 = [1] \).

By construction (see (25)), \( \langle x_0, \ldots, x_{\text{rk}_{\text{fin}}} \rangle = \text{im} (\iota_\ast) \) is of finite index in \( K_{\text{fin}} \), so that we can choose a \( \mathbb{Z} \)-basis \( \mathcal{B}_0, \ldots, \mathcal{B}_{\text{rk}_{\text{fin}}} \) of \( K_{\text{fin}} \) with the property that

\[
\langle \mathcal{B}_0, \ldots, \mathcal{B}_j \rangle = \{ x \in K_{\text{fin}} : \text{There exists } N \in \mathbb{Z}_{>0} \text{ with } Nx \in \langle \mathcal{B}_0, \ldots, \mathcal{B}_j \rangle \}
\]

for every \( 0 \leq j \leq \text{rk} (K_{\text{fin}}) - 1 = \text{rk}_{\text{fin}}^c \). In particular, we have \( \mathcal{B}_0 = [1] \). It then follows that \( \eta_c(\langle \mathcal{B}_j \rangle) \subseteq \langle \mathcal{B}_j \rangle \) and that with respect to the \( \mathbb{Z} \)-basis \( \{\mathcal{B}_j\} \), the matrix describing \( \eta_c|_{K_{\text{fin}}} \) as an endomorphism of \( K_{\text{fin}} \) is of the form

\[
\begin{pmatrix}
c^n & * \\
\vdots & \ddots & * \\
0 & \cdots & c^n
\end{pmatrix}
\]

Recall that \( ? \) can be 0 only if \( n \) is even, and in that case, the 0-th power \( c^0 \) can only appear at most once on the diagonal. Now set \( K_{\text{fin}}^f := \langle \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{\text{rk}_{\text{fin}}} \rangle \). Then we have \( K_{\text{fin}} = \langle [1] \rangle \oplus K_{\text{fin}}^f \) by construction, and \( \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{\text{rk}_{\text{fin}}} \) is a \( \mathbb{Z} \)-basis of \( K_{\text{fin}}^f \) with the desired properties. \( \square \)

This first lemma settles the \( K_{\text{fin}} \)-part.

Up to now, we have only used that \( c \) is an integer bigger than 1, the extra condition that \( \prod_{i=0}^{\text{rk}_{\text{fin}}} (1 - \zeta^i) \) divides \( c \) was not used in our arguments up to this point. But for the finite part, this condition plays a crucial role.

5.3.2. The finite part.
Moreover, $\eta$ with $\vartheta$ be a projection. “Irreducible” means that once we apply $\rho \circ cR$ out what these irreducible summands are projections of certain sizes. Of course, conjugation by a permutation matrix does not have any effect in K-theory. This means that we obtain irreducible irreducible summands which gives the class of $\rho \circ cR$. Thus up to conjugation by $\rho \circ cR$ entry of $\rho \circ cR$ is of the form

$$\eta \left( \sum_{(\mu) \neq (M) \in M} \text{im} \left( (t_M)_* \right) \right) \in [1].$$

Proof. Let $M$ be a maximal finite subgroup of $R \times \mu$, and choose a generator $(b, \zeta') \in R \times \mu$ of $M$. Our aim is to compute $\eta_c([p_{\chi} (u^b s_{\zeta}])$ (with $\chi \in \hat{\mathbb{Z}}/m \hat{\mathbb{Z}}$). By definition, $\eta_c = (\rho_{c,1})_*^{-1}(\vartheta_{c,1} \ast (\lambda_{c,1})).$ So we have to examine $(\vartheta_{c,1} \ast (\lambda_{c,1}))(u^b s_{\zeta}).$ Take two elements $d, d'$ in the system $\mathcal{R}_c$ of representatives for $R/cR$. The $(d, d')$-th entry of $(\vartheta_{c,1} \ast (\lambda_{c,1}))(u^b s_{\zeta})$ is given by

$$s_{\zeta}^* u^{-d} u^b s_{\zeta} d s_{\zeta} = \begin{cases} s_{\zeta}^* u^{-d + b + \zeta' d'} s_{\zeta} & \text{if } -d + b + \zeta' d' \notin cR, \\ 0 & \text{if } -d + b + \zeta' d' \in cR. \end{cases}$$

Therefore, the matrix $(\vartheta_{c,1} \ast (\lambda_{c,1}))(u^b s_{\zeta})$ has exactly one non-zero entry in each row and column. In other words, for fixed $d$ in $\mathcal{R}_c$, there exists exactly one $d' \in \mathcal{R}_c$ with $-d + b + \zeta' d' \in cR$, namely the element in $\mathcal{R}_c$ which represents the coset $\zeta^{-1}(d - b) + cR$. There is only one such element because $\mathcal{R}_c$ is minimal.

Moreover, $u^b s_{\zeta}$ is a cyclic element: Its $\frac{m}{\tau}$-th power is $1$.

These two observations imply that $\frac{1}{m} \sum_{j=0}^{m-1} \chi(j + \frac{m}{\tau} \mathbb{Z})(u^b s_{\zeta}) = p_{\chi}(u^b s_{\zeta})$ can be decomposed into irreducible summands, and each of these summands has to be a projection. “Irreducible” means that once we apply $\vartheta_{c,1} \ast (\lambda_{c,1})$, we obtain an irreducible matrix. Thus up to conjugation by a permutation matrix, $(\vartheta_{c,1} \ast (\lambda_{c,1}))(p_{\chi}(u^b s_{\zeta}))$ is of the form

$$\begin{pmatrix} p_1 & 0 & \cdots \\ 0 & p_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where the $p_i$ are projections of certain sizes. Of course, conjugation by a permutation matrix does not have any effect in K-theory. This means that we obtain

$$\eta_c([p_{\chi} (u^b s_{\zeta})]) = (p_{\chi})_*^{-1}([p_1] + [p_2] + \cdots)$$

where $p_{\chi}$ is the homomorphism

$$C^*(\{u^b\}, s_{\zeta}) \to \mathcal{L}(\ell^2(R/cR)) \otimes C^*(\{u^b\}, s_{\zeta}); x \mapsto e_{0,0} \otimes x.$$ 

Here $\mathcal{L}(\ell^2(R/cR)) \cong M_{\infty}(\mathbb{C})$ and $e_{0,0}$ is a minimal projection. So it remains to find out what these irreducible summands $p_i$ give in K-theory.

First of all, we look at the case $b = 0$, $i = 1$, i.e., we consider $p_{\chi}(s_{\zeta})$. Irreducible summands of size 1 must be of the form $p_{\chi}(u^b s_{\zeta})$. What we want to show now is that there is only one 1-dimensional summand which gives the class of $p_{\chi}(s_{\zeta})$. To do so, we take a 1-dimensional summand corresponding to the position $d$ (for some $d$ in $\mathcal{R}_c$). The $(d, d')$-th entry of $(\vartheta_{c,1} \ast (\lambda_{c,1}))(s_{\zeta})$ is given by $u^{\zeta^{-1}(d - d')} s_{\zeta}$. By
Theorem 4.1 the corresponding projection (i.e., \( p_\chi(u^{-1}(\zeta d - d) s_\zeta) \)) gives a \( K_0 \)-class in \( \text{im}(i_\mu)_* \) if and only if the subgroups \( \langle e^{-1}(\zeta d - d), \zeta \rangle \) and \( \langle (0, \zeta) \rangle \) of \( R \times \mu \) are conjugate. This is equivalent to \( e^{-1}(\zeta d - d) \in (\zeta - 1) \Leftrightarrow d \in cR \). But as \( R_c \) is minimal, this happens for exactly one element in \( R_c \) (by our convention, this element has to be 0, but this is not important at this point). Moreover, if \( d \) lies in \( cR \), then \( [p_\chi(u^{-1}(\zeta d - d) s_\zeta)] = [p_\chi(s_\zeta)] \) in \( K_0 \). So from the 1-dimensional summands we obtain in \( K_0 \) exactly once the class \([p_\chi(s_\zeta)]\) and some other classes in \( \sum_{(\mu) \neq (M) \in M_0} \text{im}(i_\mu)_* \).

It remains to examine higher dimensional summands. We want to show that all the higher dimensional summands give rise to \( K_0 \)-classes in \( \sum_{(\mu) \neq (M) \in M_0} \text{im}(i_\mu)_* \). Let us take a summand of size \( j \) with \( j > 1 \). This means that the \( j \)-th power of \((\vartheta_{c,1} \circ i_{1,c})(s_\zeta)\) has a non-zero diagonal entry, say at the \((d, d)\)-position. This entry is \( u^{-1}(\zeta' d - d) s_{\zeta'} \). Now we prove a result in a bit more generality than actually needed at this point. But later on, we will come back to it.

**Lemma 5.6.** Assume that for an irreducible summand of \((\vartheta_{c,1} \circ i_{1,c})(p_\chi(u^b s_{\zeta'}))\), \( j \in \mathbb{Z}_{\geq 1} \) is the smallest number such that the \( j \)-th power of this summand has non-zero diagonal entries. Let one of these non-zero diagonal entries be \( u^b s_{\zeta'_{ij}} \) for some \( b \) in \( R \). Then the \( K_0 \)-class of this summand coincides with \([p_\chi(u^b s_{\zeta'_{ij}})]\) where \( \tilde{\chi} \) is the restriction of \( \chi \in \langle \zeta' \rangle \) to \( \langle \zeta' \rangle \).

**Proof of Lemma 5.6.** Up to conjugation by a permutation matrix, the irreducible summand of \( u^b s_{\zeta'} \), we are considering is of the form

\[
\begin{pmatrix}
0 & x_j \\
x_1 & \ddots & \ddots \\
0 & \ddots & \ddots & x_{j-1} \\
& \ddots & \ddots & \ddots & x_1 x_j
\end{pmatrix}
\]

All the entries lie in \( C^* \{\{u^b\}, s_{\zeta'}\} \).

The \( j \)-th power is given by

\[
\begin{pmatrix}
x_j x_{j-1} \cdots x_2 x_1 & 0 \\
0 & x_1 x_j x_{j-1} \cdots x_2 & 0 \\
0 & \ddots & \ddots & \ddots & x_j x_{j-2} \cdots x_1 x_j
\end{pmatrix}
\]

By assumption, \( x_j x_{j-1} \cdots x_2 x_1 = u^b s_{\zeta'_{ij}} \). Then the irreducible summand of

\[
p_\chi(u^b s_{\zeta'}) = \frac{1}{m!} \sum_{k=0}^{m-1} \chi(j)(u^b s_{\zeta'})^k
\]

is given by

\[
\begin{pmatrix}
p_\chi(u^b s_{\zeta'_{ij}}) & p_\chi(u^b s_{\zeta'_{ij}}) \cdot x_1^* & \cdots \\
x_1 \cdot p_\chi(u^b s_{\zeta'_{ij}}) & x_1 \cdot p_\chi(u^b s_{\zeta'_{ij}}) \cdot x_1^* & \cdots \\
\vdots & \vdots & \ddots \\
x_{j-1} \cdots x_1 \cdot p_\chi(u^b s_{\zeta'_{ij}}) & x_{j-1} \cdots x_1 \cdot p_\chi(u^b s_{\zeta'_{ij}}) \cdot x_1^* & \cdots
\end{pmatrix}
\]
The $k$-th column is given by the product of the first column with $(x_{k-1} \cdots x_1)^*$ from the right.

But then,

$$
\begin{pmatrix}
\frac{1}{c_j} \\
\vdots \\
x_{j-1} \cdots x_1 \cdot p_{\xi}(u^b s_{\xi}) & 0 & \cdots \\
\end{pmatrix}
$$

is a partial isometry with entries in $C^*\langle \{u^b\}, s_c \rangle$ whose range projection is precisely the irreducible summand from above and whose support projection is

$$
\begin{pmatrix}
p_{\xi}(u^b s_{\xi}) & 0 & \cdots \\
\vdots & 0 \\
0 & \cdots \\
\end{pmatrix}
$$

This proves Lemma 5.6. \qed

**Corollary 5.7.** If in the situation of Lemma 5.6, we have $j = \frac{m}{s}$, i.e., $u^b s_{\xi} = 1$, then the corresponding irreducible summand of $((\theta_{c,1} \circ t_{1,c})(p_{\xi}(u^b s_{\xi})))$ gives the $K_0$-class $[1]$.

Now let us continue the proof of Lemma 5.5. We go back to the higher dimensional summands of $(\theta_{c,1} \circ t_{1,c})(s_c)$. We were considering the $(d,d)$-th position with entry $u c^{-1}(\zeta \hat{d}-d) s_{\xi}$. Lemma 5.6 tells us that this irreducible summand gives $[p_{\xi}(u c^{-1}(\zeta \hat{d}-d) s_{\xi})]$. By Theorem 4.1, this $K_0$-class lies in $\text{im} \langle (i_{\mu})_* \rangle$ if and only if the corresponding subgroup $\langle (c^{-1}(\zeta \hat{d}-d), \zeta^j) \rangle$ is conjugate to a subgroup of $\langle (0,\zeta) \rangle$. In case $\zeta^j \neq 1$, this happens if and only if $c^{-1}(\zeta \hat{d}-d) \in (\zeta^j - 1) \Leftrightarrow d \in cR \Leftrightarrow d = 0$ by our choice of $R_c$. But for $d = 0$, the summand we get is of size 1 (see above). This contradicts $j > 1$. If $\zeta^j = 1$, then we obtain a projection whose $K_0$-class is $[1]$ by Corollary 5.7.

This proves $\eta_c([p_{\xi}(s_c)]) \in [p_{\xi}(s_c)] + \sum_{(\mu) \neq (M) \in M} \text{im} \langle (i_M)_* \rangle$ for all $\chi \in \mathbb{Z}/m\mathbb{Z}$.

It remains to prove $\eta_c\left(\sum_{(\mu) \neq (M) \in M} \text{im} \langle (i_M)_* \rangle \right) \in \langle [1] \rangle$. Take a maximal finite subgroup $M$ with $(M) \neq (\mu)$. Let $(b, \zeta)$ be a generator of $M$, and consider the element $[p_{\xi}(u^b s_{\xi})]$. How do the irreducible summands of $(\theta_{c,1} \circ t_{1,c})(p_{\xi}(u^b s_{\xi}))$ look like? We claim that each of these summands must have size $\frac{m}{s}$. To show this, let $j$ be the size of such a summand. This means that there exists $d \in R_c$ such that

$$
s_{\xi}^* u^{-d} (u^b s_{\xi})^j u^d s_c = s_{\xi}^* u^{-d} u^b \zeta^{b+\cdots+\zeta^{(j-1)b}} s_{\xi} u^d s_c = s_{\xi}^* u^{-d} (1 - \zeta^{ij}) u^d s_c \neq 0.
$$

This happens if and only if $-d + \frac{1 - \zeta^{ij}}{\zeta} b + \zeta^{ij} d \in cR$. Now, if $\zeta^{ij} \neq 1$, then we can proceed as follows: As $\prod_{ijm,1 \leq i < m}(1 - \zeta^i)$ divides $c$ by assumption, we know that $1 - \zeta^i$ divides $cR$, so that $-d + \frac{1 - \zeta^{ij}}{\zeta} b + \zeta^{ij} d \in cR$ implies that $b$ lies in $(1 - \zeta^i)R$. But this is a contradiction to $(M) \neq (\mu)$. Thus we must have $\zeta^{ij} = 1$, which by minimality of $j$ implies $j = \frac{m}{s}$, as claimed.
Therefore all these irreducible summands give the $K_0$-class $[1]$ (see Corollary 5.7). This proves that

$$
\eta_c \left( \sum_{(\mu) \neq (M) \in M} \text{im } ((\iota_M)_+) \right) \in \langle [1] \rangle.
$$

\[ \Box \]

**Corollary 5.8.** With respect to the $\mathbb{Z}$-basis $y_1, \ldots, y_{rk} \in K_\text{fin}$ and $z_1, \ldots, z_{m-1}$ of $K_\mu \text{fin}$, respectively, and with respect to the $\mathbb{Z}$-basis

$$\left\{ [1], x_1, \ldots, x_{rk} \in \text{fin}, y_1, \ldots, y_{rk} \text{in, } z_1, \ldots, z_{m-1} \right\}
$$

of $K_0(C^*(\{u^b\}, s, \{e_a\})$, for every $c$ in $\mathbb{Z}_{>1}$ with the property that $\prod_{i|m, 1 \leq i < m}(1 - \zeta^i)$ divides $c$, we have that $\eta_c|_{K_{\text{fin}}^\mu \oplus K_{\text{fin}}^\nu} : K_{\text{fin}}^\mu \oplus K_{\text{fin}}^\nu \to K_0(C^*(\{u^b\}, s, \zeta))$ is of the form

$$
\begin{pmatrix}
* & * & & \\
* & * & & \\
0 & 1 & * & \\
0 & \ddots & & 1
\end{pmatrix}.
$$

**Proof.** This follows from Lemma 5.5 and the way the basis elements $y_1, \ldots, y_{rk} \in \text{fin}$ and $z_1, \ldots, z_{m-1}$ were chosen (see Section 4). \[ \Box \]

With this corollary, together with Lemma 5.4, we have completed the proof of Proposition 5.3.

### 5.4. K-theory for a sub-C*-algebra

Now we can compute the K-theory of $C^*(\{u^b\}, s, \zeta, \{e_a\})$. We can also determine $\text{Ad}(s_c) = s_c \cup s_c^*$ on K-theory.

**Proposition 5.9.** We have

$$K_0(C^*(\{u^b\}, s, \zeta, \{e_a\})) \cong Q^{(K_\text{inf})^{-\delta}} \oplus \mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1}.$$

Here $\delta$ is 1 if for all $c$ in $\mathbb{Z}_{>1}$ divisible by $\prod_{i|m, 1 \leq i < m}(1 - \zeta^i)$, the least exponent of $c$ in the matrix describing $\eta_c$ (see Proposition 5.3) is 0. Otherwise $\delta$ is 1.

$K_1(C^*(\{u^b\}, s, \zeta, \{e_a\}))$ vanishes.

Moreover, there exists a $Q$-basis of $Q^{(K_\text{inf})^{-\delta}}$ and a $\mathbb{Z}$-basis of $\mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1}$ such that, for all $c$ in $\mathbb{Z}_{>1}$ with $\prod_{i|m, 1 \leq i < m}(1 - \zeta^i)$ dividing $c$, the homomorphism $\text{Ad}(s_c)$
is of the form

\[
\begin{pmatrix}
  c^{-n} & * & 0 & 0 \\
  \vdots & \ddots & \ddots & \ddots \\
  0 & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & 1 & * & * \\
  0 & 0 & 0 & \ddots & 1 \\
 0 & 0 & 0 & & 1
\end{pmatrix}
\]

if \( \delta = 1 \)

and

\[
\begin{pmatrix}
  c^{-n} & * & 0 \\
  \vdots & \ddots & \ddots \\
  0 & \ddots & \ddots & \ddots & \ddots \\
  0 & 1 & * & * \\
  0 & 0 & \ddots & 1 \\
 0 & 0 & 0 & & 1
\end{pmatrix}
\]

if \( \delta = 0 \).

Note that in the first box on the diagonal of these matrices, all the diagonal entries are always strictly less than 1.

Proof. We know that \( C^*(\{ u^b \}, s_\zeta, \{ e_a \}) \) can be identified with the inductive limit obtained from the \( C^* \)-algebras \( C^*(\{ u^b \}, s_\zeta, e_a) \), for \( a \in \mathbb{R}^\times \) and the inclusion maps \( \iota_{a,ac} : C^*(\{ u^b \}, s_\zeta, e_a) \to C^*(\{ u^b \}, s_\zeta, e_{ac}) \). Using continuity of K-theory, together with Lemma 5.1 and Lemma 5.2, we obtain for \( i = 0, 1 \):

\[
K_i(C^*(\{ u^b \}, s_\zeta, \{ e_a \})) \cong \lim_{\longrightarrow} K_i(C^*(\{ u^b \}, s_\zeta)), \eta_c).
\]

From this, it is immediate that \( K_1(C^*(\{ u^b \}, s_\zeta, \{ e_a \})) \) vanishes by (*** in Theorem 4.1). Moreover, the description for \( K_0(C^*(\{ u^b \}, s_\zeta, \{ e_a \})) \) can be deduced from the description of \( \eta_c \) in Proposition 5.3.

Concerning the description of \( (\text{Ad } (s_c))_* \), let us explain the case \( \delta = 1 \). The case \( \delta = 0 \) is similar. We observe that \( (\text{Ad } (s_c))_* \) is given by the inverse of the homomorphism on

\[
K_0(C^*(\{ u^b \}, s_\zeta, \{ e_a \})) \cong \lim_{\longrightarrow} K_0(C^*(\{ u^b \}, s_\zeta)), \eta_c).
\]

induced by \( \eta_c \). This follows from (26). We obtain that there exists a \( \mathbb{Q} \)-basis of \( Q^l(K_{i,\alpha})^{-\delta} \) and a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1} \) such that \( (\text{Ad } (s_c))_* \) is of the form

\[
\begin{pmatrix}
  c^{-n} & * & * \\
  \vdots & \ddots & \ddots \\
  0 & \ddots & \ddots & \ddots \\
  0 & 0 & 1 & * \\
  0 & 0 & 0 & \ddots \\
 0 & 0 & 0 & 1
\end{pmatrix}
\]
Modifying the $\mathbb{Z}$-basis if necessary, we can find a new $\mathbb{Z}$-basis for $\mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1}$ such that the two boxes in the upper right corner of (37) vanish. □

5.5. Passing over to the infinite adele space. We can now compute K-theory for certain crossed products involving the profinite completion of $\mathbb{R}$. Using the duality theorem, we are then able to pass over to the infinite adele space.

Corollary 5.10. We have

\begin{align*}
K_0(C(\mathbb{R}) \rtimes R \rtimes \mu) &\cong \mathbb{Q}^{rk(K_{i,n})-\delta} \oplus \mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1} \\
K_1(C(\mathbb{R}) \rtimes R \rtimes \mu) &\cong \{0\} \\
K_0(C_0(\mathcal{A}_\infty) \rtimes K \rtimes \mu) &\cong \mathbb{Q}^{rk(K_{i,n})-\delta} \oplus \mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1} \\
K_1(C_0(\mathcal{A}_\infty) \rtimes K \rtimes \mu) &\cong \{0\}.
\end{align*}

Proof. These results follow from the duality theorem (see Paragraph 2.3) and Proposition 5.9. □

For the next result, we need Proposition 6.1.

Corollary 5.11. With respect to the same bases as in Proposition 5.9, we must have that $\text{Ad}(s_c)$ is of the form

\[
\begin{pmatrix}
 c^{-n} & * & & \\
 & \ddots & & \\
 & & & 0 \\
0 & & & \text{id}_{\mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1}}
\end{pmatrix}
\]

for all $c \in \mathbb{Z}_{>1}$ divisible by $\prod_{i|m, 1 \leq i < m}(1-\zeta^i)$. Moreover, if $n$ is even, $\delta$ must be 1.

Proof. If $n = [K : \mathbb{Q}]$ is odd, $\delta$ must vanish and $m$ must be 2 as $K$ admits an embedding into $\mathbb{R}$ so that $\mu$ must be $\{\pm 1\}$. So in that case, there is nothing to prove.

Now let us consider the case that $n = [K : \mathbb{Q}]$ is even. First of all, we know that under the canonical isomorphism $C(\mathbb{R}) \rtimes R \rtimes \mu \cong C^*\left(\{u^b\}, s_\zeta, \{e_a\}\right)$, the endomorphism $\beta_c^{(fin)}$ of $C(\mathbb{R}) \rtimes R \rtimes \mu$ induced by multiplication with $c$ corresponds to $\text{Ad}(s_c)$. From Proposition 5.9, we see that $\text{rk}(\ker(\text{id} - (\text{Ad}(s_c))) \leq m$ and that we can only have equality if $\delta = 1$ and $\text{Ad}(s_c)$ is the identity on $\mathbb{Z}^\delta \oplus \mathbb{Z}^{m-1}$ in Proposition 5.9. Thus the same holds for $\text{rk}(\ker(\text{id} - (\beta_c^{(fin)})))$.

As against that, we know by Corollary 3.28 that we can identify $K_*(C_0(\mathcal{A}_\infty) \rtimes K \rtimes \mu)$ with $K_*(C(\mathbb{R}) \rtimes R \rtimes \mu)$ so that $(\beta_c)_*$ $(\beta_c$ is the automorphism of $C_0(\mathcal{A}_\infty) \rtimes K \rtimes \mu$ induced by multiplication with $c$) corresponds to $(\beta_c^{(fin)})_*$. Thus $\text{rk}(\ker(\text{id} - (\beta_c^{(fin)})) = \text{rk}(\ker(\text{id} - (\beta_c)_*))$. But the canonical homomorphism $C_0(\mathcal{A}_\infty) \rtimes \mu \to C_0(\mathcal{A}_\infty) \rtimes K \rtimes \mu$ maps into $\ker(\text{id} - (\beta_c)_*)$ in $K_0$ since $\beta_c$ is homotopic to the
identity on $C_0(\mathcal{A}_\infty) \times \mu$ (recall that $n$ is even). Moreover, by Proposition 6.1, we know that this canonical homomorphism is injective on $K_0$. As $K_0(C_0(\mathcal{A}_\infty) \times \mu) \cong K_0(C^*(\mu)) \cong \mathbb{Z}^m$ by equivariant Bott periodicity (see Theorem 20.3.2 in [Bla1], $n$ is even), we conclude that $\text{rk} \left( \text{ker} \left( \text{id} - (\beta_\mu)_* \right) \right) \geq m$. So $\text{rk} \left( \text{ker} \left( \text{id} - (\beta^{(fin)}_\mu)_* \right) \right)$ must be $m$, and our assertion follows. 

**Corollary 5.12.** If $K$ has higher roots of unity ($|\mu| > 2$), then

$$K_0(C_0(\mathcal{A}_\infty) \times K \times \mu) \cong \mathbb{Q}^{\text{rk}(K_{\infty}) - 1} \oplus \mathbb{Z}^m \quad \text{and} \quad K_1(C_0(\mathcal{A}_\infty) \times K \times \mu) \cong \{0\}.$$ 

Moreover, there exists a $\mathbb{Q}$-basis for $\mathbb{Q}^{\text{rk}(K_{\infty}) - 1}$ and a $\mathbb{Z}$-basis for $\mathbb{Z}^m$ such that for $c$ in $\mathbb{Z}_{\geq 1}$ divisible by $\prod_{i|m, 1 \leq i < m}(1 - \zeta^i)$, $(\beta_\mu)_* : K_0(C_0(\mathcal{A}_\infty) \times K \times \mu) \to K_0(C_0(\mathcal{A}_\infty) \times K \times \mu)$ is of the form

$$
\begin{pmatrix}
n & * & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \text{id}_{\mathbb{Z}^m}
\end{pmatrix}
$$

(42)

**Proof.** $|\mu| > 2$ implies that $n$ is even. So the second statement in Corollary 5.11 tells us that $\delta = 1$. Thus our first two statements about $K_0$ and $K_1$ follow from (40) and (41). Furthermore, we can identify $K_*(C_0(\mathcal{A}_\infty) \times K \times \mu)$ with $K_*(C(\mathring{R}) \times R \times \mu)$ so that $(\beta_\mu)_*$ corresponds to $(\beta^{(fin)}_\mu)_*$. We can also identify $C(\mathring{R}) \times R \times \mu$ with $C^*(\{u^b\}, s_c, \{e_a\})$ so that $\beta^{(fin)}_\mu$ corresponds to $\text{Ad}(s_c)$. Thus our second statement about $(\beta_\mu)_*$ follows from Corollary 5.11. 

**Corollary 5.13.** If $K$ has higher roots of unity ($|\mu| > 2$), then we have for all $c$ in $\mathbb{Z}_{\geq 1}$ divisible by $\prod_{i|m, 1 \leq i < m}(1 - \zeta^i)$ that $K_i(C_0(\mathcal{A}_\infty) \times K \times (\mu \times \langle c \rangle)) \cong \mathbb{Z}^m$ ($i = 0, 1$).

**Proof.** Just plug in the results from the last corollary into the Pimsner-Voiculescu sequence for $C_0(\mathcal{A}_\infty) \times K \times (\mu \times \langle c \rangle) \cong (C_0(\mathcal{A}_\infty) \times K \times \mu) \rtimes_{\beta_\mu} \mathbb{Z}$. 

5.6. **End of proof.** We are now ready to prove our main result. First of all, let us fix one integer $c > 1$ with the property that $\prod_{i|m, 1 \leq i < m}(1 - \zeta^i)$ divides $c$. In addition, we can choose $c_1, c_2, \ldots$ in $K^\times$ such that $c, c_1, c_2, \ldots$ are free generators of a free abelian subgroup $\Gamma$ of $K^\times$ with $K^\times = \mu \rtimes \Gamma$. Let $\Gamma_j := \langle c, c_1, \ldots, c_j \rangle$ ($\Gamma_0 = \langle c \rangle$).

**Proposition 5.14.** The canonical homomorphism

$$C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_j) \to C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_j)$$

is a rational isomorphism for all $i \geq 0$. Moreover, $\beta_{i+1}$, the automorphism induced by multiplication with $c_{i+1}$, is the identity on $K_*(C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_j))$. 

Here $K_*$ stands for the $\mathbb{Z}/2\mathbb{Z}$-graded abelian group $K_0 \oplus K_1$. 

Proof. We proceed inductively on \( j \). Let us start with \( j = 0 \). The Pimsner-Voiculescu exact sequence gives the following exact sequences:

\[
\begin{array}{c}
\{0\} & \rightarrow & K_1(C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_0)) & \rightarrow & K_0(C_0(\mathcal{A}_\infty) \times \mu) \\
\text{id} - (\beta_\ast) = 0 & K_0(C_0(\mathcal{A}_\infty) \times \mu) & \rightarrow & K_0(C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_0)) & \rightarrow & \{0\}
\end{array}
\]

and

\[
\begin{array}{c}
\{0\} & \rightarrow & K_1(C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_0)) & \rightarrow & K_0(C_0(\mathcal{A}_\infty) \times K \times \mu) \\
\text{id} - (\beta_\ast) = \begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} e & \ast \\ \ast & \ast \end{pmatrix} \end{pmatrix} & K_0(C_0(\mathcal{A}_\infty) \times K \times \mu) & \rightarrow & K_0(C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_0)) & \rightarrow & \{0\}
\end{array}
\]

In the first sequence, we have \((\beta_\ast) = \text{id}\) because \(\beta_\ast\) is homotopic to the identity on \(C_0(\mathcal{A}_\infty) \times \mu\). In the second sequence, we have plugged in the matrix describing \((\beta_\ast)\) given in (42).

Moreover, naturality of the Pimsner-Voiculescu sequence allows us to connect these two sequences by homomorphisms on K-theory induced by the canonical maps

\[
C_0(\mathcal{A}_\infty) \times \mu \rightarrow C_0(\mathcal{A}_\infty) \times K \times \mu
\]

and

\[
C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_0) \rightarrow C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_0).
\]

By Proposition 6.1, we know that \(C_0(\mathcal{A}_\infty) \times \mu \rightarrow C_0(\mathcal{A}_\infty) \times K \times \mu\) induces an injective map on \(K_0\) whose image is contained in the copy of \(\mathbb{Z}^m\) in \(K_0(C_0(\mathcal{A}_\infty) \times K \times \mu)\). Comparing the ranks, we deduce that this injective map must be a rational isomorphism when we restrict its image to the copy of \(\mathbb{Z}^m\). Since this copy is isomorphic to \(K_1(C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_0))\) (for \(i = 0, 1\)) via the maps in the second exact sequence (43), we obtain that the canonical homomorphism \(C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_0) \rightarrow C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_0)\) induces a rational isomorphism on K-theory.

Now we know that \(\beta_{c_1}\) is homotopic to the identity on \(C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_0)\). This implies \((\beta_{c_1})_\ast = \text{id}\) on \(K_*(C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_0))\). As the canonical homomorphism \(C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_0) \rightarrow C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_0)\) induces a rational isomorphism on K-theory (see above), we know that

\[
(\beta_{c_1})_\ast \otimes \mathbb{Z} \text{id}_\mathbb{Q} = \text{id} \otimes \mathbb{Z} \text{id}_\mathbb{Q} \text{ on } K_*(C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_0)) \otimes \mathbb{Z} \mathbb{Q}.
\]

But as \(K_*(C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_0))\) is free abelian (see Corollary 5.13, \(\Gamma_0 = \langle c \rangle\)), it follows that \((\beta_{c_1})_\ast\) must be the identity on \(K_*(C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_0))\). This settles the case \(j = 0\).

The remaining induction step is proven in a similar way. We just have to use that \(\beta_{c_{j+1}}\) is homotopic to the identity on \(C_0(\mathcal{A}_\infty) \times (\mu \times \Gamma_j)\).

\[\square\]

**Corollary 5.15.** For every \(i \in \mathbb{Z}_{\geq 0}\), we can identify

\[
K_*(C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_j)) \text{ with } K_0(C_\ast(\mu)) \otimes \Lambda^\ast(\Gamma_j)
\]

in such a way that the map on K-theory induced by the canonical homomorphism

\[
C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_j) \rightarrow C_0(\mathcal{A}_\infty) \times K \times (\mu \times \Gamma_{j+1})
\]

corresponds to the canonical map

\[
K_0(C_\ast(\mu)) \otimes \mathbb{Z} \Lambda^\ast(\Gamma_j) \rightarrow K_0(C_\ast(\mu)) \otimes \mathbb{Z} \Lambda^\ast(\Gamma_{j+1})
\]
induced by the inclusion $\Gamma_j \hookrightarrow \Gamma_{j+1}$ for all $j \in \mathbb{Z}_{\geq 0}$.

**Proof.** This follows inductively on $j$ using $(\beta_{(j+1)^*})_* = \text{id}$ on $K_*(C_0(\mathcal{A}_{\infty}) \rtimes K \rtimes (\mu \times \Gamma_j))$ (see Proposition 5.14) and the Pimsner-Voiculescu exact sequence. The induction starts with Corollary 5.13. □

Finally, we arrive at

$$K_*(C_0(\mathcal{A}_{\infty}) \rtimes K \rtimes K^\times \cong \lim_{j} K_*(C_0(\mathcal{A}_{\infty}) \rtimes K \rtimes (\mu \times \Gamma_j))$$

$$\cong \lim_{j} K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^* (\Gamma_j) \cong K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^* (\Gamma).$$

This completes the proof of our main result, Theorem 1.1.

**Proof of Corollary 1.3.** By [Cu-Li1], Theorem 3.6, the ring $C^*$-algebras of rings of integers are simple and purely infinite, and by Corollaries 3 and 4 in [Li], these ring $C^*$-algebras are nuclear and satisfy the UCT. Moreover, these ring $C^*$-algebras are obviously unital and separable, and it is easy to see that for these algebras, the class of the unit in $K_0$ vanishes. Thus [Ror], Theorem 8.4.1 (iv) tells us that two such ring $C^*$-algebras are isomorphic if and only if their $K$-groups are isomorphic. Now our corollary follows from Theorem 1.2. □

**Remark 5.16.** In [Cu-Li2], Remark 6.6, it was observed that the same ideas which lead to the duality theorem also yield

$$C_0(\mathcal{A}) \rtimes K \rtimes K^\times \sim_{\mathcal{M}} C^*(K \rtimes K^\times).$$

We would like to add comments on this. First, it is even possible to implement (44) by an Aut $(K/Q)$- or Aut $(K/F_p(T))$-equivariant imprimitivity bimodule, for the same reason as in Remark 3.32. Secondly, we can compute the $K$-theory of $C_0(\mathcal{A}) \rtimes K \rtimes K^\times$ using (44) and Theorem 4.1 in the case of a number field $K$. The strategy is the same as the one we used to prove Theorem 1.1. (44) reduces our problem to computing $K_*(C^*(K \rtimes K^\times))$. We start with computing $K_*(C^*(K \rtimes \mu))$. This to end, we write $C^*(K \rtimes \mu)$ as an inductive limit where all the $C^*$-algebras are given by $C^*(R \rtimes \mu)$ and the connecting maps are induced by multiplication with elements from $R^\times$. We can then use Theorem 4.1 to determine the corresponding inductive limit in $K$-theory. To complete our computation, we proceed in an analogous manner as in Paragraph 5.6. We choose a free abelian subgroup $\Gamma$ of $K^\times$ such that $K^\times = \mu \times \Gamma$ and then use the Pimsner-Voiculescu sequence iteratively. As a final result, we obtain that the canonical homomorphism $C^*(K^\times) \to C^*(K \rtimes K^\times)$ induces an isomorphism on $K$-theory. Thus, for every number field $K$, we obtain $K_*(C_0(\mathcal{A}) \rtimes K \rtimes K^\times) \cong K_*(C^*(K \rtimes K^\times)) \cong K_*(C^*(K)) \cong K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^* (\Gamma)$.

6. INJECTIVITY OF CERTAIN INCLUSIONS ON $K$-THEORY

We want to prove

**Proposition 6.1.** For every number field $K$, the homomorphism

$$K_0(C_0(\mathcal{A}_{\infty}) \rtimes \mu) \to K_0(C_0(\mathcal{A}_{\infty}) \rtimes K \rtimes \mu)$$

induced by the canonical map $C_0(\mathcal{A}_{\infty}) \rtimes \mu \to C_0(\mathcal{A}_{\infty}) \rtimes K \rtimes \mu$ is injective.
Recall that $K \rtimes \mu$ acts on $C_0(\mathbb{R})$ via affine transformations as in Section 2.3. This proposition is needed in the proof of Theorem 1.1; more precisely, it is needed in the proofs of Corollary 5.11 and Proposition 5.14. We have postponed the proof of this proposition until now because it is independent from the previous sections.

6.1. **Induction and restriction.** In this section let $G$ be a discrete group and let $A$ be a $G$-$C^*$-algebra, i.e., a $C^*$-algebra $A$ with left $G$-action. We write $g \cdot a$ for this action. We write elements in $C_c(G, A)$ as finite sums of the form $\sum_g a_g \cdot g$. Let us present some elementary facts about induction and restriction which hold for reduced as well as full crossed products. But as we will consider amenable groups anyway later on, we only treat the case of reduced crossed products and remark that full crossed products can be studied in a similar way.

Let $\iota : H \hookrightarrow G$ be an injective group homomorphism. Then we obtain a homomorphism $\text{id}_A \rtimes \iota : A \rtimes_r H \rightarrow A \rtimes_r G$ which induces the map called induction with $\iota$,

$$
\tau_\iota = \text{ind}_\iota : K_i(A \rtimes_r H) \rightarrow K_i(A \rtimes_r G).
$$

(45)

Now suppose that the index of the image of $\iota$ in $G$ is finite. We want to construct maps in the “wrong” direction, i.e., a map $K_i(A \rtimes_r G) \rightarrow K_i(A \rtimes_r H)$. To simplify notations, we think of $H$ as a subgroup of $G$ via $\iota$. On $K_0$, we proceed as follows:

We obtain an isomorphism of (left) $A \rtimes_r H$-modules

$$
\bigoplus_{\gamma \in G/H} A \rtimes_r H \cong \text{res}_{A \rtimes_r G}^{A \rtimes_r H}(A \rtimes_r G)
$$

(46)

sending $(x_{\gamma} h)_{\gamma H}$ to $\sum_{\gamma H \in G/H} x_{\gamma H} \cdot \gamma^{-1}$ after choices of representatives $\gamma \in \gamma H$ for every $\gamma H \in G/H$. Hence $A \rtimes_r G$ is a finitely generated free $A \rtimes_r H$-module. This implies that the restriction of every finite generated projective $A \rtimes_r G$-module to $A \rtimes_r H$ is again a finitely generated projective $A \rtimes_r H$-module. Hence we obtain a homomorphism $\tau^* = \text{res}_\iota : K_0(A \rtimes_r G) \rightarrow K_0(A \rtimes_r H)$ which is called restriction with $\iota$.

Here is an alternative construction which has the advantage that it works for $K_1$ as well: First of all, we represent $A$ faithfully on a Hilbert space $\mathcal{H}$. Then $A \rtimes_r G$ is faithfully represented on $\mathcal{H} \otimes \ell^2(G)$ via $(a \cdot g)(\xi \otimes \varepsilon_\gamma) = ((g \gamma)^{-1} \cdot a)\xi \otimes \varepsilon_{g\gamma}$. We identify $A \rtimes_r G$ with concrete operators on $\mathcal{H} \otimes \ell^2(G)$ via this representation. Now fix representatives $\gamma \in \gamma H$ for every $\gamma H \in G/H$. From the (set-theoretical) bijection $G = \bigcup_{\gamma} \gamma H \cong \bigcup_{\gamma H} G/H$ we obtain a unitary

$$
\mathcal{H} \otimes \ell^2(G) \cong \bigoplus_{G/H} \mathcal{H} \otimes \ell^2(H)
$$

$$
\sum_{\gamma} \sum_h \lambda_{\gamma h} \xi_{\gamma h} \otimes \varepsilon_{\gamma h} \mapsto \left( \sum_h \lambda_{\gamma h} \xi_{\gamma h} \otimes \varepsilon_{h} \right)_{\gamma}.
$$

Conjugation by this unitary yields the identification

$$
\mathcal{L}(\mathcal{H} \otimes \ell^2(G)) \cong M_{|G/H|}(\mathcal{L}(\mathcal{H} \otimes \ell^2(H))), \quad T \mapsto (P_{\ell^2(\mathcal{H})} \gamma^{-1} \gamma^* T P_{\ell^2(\mathcal{H})})_{\gamma, \gamma'}
$$

where $P_{\ell^2(\mathcal{H})}$ is the orthogonal projection onto the subspace $\ell^2(G)$ of $\ell^2(G)$. 


A straightforward computation shows that this isomorphism sends the operator $a \cdot g$ to the matrix whose $(\gamma, \gamma')$-th entry is $(\gamma^{-1} \cdot a) \cdot (\gamma^{-1} g \gamma')$ if $\gamma^{-1} g \gamma'$ lies in $H$ and $0$ if $\gamma^{-1} g \gamma' \notin H$. In particular, $A \rtimes_r G$ is mapped to $M_{[G,H]}(A \rtimes_r H)$. This homomorphism induces the desired map $\iota^*: \text{K}_i(A \rtimes_r G) \to \text{K}_i(M_{[G,H]}(A \rtimes_r H)) \cong \text{K}_i(A \rtimes_r H)$.

Moreover, these restriction maps do not depend on the choices of the representatives $\gamma$ in $\gamma H \in G/H$. The reason is that for two different choices, the constructed homomorphisms $A \rtimes_r G \to M_{[G,H]}(A \rtimes_r H)$ turn out to be unitarily equivalent, hence they induce the same map in K-theory.

If $\iota: H \to G$ is an inclusion of subgroups, one often writes $\text{res}_i = \text{res}_i^G$ and $\text{ind}_i = \text{ind}_i^G$. For $g \in G$ conjugation defines an isomorphism of C*-algebras $c(g): A \rtimes G \to A \rtimes G$ sending $x \in A \rtimes G$ to $g x g^{-1}$. The two endomorphisms $\text{ind}_{c(g)}$ and $\text{res}_{c(g)}$ of $K_0(A \rtimes G)$ are the identity. Hence in the next lemma the choice of representatives $\gamma \in H \gamma K$ for an element $H \gamma K \in H \backslash G/K$ does not matter. It is a variation of the classical Double Coset Formula.

**Lemma 6.2.** Let $H, K \subseteq A$ be two subgroups of $G$. Suppose that $H$ has finite index in $G$. Then, for $i = 0, 1$, we get the following equality of homomorphisms $K_i(A \rtimes_r K) \to K_i(A \rtimes_r G)$:

$$\text{res}_i^G \circ \text{ind}_i^G = \sum_{H \gamma K \in H \backslash G/K} \text{ind}_{c(\gamma)^*}: K_i(H \gamma K) \to K_i(G/H) \circ \text{res}_i^{|K \cap \gamma^{-1} H \gamma|},$$

where $c(\gamma)$ is conjugation with $\gamma$, i.e., $c(\gamma)(k) = \gamma k \gamma^{-1}$.

**Proof.** Since $H$ has finite index in $G$, $K \cap \gamma^{-1} H \gamma$ has finite index in $K$ and $H \backslash G/K$ is finite. Hence the expression appearing in Lemma 6.2 makes sense.

On $K_0$, we can proceed as follows: Let $P$ be a finitely generated projective $A \rtimes_r K$-module. Fix choices of representatives $\gamma \in H \gamma K$ for every $H \gamma K \in H \backslash G/K$. Next we claim that the following homomorphism

$$\bigoplus_{H \gamma K \in H \backslash G/K} \text{ind}_{c(\gamma)^*} \left( \text{res}_i^{K \cap \gamma^{-1} H \gamma} P \right) = \bigoplus_{H \gamma K \in H \backslash G/K} A \rtimes_r H \otimes_{A \rtimes_r (K \cap \gamma^{-1} H \gamma)} P \cong \text{res}_i^G \circ \text{ind}_i^G P = A \rtimes_r G \otimes_{A \rtimes_r K} P,$$

is an isomorphism of $A \rtimes_r K$-modules. Its restriction to the summand for $H \gamma K \in H \backslash G/K$ sends $x \otimes p$ for $x \in A \rtimes_r K$ and $p \in P$ to $x \gamma \otimes p$. Since it is natural and compatible with direct sums, it suffices to show bijectivity for $P = A \rtimes_r H$ what is straightforward.

Again, we present an alternative proof which works in general (i.e., for $i = 1$ as well). Choose representatives $\gamma \in H \gamma K$ for every $H \gamma K \in H \backslash G/K$. For every such $\gamma$, choose representatives $\kappa_\gamma \in \kappa_\gamma(K \cap \gamma^{-1} H \gamma)$ for every $\kappa_\gamma(K \cap \gamma^{-1} H \gamma) \in K/(K \cap \gamma^{-1} H \gamma)$. The first observation is that the products $\kappa_\gamma \gamma^{-1}$ form a full set of representatives for $G/H$, i.e., we can write $G$ as a disjoint union as follows:

$$G = \bigcup_{\gamma, \kappa_\gamma} (\kappa_\gamma \gamma^{-1} H).$$

Now we use the representatives $\{\kappa_\gamma \gamma^{-1}\}$ of $G/H$ to construct as above the homomorphism $A \rtimes_r G \to M_{[G,H]}(A \rtimes_r H)$ which induces $\text{res}_i^G$ on K-theory. The
composition of this map with the canonical map $A \rtimes_r K \rightarrow A \rtimes_r G$ is given by

$$C_c(K, A) \ni a \cdot k \mapsto (x_{\kappa, \gamma^{-1}, \kappa', \gamma'^{-1}}) \in M_{[G:H]}(A \rtimes_r H)$$

with

$$x_{\kappa, \gamma^{-1}, \kappa', \gamma'^{-1}} = \begin{cases} ((\kappa_1^{-1}) \cdot a) \cdot (\kappa_2^{-1}kk\gamma\gamma'^{-1}) & \text{if } \gamma\kappa_1^{-1}kk\gamma\gamma'^{-1} \in H, \\ 0 & \text{else.} \end{cases}$$

The second observation is that the matrix $(x_{\kappa, \gamma^{-1}, \kappa', \gamma'^{-1}})$ can be decomposed into smaller matrix blocks since for $\gamma \neq \gamma'$, $\gamma\kappa_1^{-1}kk\gamma\gamma'^{-1}$ does not lie in $H$ no matter which $k \in K$ we take. This holds because $\gamma \neq \gamma'$ implies $(H\gamma K) \cap (H\gamma' K) = \emptyset$ by our choice of the $\gamma$s. Hence in K-theory, we obtain that the class of $(x_{\kappa, \gamma^{-1}, \kappa', \gamma'^{-1}})$ is the sum over $\gamma$ of the classes of $(x_{\kappa, \gamma^{-1}, \kappa', \gamma'^{-1}})_{\kappa, \kappa'}$.

Now the third observation is that

$$x_{\kappa, \gamma^{-1}, \kappa', \gamma'^{-1}} = \begin{cases} ((\kappa_1^{-1}) \cdot a) \cdot (\kappa_2^{-1}kk\gamma\gamma'^{-1}) & \text{if } \gamma\kappa_1^{-1}kk\gamma\gamma'^{-1} \in H, \\ 0 & \text{else.} \end{cases}$$

This means that the map

$$a \cdot k \mapsto (x_{\kappa, \gamma^{-1}, \kappa', \gamma'^{-1}})_{\kappa, \kappa'}$$

is precisely the composition with $c(\gamma)$ (or rather the extension of $c(\gamma)$ to matrices) of one of the maps $A \rtimes_r K \rightarrow M_{[K:K\cap \gamma^{-1}H\gamma]}(A \rtimes_r (K \cap \gamma^{-1}H\gamma))$ which induce $\res_K^{K\cap \gamma^{-1}H\gamma}$. This proves the Double Coset Formula. \hfill \Box

Let $\iota : H \rightarrow G$ be the inclusion of a normal subgroup of finite index. Denote by $N_{G/H} \in \mathbb{Z}[G/H]$ the norm element, i.e., $N_{G/H} = \sum_{H \gamma H} \gamma H$. If $M$ is any $\mathbb{Z}[G/H]$-module, then multiplication with $N_{G/H}$ induces a map $\mathbb{Z} \otimes \mathbb{Z}[G/H] M \rightarrow M^{G/H}$ whose kernel and whose cokernel are annihilated by multiplication with $[G : H]$. Denote by $c(g) : A \rtimes_r H \rightarrow A \rtimes_r H$ and by $c(g) : A \rtimes_r G \rightarrow A \rtimes_r G$ the ring homomorphisms obtained by conjugation with $g$, i.e., they send $x$ to $gxg^{-1}$. The induction homomorphism $\ind_{\iota(g)} : K_\iota(A \rtimes_r G) \rightarrow K_\iota(A \rtimes_r H)$ is the identity. The induction homomorphism $\ind_{\iota(g)} : K_\iota(A \rtimes_r H) \rightarrow K_\iota(A \rtimes_r H)$ is the identity provided that $g \in H$. Since $c(g_1) \circ c(g_2) = c(g_1g_2)$ holds for $g_1, g_2 \in G$ and $c(1) = \id$, we obtain a $G/H$-action on $K_\iota(A \rtimes_r H)$. The group homomorphisms $c(g) \circ \iota$ and $i \circ c(g)$ agree. Hence the map $\iota_* = \ind_\iota : K_\iota(A \rtimes_r H) \rightarrow K_\iota(A \rtimes_r H)$ factors over the canonical projection $K_\iota(A \rtimes_r H) \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G/H]} K_\iota(A \rtimes_r H)$ to a homomorphism $\iota^* : G \otimes_{[G:H]} K_\iota(A \rtimes_r H) \rightarrow K_\iota(A \rtimes_r H)$. In addition, the homomorphism $A \rtimes_s G \rightarrow M_{[G:H]}(A \rtimes_s H)$ which induces $\res$, commutes with $c(g)$ (the extended map on matrices) up to unitary equivalence, and therefore the map $\iota^* = \res : K_\iota(A \rtimes_r G) \rightarrow K_\iota(A \rtimes_r H)$ factors over the inclusion $K_\iota(A \rtimes_r H)^{G/H} \hookrightarrow K_\iota(A \rtimes_r H)$ to a map $\iota^* : K_\iota(A \rtimes_r G) \rightarrow K_\iota(A \rtimes_r H)^{G/H}$. These considerations and Lemma 6.3 imply

**Lemma 6.3.** Let $\iota : H \rightarrow G$ be the inclusion of a normal subgroup of finite index. Then we obtain a commutative diagram such that the kernel and cokernel of the
lower horizontal map are annihilated by $[G : H]$.

Moreover, we have:

**Corollary 6.4.** Let $F$ be a finite group and $A$ be a $F$-C*-algebra. Then the inclusion induces a map $K_i(A) \to K_i(A \rtimes F)$ that factors over the canonical projection $K_i(A) \to \mathbb{Z} \otimes_{\mathbb{Z}F} K_i(A)$ to a map $\mathbb{Z} \otimes_{\mathbb{Z}F} K_i(A) \to K_i(A \rtimes F)$ whose kernel is annihilated by multiplication with $|F|$.

**Proof.** This follows directly from Lemma 6.3 applied to $H = \{1\}$ and $G = F$. Of course, since $F$ is finite, we do not need to distinguish between reduced and full crossed products. □

6.2. **Injectivity after inverting orders.** Now we consider the case $G = R \rtimes \mu$.

Actually, we can treat a slightly more general situation, namely that $G = \mathbb{Z}_n \rtimes \mu$ for a finite cyclic group $\mu$ where the conjugation action of $\mu$ on $\mathbb{Z}_n$ is free outside the origin $0 \in \mathbb{Z}_n$. Let $A$ be a $G$-C*-algebra. Since $G$ is amenable, we do not have to distinguish between full and reduced crossed products. It is clear that $A \rtimes G \cong (A \rtimes \mathbb{Z}_n) \rtimes F$ where $\mathbb{Z}_n$ acts on $A$ via the restricted action and $F$ acts on $A \rtimes \mathbb{Z}_n$ via $c(f)$, i.e., $f \cdot (a \cdot g) = (f \cdot a) \cdot (fgf^{-1})$ for $a \in A$ and $g \in \mathbb{Z}_n$.

**Theorem 6.5.** Suppose that the map $K_i(A) \to K_i(A \rtimes \mathbb{Z}_n)$ is injective after inverting $|F|$. Then the map $K_i(A \rtimes F) \to K_i(A \rtimes G)$ induced by the canonical homomorphism $A \rtimes F \to A \rtimes G$ is injective after inverting $|F|$.

The proof of this theorem needs some preparation.

**Lemma 6.6.** Let $\mathcal{M}$ be a complete system of representatives of conjugacy class of maximal finite subgroups of $G$. Let $m$ be the least common multiple of the orders of the subgroups $M \in \mathcal{M}$.

Then for every $M \in \mathcal{M}$ the canonical homomorphisms $A \to A \rtimes M \to A \rtimes G$ induce a map

$$K_i(A \rtimes M) \to K_i(A \rtimes G)$$

which is bijective after inverting $m$.

**Proof.** We have the following $G$-pushout (compare [La-Lü], (4.5))

$$\begin{array}{ccc}
\Pi_{M \in \mathcal{M}} G \times_M EM & \xrightarrow{f} & EG \\
\Pi_{M \in \mathcal{M}} \text{id}_G \times_M f_M & & \\
\Pi_{M \in \mathcal{M}} G \times_M \{\bullet\} & \xrightarrow{f} & EG
\end{array}$$
Applying equivariant K-homology with coefficients in $A$, we obtain a long exact sequence (the Mayer-Vietoris sequence associated with the pushout)

$$
\cdots \rightarrow \bigoplus_{M \in \mathcal{M}} K_i^G(G \times_M EM; A) \rightarrow K_i^G(EG; A) \oplus \bigoplus_{M \in \mathcal{M}} K_i^G(G \times_M \{\bullet\}; A) \\
\hspace{2cm} \rightarrow K_i^G(EG; A) \rightarrow \bigoplus_{M \in \mathcal{M}} K_i^G(G \times_M EM; A) \\
\hspace{2cm} \rightarrow K_{i-1}^G(EG; A) \oplus \bigoplus_{M \in \mathcal{M}} K_{i-1}^G(G \times_M \{\bullet\}; A) \rightarrow K_{i-1}^G(EG; A) \rightarrow \cdots
$$

There is a spectral sequence converging to $K_i^G(G \times_M EM; A)$ whose $E^2$-term is $H_p^G(G \times_M EM; K_q(A))$. Since $M$ is finite, we know that $H_p^G(G \times_M EM; K_q(A)) \cong H_p^M(EM; K_q(A))$ is annihilated by multiplication with $|M|$ for $p \geq 1$. Since $H_0^G(G \times_M EM; K_i(A))$ can be identified with $\mathbb{Z} \otimes_{\mathbb{Z}M} K_i(A)$, the edge homomorphism

$$
\mathbb{Z} \otimes_{\mathbb{Z}M} K_i(A) \rightarrow K_i^G(G \times_M EM; A)
$$

is bijective after inverting $|M|$. Its composite with

$$
K_i^G(id_G \times f_M) : K_i^G(G \times_M EM; A) \rightarrow K_i^G(G \times_M \{\bullet\}; A) \cong K_i(A \times M)
$$

is the map $\mathbb{Z} \otimes_{\mathbb{Z}M} K_i(A) \rightarrow K_i(A \times M)$ induced by the canonical homomorphism $A \rightarrow A \times M$. The kernel of this map $\mathbb{Z} \otimes_{\mathbb{Z}M} K_i(A) \rightarrow K_i(A \times M)$ is annihilated by multiplication with $|M|$ by Corollary 6.4. This already implies injectivity of the map in (47) after inverting $m$. But this also implies that the long exact sequence (48) yields after inverting $m$ the short exact sequence

$$
0 \rightarrow \bigoplus_M \mathbb{Z} \otimes_{\mathbb{Z}M} K_i(A)[\frac{1}{m}] \\
\hspace{2cm} \rightarrow K_i^G(EG; A)[\frac{1}{m}] \oplus \bigoplus_{M \in \mathcal{M}} K_i(A \times M)[\frac{1}{m}] \\
\hspace{4cm} \rightarrow K_i(A \times G)[\frac{1}{m}] \rightarrow 0
$$

where we used that $G$ is amenable, hence satisfies the Baum-Connes conjecture with coefficients, i.e., $\text{asub} : K_i^G(EG; A) \rightarrow K_i(A \times G)$ is an isomorphism (see [Hig-Kas]). Exactness of (49) immediately yields surjectivity of the map in (47) because $\bigoplus_M \mathbb{Z} \otimes_{\mathbb{Z}M} K_i(A)[\frac{1}{m}] \rightarrow \bigoplus_{M \in \mathcal{M}} K_i(A \times M)[\frac{1}{m}]$ is induced by the canonical homomorphisms $A \rightarrow A \times M$ as explained above and $\bigoplus_{M \in \mathcal{M}} K_i(A \times M)[\frac{1}{m}] \rightarrow K_i(A \times G)[\frac{1}{m}]$ is induced by the canonical homomorphisms $A \times M \rightarrow A \times G$. This proves our claim. □

**Lemma 6.7.** For $i = 0$ or $1$, suppose that the map $K_i(A) \rightarrow K_i(A \times \mathbb{Z}^n)$ is injective after inverting $|F|$. Then the map $\mathbb{Z} \otimes_{\mathbb{Z}F} K_i(A) \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}F} K_i(A \times \mathbb{Z}^n)$ coming from the canonical homomorphism $A \rightarrow A \times \mathbb{Z}^n$ is injective after inverting $|F|$.

**Proof.** Consider the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z} \otimes_{\mathbb{Z}F} K_i(A) & \rightarrow & \mathbb{Z} \otimes_{\mathbb{Z}F} K_i(A \times \mathbb{Z}^n) \\
\downarrow N_F & & \downarrow N_F \\
K_i(A)^F & \rightarrow & K_i(A \times \mathbb{Z}^n)^F \\
\downarrow & & \downarrow \\
K_i(A) & \rightarrow & K_i(A \times \mathbb{Z}^n)
\end{array}
$$
The vertical maps denoted by $N_F$ are given by multiplication with the norm element $N_F$ and are isomorphisms after inverting $|F|$. The two lower vertical arrows are the canonical inclusions. The horizontal arrows are induced by the canonical homomorphism $A \to A \rtimes \mathbb{Z}^n$. Since the lower horizontal arrow is injective after inverting $|F|$ by assumption, the same is true for the upper horizontal arrow. □

Proof of Theorem 6.5. By assumption, the canonical homomorphism $A \to A \rtimes \mathbb{Z}^n$ is injective in $K$-theory once we invert $|F|$. Hence Lemma 6.7 implies that $\mathbb{Z} \otimes_{\mathbb{Z}F} K_i(A) \to \mathbb{Z} \otimes_{\mathbb{Z}F} K_i(A \rtimes \mathbb{Z}^n)$ is injective after inverting $|F|$. Now Corollary 6.4 tells us that the map

$$\mathbb{Z} \otimes_{\mathbb{Z}F} K_i(A \rtimes \mathbb{Z}^n) \to K_i((A \rtimes \mathbb{Z}^n) \rtimes F) \cong K_i(A \rtimes G)$$

is injective after inverting $|F|$. Hence $\mathbb{Z} \otimes_{\mathbb{Z}F} K_i(A) \to K_i(A \rtimes G)$ is injective after inverting $|F|$. Finally apply Lemma 6.6 in the case $M = F$. □

6.3. Injectivity. Finally, we are ready for the

Proof of Proposition 6.1. Let $c$ be some element in $R^\times$. Since the additive action of $c^{-1}R$ is homotopic to the trivial action, an iterative application of the Pimsner-Voiculescu sequence implies that the canonical homomorphism $C_0(\mathbb{A}_\infty) \to C_0(\mathbb{A}_\infty) \rtimes (c^{-1}R)$ is injective on $K_0$. Thus Theorem 6.5 yields that $C_0(\mathbb{A}_\infty) \rtimes \mu \to C_0(\mathbb{A}_\infty) \rtimes (c^{-1}R) \rtimes \mu$ is injective on $K_0$ after inverting $|\mu|$. By equivariant Bott periodicity (see Theorem 20.3.2 in [Bla1], $n$ is even), we know that $K_0(C_0(\mathbb{A}_\infty) \rtimes \mu) \cong K_0(C^*(\mu))$ and the latter group is free abelian. Thus the canonical homomorphism $C_0(\mathbb{A}_\infty) \rtimes \mu \to C_0(\mathbb{A}_\infty) \rtimes (c^{-1}R) \rtimes \mu$ itself must be injective on $K_0$. But then, since

$$C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu = \bigcup_{c \in R^\times} C_0(\mathbb{A}_\infty) \rtimes (c^{-1}R) \rtimes \mu,$$

the canonical homomorphism $C_0(\mathbb{A}_\infty) \rtimes \mu \to C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu$ must be injective on $K_0$ as well by continuity of $K_0$. □

References


