Descriptive set theory, dichotomies and graphs

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Introduction
Descriptive Set Theory is the study of definable subsets of Polish spaces, i.e. separable completely metrizable spaces.

Preliminaries
Familiarity with set theoretic operations and topological notions is assumed.

Notions
Throughout 2 denotes the set \( \{0, 1\} \)
Theorem (Silver)

Let $E$ be a definable equivalence relation on a Hausdorff space (e.g. isomorphism relation on a suitable class of objects) then exactly one of following holds:

- There are at most countably many equivalence classes.
- There exists a perfect set of inequivalent elements.

We will prove this from a graph theoretic dichotomy.
In fact there exist variants of that dichotomy which can be used to obtain similar results.

By slightly strengthening definable we can show: Either there is a Borel reduction from $E$ to $=\text{ on } \mathcal{P}(\mathbb{N})$ or there is a continuous embedding of $=^*$ on $\mathcal{P}(\mathbb{N})$ into $E$.

Silvers result for pregeometries (e.g. vector spaces: either the dimension is countable or there exists a perfect set of linear independent elements).

The minimal subset of a Polish space that is not intersection of countable many open sets, is the rationals.
What does definable mean?

Consider the following spaces:

\(2^\mathbb{N}\), 0-1-sequences, and \(\mathbb{N}^\mathbb{N}\), sequences of natural numbers. Endowed with the product topology of the discrete topology on \(2\), \(\mathbb{N}\) respectively.

Definition

A set is **analytic** if it is the continuous image of closed subset of \(\mathbb{N}^\mathbb{N}\). The complement of an analytic set is **co-analytic**. A set that is both is **bi-analytic**.

Note:

If you close the class of Borel sets of a Polish space under projections then you will get the analytic sets.
Definition

A set $A \subseteq X$ is **co-meager** if it contains a set that is the intersection of countable many dense open sets. The complement of a co-meager set is **meager**.

Definition

$A$ is **Baire measurable** if $A \Delta U$ is meager for some open set $U$.

Note:

Analytic and Borel sets are Baire measurable.

Theorem (Mycielski)

*Let* $R \subseteq (2^\mathbb{N})^2$ *be comeager then there exists a perfect set of pairwise* $R$-*related elements.*
Note:
In a Polish space the meager sets form a $\sigma$-Ideal (co-meager sets form a $\sigma$-complete filter). In this respect they behave similar to Lebesgue null sets (sets of full measure) in a probability space.

But
There are co-meager sets with Lebesgue measure zero and thus meager sets of full measure.

However we have one other important similarity:

Theorem (Kuratowski-Ulam)
Let $X, Y$ be analytic Hausdorff spaces and $A \subseteq X \times Y$ Baire measurable. Then $A$ is co-meager iff

$$A_x := \{x \in X \mid \text{For co-meagerly many } y \in Y, (x, y) \in A\}$$

is co-meager in $X$. 
Definition

A Graph $G$ on a set $X$ is a symmetric, irreflexive relation on $X \times X$.

Definition

A set $A$ is $G$-independent if $G \upharpoonright (A \times A) = \emptyset$.

Definition

A function $c: X \mapsto \mathbb{N}$ is a countable Borel coloring of $G$ if $c^{-1}(n)$ is Borel and $G$-independent for all $n \in \mathbb{N}$.
Fix $S \subseteq 2^{\mathbb{N}}$ with the following properties:

1. $\forall n \in \mathbb{N} \mid |S \cap 2^n| = 1$, we denote that element by $s_n$

2. $\forall s \in 2^{\mathbb{N}} \exists n \in \mathbb{N} \mid s \subseteq s_n$

**Definition**

$G_0$ is the following Graph on $2^{\mathbb{N}}$

$$G_0 = \{(s_n(0)x, s_n(1)x) \mid n \in \mathbb{N}, x \in 2^{\mathbb{N}}\}$$
Proposition

Let $A \subseteq 2^\mathbb{N}$ be $G_0$-independent and Baire measurable then $A$ is meager.

Theorem

Let $X$ be a Hausdorff space and $G$ an analytic graph on $X$. Then exactly one of the following holds:

- There is a countable Borel coloring of $G$
- There is a continuous homomorphism from $G_0$ to $G$
Theorem (Silver)

Let $E$ be a co-analytic equivalence relation on a Hausdorff space then exactly one of following holds:

- There are at most countably many equivalence classes.
- There exists a perfect set of inequivalent elements.