Games in Set Theory

Why logicians like to play games.

Sandra Uhlenbrock
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Games in Set Theory

David (I) 1 0 1 1
Yizheng (II) 0 0 1 0
Definition (Gale/Stewart 1953)

Let $A \subset 2^\mathbb{N}$. With $G(A)$ we denote the following game

\[
\begin{array}{c|ccc}
I & i_0 & i_2 & \cdots \\
II & i_1 & i_3 & \cdots \\
\end{array}
\]

for $i_n \in \{0, 1\}$ and $n \in \mathbb{N}$.

We say player I wins the game iff $(i_n)_{n \in \mathbb{N}} \in A$. Otherwise player II wins.

We say $G(A)$ (or $A$ itself) is **determined** iff one of the players has a winning strategy (short: ws) (in the obvious sense).
Which games are determined?

Theorem (Gale/Stewart)

(AC) Let \( A \subset 2^\mathbb{N} \) be open or closed. Then \( G(A) \) is determined.

For any \( B \subset [0, 1] \) and \( s \in 2^{<\mathbb{N}} \), let

\[
B/s = \{ x \in 2^\mathbb{N} | s \upharpoonright x \in B \}.
\]

Claim

If I has no ws in \( G(B/s) \), then for any move \( i \) of I there is a move \( j \) of II such that I has no ws in \( G(B/s \upharpoonright (i,j)) \).

Proof.

Otherwise, let \( i \) be s.t. for every \( j \), I has a winning strategy \( \sigma \) in \( G(B/s \upharpoonright (i,j)) \). Then I would also have a ws in \( G(B/s) \): Initially play \( i \) and then follow the ws \( \sigma \).
Which games are determined?

Proof.

Suppose now that $A$ is open and assume that $I$ has no winning strategy in $G(A)$.
We can recursively define a strategy $\tau$ for $II$ using the claim. Therefore for any partial play $s$ according to $\tau$, $I$ has no ws in $G(A/s)$.

$\tau$ is in fact a winning strategy for $II$ in $G(A)$:
Assume towards a contradiction that $x$ is a play according to $\tau$ s.t. $x \in A$.
Since $A$ is open, there is an $n \in \mathbb{N}$ s.t. for every $y \in O(x \upharpoonright n)$ in fact $y \in A$.
But then any strategy for $I$ in $G(A/(x \upharpoonright n))$ would be a winning one, contradicting the definition of $\tau$.

The argument for closed $A$ is analogous, with the roles of $I$ and $II$ interchanged.
Are all games determined?

Theorem (Gale/Stewart)
Assuming AC there is a set of reals which is not determined.

Proof.

Let $(\sigma_\alpha | \alpha < 2^\mathbb{N})$ resp. $(\tau_\alpha | \alpha < 2^\mathbb{N})$ enumerate the strategies for I resp. II.

Recursively choose $a_\alpha, b_\alpha$ for $\alpha < 2^\mathbb{N}$ as follows:

Having chosen $a_\beta$ and $b_\beta$ for $\beta < \alpha$, choose $b_\alpha$ s.t.

$$b_\alpha = \sigma_\alpha * y$$

for some $y \in 2^\mathbb{N}$ and $b_\alpha \notin \{a_\beta | \beta < \alpha\}$. This is possible since $|\{\sigma_\alpha * y | y \in 2^\mathbb{N}\}| = 2^\mathbb{N}$.

Similarly choose $a_\alpha$ s.t. $a_\alpha = z * \tau_\alpha$ for some $z$ and $a_\alpha \notin \{b_\beta | \beta < \alpha\}$. 
Are all games determined?

Proof.

Set \( A = \{ a_\alpha \mid \alpha < 2^\mathbb{N} \} \) and \( B = \{ b_\alpha \mid \alpha < 2^\mathbb{N} \} \).

Claim

\( A \) and \( B \) are disjoint.

Assume \( a_\alpha = b_\beta \in A \cap B \) with \( a_\alpha = \sigma_\alpha \ast y \) and \( b_\beta = z \ast \tau_\beta \). Wlog \( \alpha < \beta \).

\( b_\beta \) is constructed s.t. \( b_\beta \notin \{ a_\gamma \mid \gamma < \beta \} \), so in particular \( b_\beta \neq a_\alpha \). Contradiction.

Claim

Neither I nor II has a winning strategy for \( G(A) \).

Assume I has a ws for \( G(A) \), say \( \sigma_\alpha \). That means \( \sigma_\alpha \ast x \in A \) for all \( x \).

But \( b_\alpha = \sigma_\alpha \ast y \in B \), so by the above claim \( b_\alpha \notin A \). Contradiction.

The argument for II is analogous.
What’s about sets “in between”?

- For Borel sets determinacy can be shown in ZFC.
- For Analytic sets and Projective sets determinacy is not provable in ZFC.
- For arbitrary sets determinacy contradicts Choice (and therefore ZFC).
Another approach

Definition

We say $A$ is Wadge-reducible to $B$ ($A \leq_W B$) iff

$$x \in A \iff f(x) \in B$$

for some continuous $f : 2^\mathbb{N} \to 2^\mathbb{N}$.

We say $A$ and $B$ are Wadge-comparable iff either $A \leq_W B$ or else $B \leq_W 2^\mathbb{N} \setminus A$.

Theorem (Wadge)

Assume AD. Suppose that $A, B \subseteq 2^\mathbb{N}$. Then $A$ and $B$ are Wadge-comparable.
Proof.

Consider the Wadge game $\mathcal{W}G(A, B)$ which is defined as follows:

\[
\begin{array}{c|ccc}
I & x_0 & x_2 & \cdots \\
II & x_1 & x_3 & \cdots \\
\end{array}
\]

for $x_n \in \{0, 1\}$ and $n \in \mathbb{N}$.

II wins exactly when

$$x_I \in A \text{ iff } x_{II} \in B.$$  

If $\tau$ is a winning strategy for II, then for any $z$

$$z \in A \text{ iff } (z \ast \tau)_{II} \in B.$$  

So $A \leq_W B$.

On the other hand, if $\sigma$ is a winning strategy for I, then for any $y$

$$(\sigma \ast y)_I \notin A \text{ iff } y \in B.$$  

So $B \leq_W 2^\mathbb{N} \setminus A$.  

\[\square\]
Let’s look at the case where we only consider projective sets.

- The last proof shows that PD implies Wadge-comparability for projective sets.
- The other direction is a well-known open problem.
- Why is this interesting?
  It would be the first proof of PD from a statement that only involves projective sets.
Questions in Modern Set Theory

- Prove equiconsistency of statements that are not provable in ZFC
- Analyze how the “universe” changes, if statements are added to ZF or ZFC.

For example: Which statements are true under ZF + AD?
Ready to play?

For reference see Akihiro Kanamori “The higher infinite”.