Increasing u_2 by a stationary set preserving forcing

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Abstract

We show that if I is a precipitous ideal on ω_1 and if $\theta > \omega_1$ is a regular cardinal, then there is a forcing $\mathbb{P} = \mathbb{P}(I, \theta)$ which preserves the stationarity of all I-positive sets such that in $V^{\mathbb{P}}$, $\langle H_\theta; \in, I\rangle$ is a generic iterate of a countable structure $\langle M; \in, \overline{I}\rangle$. This shows that if the nonstationary ideal on ω_1 is precipitous and $H_\theta^{\#}$ exists, then there is a stationary set preserving forcing which increases $\underset{\sim}{\delta}^1_2$. Moreover, if Bounded Martin's Maximum holds and the nonstationary ideal on ω_1 is precipitous, then $\underset{\sim}{\delta}^1_2 = u_2 = \omega_2$.

In this paper we modify Jensen's $\mathcal{L}$-forcing (cf. [Jen90a] and [Jen90b]) and apply this to the theory of precipitous ideals and the question about the size of u_2. Forcings which increase the size of u_2 were already presented in the past. After Steel and van Wesep had shown that $u_2 = \omega_2$ is consistent in the presence of large cardinal hypotheses (cf. [SVW82]), Woodin proved that if the nonstationary ideal on ω_1 is ω_2-saturated and $\mathcal{P}(\omega_1)^{\#}$ exists, then $u_2 = \omega_2$ (cf. [Woo99, Theorem 3.17]; in particular, $u_2 = \omega_2$ follows from Martin's Maximum by work of Foreman, Magidor and Shelah, cf. [FS88].) More recently, Ketchersid, Larson, and Zapletal also constructed forcings which increase u_2 (cf. [KLZ07]).

Recall that $\underset{\sim}{\delta}^1_2$ is the supremum of the lengths of all $\underset{\sim}{\Delta}^1_2$ well-orderings of the reals, and that if the reals are closed under sharps, then u_2, the second uniform indiscernible, is defined to be the least ordinal above ω_1

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which is an x-indiscernible for every $x \in \mathbb{R}$. By the Kunen-Martin Theorem (cf. [Mos80, Theorem 2G.2]), if $\leq$ is a $\Delta^1_2(x)$ prewellordering of $\mathbb{R}$, then the length of $\leq$ is less than $\omega_1^{+L[x]}$. Moreover, if $x^\#$ exists, then there is a $\Delta^1_2(x^\#)$-prewellordering of $\mathbb{R}$ of length $\omega_1^{+L[x]}$, which implies $\omega_1^{+L[x]} < \underset{\sim}{\delta}^1_2$. Also, $\omega_1^{+L[x^\#]} < u_2^x$, the least x-indiscernible above ω_1. Therefore, if the reals are closed under sharps, then

$$u_2 = \sup\left\{\omega_1^{+L[x]}; x \in \mathbb{R}\right\} = \underset{\sim}{\delta}^1_2.$$

In this paper we'll consider generic iterations of structures of the form $\langle M; \in, I\rangle$, where M is a transitive model of ZFC^* + "ω_1 exists" and inside M, I is a uniform and normal ideal on ω_1^M. Here, ZFC^* is a reasonable weak fragment of ZFC such that ZFC^* + "ω_1 exists" is suitable for taking generic ultrapowers by ideals on ω_1 (cf. [Woo99]). For a set X, we let $X^\#$ denote the least X-mouse, i.e., the least X-premouse $\mathcal{P} = (J_\alpha(X); \in, X, E_\alpha)$, such that $E_\alpha \neq \emptyset$, $\mathcal{P}$ is sound above X, and $\mathcal{P}$ is iterable. The universe of any $X^\#$ is a model of ZFC^* + "ω_1 exists."

Let I be an ideal on ω_1. We shall write $I^+ = \{x \subseteq \omega_1; x \notin I\}$ for the set of the I-positive sets. We shall also write $X \leq_I Y$ iff $X \setminus Y \in I$. Forcing with $\langle I^+, \leq_I\rangle$ adds a V-measure G and thereby a generic embedding $\pi\colon V \to \mathrm{Ult}(V; G)$. The ideal I is *precipitous* iff $\mathrm{Ult}(V; G)$ is always well-founded. (Cf. [Jec03, pp. 424ff.].)

Definition 1. Let M be a transitive model of ZFC^* + "ω_1 exists," and let $I \subseteq \mathcal{P}(\omega_1^M)$ be such that $\langle M; \in, I\rangle \models$ "I is a uniform and normal ideal on ω_1^M." Let $\gamma \leq \omega_1$. Then

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \gamma\rangle, \langle G_i; i < \gamma\rangle\rangle$$

is called a *putative generic iteration of* $\langle M; \in, I\rangle$ (*of length* $\gamma + 1$) iff the following hold true.

i. $M_0 = M$ and $I_0 = I$.

ii. For all $i \leq j \leq \gamma$, $\pi_{i,j}\colon \langle M_i; \in, I_i\rangle \to \langle M_j; \in, I_j\rangle$ is elementary, $I_i = \pi_{0,i}(I)$, and $\kappa_i = \pi_{0,i}(\omega_1^M) = \omega_1^{M_i}$.

iii. For all $i < \gamma$, M_i is transitive and G_i is $\langle I_i, \leq_{I_i}\rangle$-generic over M_i.

iv. For all $i + 1 \leq \gamma$, $M_{i+1} = \mathrm{Ult}(M_i; G_i)$ and $\pi_{i,i+1}$ is the associated ultrapower map.

v.

$pi_{j,k} \circ pi_{i,j} = \pi_{i,k}$ for $i \leqslant j \leqslant k$.

vi. If $\lambda \leq \gamma$ is a limit ordinal, then $\langle M_\lambda, \pi_{i,\lambda}, i < \lambda\rangle$ is the direct limit of $\langle M_i, \pi_{i,j}, i \leqslant j < \lambda\rangle$.

We call

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \gamma\rangle, \langle G_i; i < \gamma\rangle\rangle$$

a *generic iteration of* $\langle M; \in, I\rangle$ (*of length* $\gamma+1$) iff it is a putative generic iteration of $\langle M; \in, I\rangle$ and M_γ is transitive. $\langle M; \in, I\rangle$ is *generically* $\gamma+1$ *iterable* iff every putative generic iteration of $\langle M; \in, I\rangle$ of length $\gamma+1$ is an iteration.

Notice that we want (putative) iterations of a given model $\langle M; \in, I\rangle$ to exist in V, which amounts to requiring that the relevant generics G_i may be found in V. The following lemma is therefore only interesting in situations in which M (or a large enough initial segment thereof) is countable so that we may actually find generics in V.

Lemma 2 (Woodin). *Let M be a transitive model of* ZFC*, and let $I \subseteq \mathcal{P}(\omega_1^M)$ be such that $\langle M; \in, I\rangle \models$ "I is a uniform and normal precipitous ideal on ω_1^M." Then $\langle M; \in, I\rangle$ is generically $\gamma+1$ iterable whenever $\gamma < \min(M \cap \mathrm{OR}, \omega_1^V + 1)$.*

Proof. The proof is taken from [Woo99, Lemma 3.10, Remark 3.11]. By absoluteness, if $\langle M; \in, I\rangle$ is not generically $\gamma+1$ iterable, then $\langle H_\kappa^M; \in, I\rangle$ is not generically $\gamma+1$ iterable inside $M^{\mathrm{Col}(\omega,\delta)}$ for some κ and δ such that κ is regular in M, $H_\kappa^M \models \mathsf{ZFC}^* +$ "ω_1 exists," and $\delta \geq \gamma$ (cf. [Woo99, Lemma 3.8]). Let $\langle \kappa_0, \eta_0, \gamma_0\rangle$ be the least triple in the lexicographical order such that:

- $\kappa_0 > \omega_1^M$ is regular in M,
- $\eta_0 < \kappa_0$, and
- for some δ, inside $M^{\mathrm{Col}(\omega,\delta)}$, there is a putative iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \gamma_0\rangle, \langle G_i; i < \gamma_0\rangle\rangle$$

  of $\langle H_{\kappa_0}^M; \in, I\rangle$ such that $\pi_{0,\gamma_0}(\eta_0)$ is ill-founded.

As I is precipitous in M, γ_0 and η_0 are limit ordinals. Choose some $i^* < \gamma_0$ and $\eta^* < \pi_{0,i^*}(\eta_0)$ such that $\pi_{i^*,\gamma_0}(\eta^*)$ is ill-founded. We may construe

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i^* \leqslant i \leqslant j \leqslant \gamma_0\rangle, \langle G_i; i^* \leqslant i < \gamma_0\rangle\rangle$$

as a putative generic iteration of $H^{M_{i^*}}_{\pi_{0,i^*}(\kappa_0)}$. By elementarity, the triple $\langle \pi_{0,i^*}(\kappa_0), \pi_{0,i^*}(\eta_0), \pi_{0,i^*}(\gamma_0)\rangle$ is the least triple $\langle \kappa, \eta, \gamma\rangle$ such that

- $\kappa > \omega_1^{M_{i^*}}$ is regular in M_{i^*},
- $\eta < \kappa$, and
- for some δ, inside $M_{i^*}^{\mathrm{Col}(\omega,\delta)}$, there is a putative iteration

$$\langle\langle M'_i, \pi'_{i,j}, I'_i, \kappa'_i; i \leqslant j \leqslant \gamma\rangle, \langle G'_i; i < \gamma\rangle\rangle$$

of $\langle H^{M_{i^*}}_{\pi_{0,i^*}(\kappa)}; \in, I_{i^*}\rangle$ such that $\pi'_{0,\gamma}(\eta)$ is ill-founded.

However, by the existence of

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i^* \leqslant i \leqslant j \leqslant \gamma_0\rangle, \langle G_i; i^* \leqslant i < \gamma_0\rangle\rangle$$

and by absoluteness, the triple $\langle \pi_{0,i^*}(\kappa_0), \eta^*, \gamma_0 - i^*\rangle$ contradicts the alleged characterization of the triple $\langle \pi_{0,i^*}(\kappa_0), \pi_{0,i^*}(\eta_0), \pi_{0,i^*}(\gamma_0)\rangle$ inside M_{i^*}. □

By NS_{ω_1} we shall denote the nonstationary ideal on ω_1.
We may now state and prove our main result.

Theorem 3. *Let I be a precipitous ideal on ω_1, and let $\theta > \omega_1$ be a regular cardinal. There is a poset $\mathbb{P}(I, \theta)$, preserving the stationarity of all sets in I^+, such that if G is $\mathbb{P}(I, \theta)$-generic over V, then in $V[G]$ there is a generic iteration*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that if $i < \omega_1$, then M_i is countable and $M_{\omega_1} = \langle H_\theta; \in, I\rangle$. If $I = \mathrm{NS}_{\omega_1}$, then $\mathbb{P}_{\mathrm{NS}_{\omega_1}}$ is stationary set preserving.

It is easy to see that every set in I^+ has to be stationary in V. The most difficult part of the construction is to arrange that every set in I^+ will remain stationary in the forcing extension.

The proof of Theorem 3 stretches over several lemmas and builds upon Jensen's [Jen90a] and [Jen90b]. Fixing I and θ, let us pick a regular cardinal ρ such that $2^{2^{<\theta}} < \rho$. Therefore, $H_\theta \in H_\rho$, and in fact every subset of

$\mathcal{P}(H_\theta)$ is in H_ρ as well. In particular, the forcing $\mathbb{P}(I, \theta)$ we are about to define will be an element of H_ρ. It is easy to verify that if a forcing $\mathbb{Q} \in V$ is ω_1-distributive, then I is still precipitous in $V^{\mathbb{Q}}$. We may and shall therefore assume that $2^{<\theta} = \theta$ and $2^{<\rho} = \rho$, i.e., that $\mathrm{Card}(H_\theta) = \theta$ and $\mathrm{Card}(H_\rho) = \rho$, because if this were not true in V, then we may first force with $\mathbb{Q} = \mathrm{Col}(\rho, \rho) \times \mathrm{Col}(\theta, \theta)$ and work with $V^{\mathbb{Q}}$ rather than V as our ground model in what follows.

Our starting point is thus that in V, I is a precipitous ideal on ω_1 and θ and ρ are regular cardinals such that $\omega_2 \leq \theta = 2^{<\theta} < 2^\theta < \rho = 2^{<\rho}$. Let us fix a well-order, denoted by $<$, of H_ρ of order type ρ such that $< \restriction H_\theta$ is an initial segment of $<$ of order type θ. (In what follows, we shall also write $<$ for $< \restriction H_\theta$.) We shall write

$$\mathcal{H} = \langle H_\rho; \in, H_\theta, I, < \rangle,$$

and we shall also write

$$\mathcal{M} = \langle H_\theta; \in, I, < \rangle.$$

In what follows, models will always be models of the languange of set theory. We shall tacitly assume that if $\mathfrak{A}$ is a model, then the well-founded part $\mathrm{wfp}(\mathfrak{A})$ of $\mathfrak{A}$ is transitive.

Let us now define our forcing $\mathbb{P}(I, \theta)$.

Definition 4. Conditions p in $\mathbb{P}(I, \theta)$ are triples

$$p = \langle \langle \kappa_i^p ; i \in \mathrm{dom}(p) \rangle, \langle \pi_i^p ; i \in \mathrm{dom}(p) \rangle, \langle \tau_i^p ; i \in \mathrm{dom}_-(p) \rangle \rangle$$

such that the following hold true.

i. Both $\mathrm{dom}(p)$ and $\mathrm{dom}_-(p)$ are finite, and $\mathrm{dom}_-(p) \subseteq \mathrm{dom}(p) \subseteq \omega_1$.

ii. $\langle \kappa_i^p ; i \in \mathrm{dom}(p) \rangle$ is a sequence of countable ordinals.

iii. $\langle \pi_i^p ; i \in \mathrm{dom}(p) \rangle$ is a sequence of finite partial maps from ω_1 to θ.

iv. $\langle \tau_i^p ; i \in \mathrm{dom}_-(p) \rangle$ is a sequence of complete $\mathcal{H}$-types over H_θ, i.e., for each $i \in \mathrm{dom}_-(p)$ there is some $x \in H_\rho$ such that, having φ range over $\mathcal{H}$-formulae with free variables $u, \vec{v}$,

$$\tau_i^p = \{ \langle \ulcorner \varphi \urcorner, \vec{z} \rangle \ ; \ \vec{z} \in H_\theta \wedge \mathcal{H} \models \varphi[x, \vec{z}] \}.$$

v. If i, $j \in \mathrm{dom}_-(p)$, where $i < j$, then there is some $n < \omega$ and some $\vec{u} \in \mathrm{ran}(\pi_j^p)$ such that

$$\tau_i^p = \{ (m, \vec{z}) \ ; \ (n, \vec{u} ^\frown m ^\frown \vec{z}) \in \tau_j^p \}.$$

vi. In $V^{\mathrm{Col}(\omega,\theta)}$, there is a *model which certifies* p *with respect to* $\mathcal{M}$, by which we mean a model $\mathfrak{A}$ such that $\theta+1 \subset \mathrm{wfp}(\mathfrak{A})$, in fact $H_\theta \in \mathfrak{A}$, $\mathfrak{A} \models \mathsf{ZFC}^-$ ($= \mathsf{ZFC} \setminus$ Power Set), for all $S \in I^+$, $\mathfrak{A} \models$ "S is stationary," and inside $\mathfrak{A}$, there is a generic iteration

$$\langle\langle M_i^{\mathfrak{A}}, \pi_{i,j}^{\mathfrak{A}}, I_i^{\mathfrak{A}}, \kappa_i^{\mathfrak{A}}; i \leqslant j \leqslant \omega_1\rangle, \langle G_i^{\mathfrak{A}}; i < \omega_1\rangle\rangle$$

such that

(a) if $i < \omega_1$, then $M_i^{\mathfrak{A}}$ is countable,

(b) if $i < \omega_1$ and if $\xi < \theta$ is definable over $\mathcal{M}$ from parameters in $\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$, then $\xi \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$,

(c) $M_{\omega_1}^{\mathfrak{A}} = \langle H_\theta; \in, I\rangle$,

(d) if $i \in \mathrm{dom}(p)$, then $\kappa_i^p = \kappa_i^{\mathfrak{A}}$ and $\pi_i^p \subseteq \pi_{i,\omega_1}^{\mathfrak{A}}$,

(e) if $i \in \mathrm{dom}_-(p)$, then for all $n < \omega$ and for all $\vec{z} \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$,

$$\exists y \in H_\theta \; (n, y^\frown \vec{z}) \in \tau_i^p \Longrightarrow \exists y \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}}) \; (n, y^\frown \vec{z}) \in \tau_i^p.$$

If p, $q \in \mathbb{P}$, then we write $p \leqslant q$ iff $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$, $\mathrm{dom}_-(q) \subseteq \mathrm{dom}_-(p)$, for all $i \in \mathrm{dom}(q)$, $\kappa_i^p = \kappa_i^q$ and $\pi_i^q \subseteq \pi_i^p$, and for all $i \in \mathrm{dom}_-(q)$, $\tau_i^q = \tau_i^p$.

Conditions p should be seen as finite attempts to describe the iteration leading to $\langle H_\theta; \in, I\rangle$, the first component being finitely many critical points κ_i^p of the iteration, and the second component being finite attempts π_i^p to describe the iteration maps restricted to the ordinals. The presence of $<$ will guarantee that knowing the action of these maps on the ordinals means knowing the maps themselves. The third components τ_i^p will guarantee that the iteration maps extend to elementary maps into $\mathcal{H}$ with some $x \in H_\rho$ of interest in their range (cf. Lemma 16 below), which will be relevant in the verification that $\mathbb{P}(I,\theta)$ preserves the stationarity of all sets in I^+.

It should be stressed that $\omega_1^V \in I^+$, so that if $\mathfrak{A}$ certifies any condition p with respect to $\mathcal{M}$, then $\omega_1^{\mathfrak{A}} = \omega_1^V$. It is also clear that

$$\mathfrak{A} \models \mathrm{Card}(H_\theta) = \aleph_1.$$

Let us start the discussion of $\mathbb{P}(I,\theta)$. Let us write $\mathbb{P} = \mathbb{P}(I,\theta)$ from now on.

Lemma 5. $\mathbb{P} \neq \emptyset$.

Proof. We need to verify that in $V^{\mathrm{Col}(\omega,\theta)}$ there is a model which certifies the trivial condition $\langle\langle\rangle,\langle\rangle,\langle\rangle\rangle$ with respect to $\mathcal{M}$.

Let g be $\mathrm{Col}(\omega,<\rho)$-generic over V. Notice that inside $V[g]$, $\langle V;\in,I\rangle$ is generically $\rho+1$ iterable by Lemma 2. Let us work inside $V[g]$ until further notice.

Let us choose a bijection $\varphi:[\rho]^{<\rho}\to\rho$, and let $\langle S_\nu;\nu<\rho\rangle$ be a partition of ρ into pairwise disjoint stationary subsets of ρ. Define $f:\rho\to[\rho]^{<\rho}$ by

$$f(i)=s\Longleftrightarrow i\in S_{\varphi(s)}.$$

In other words, $f"S_{\varphi(s)}=\{s\}$ for every $s\in[\rho]^{<\rho}$.

Let us recursively construct a generic iteration

$$\langle\langle M_i,\pi_{i,j},I_i,\kappa_i;i\leqslant j\leqslant\rho\rangle,\langle G_i;i<\rho\rangle\rangle$$

of $M_0=\langle H_\theta;\in,I\rangle$. Suppose $\langle\langle M_k,\pi_{k,j},I_k,\kappa_k;k\leqslant j\leqslant i\rangle,\langle G_k;k<i\rangle\rangle$ has already been constructed, where $i<\rho$. If there is a (unique) $j\leq i$ such that $f(i)\in I_j^+$, i.e., $\pi_{j,i}(f(i))\in I_i^+$, then let us choose G_i such that $\pi_{j,i}(f(i))\in G_i$. If there is no such $j\leq i$, then we choose G_i arbitrarily. This defines the generic iteration.

Now let $S\in I_\rho^+$. Let $j<\rho$ and $s\in M_j$ be such that $\pi_{j,\rho}(s)=S$. Whenever $j\leq i<\rho$ and $f(i)=s$, then $\pi_{j,i}(s)\in G_i$, i.e., $\kappa_i\in\pi_{i,i+1}(\pi_{j,i}(s))=\pi_{j,i+1}(s)\subseteq\pi_{j,\rho}(s)=S$. This shows that

$$S_{\varphi(s)}\setminus j\subseteq\{i<\rho\ ;\ \kappa_i\in S\},$$

so that S is in fact stationary.

Let us now leave $V[g]$ and pick some h which is $\mathrm{Col}(\omega,\pi_{0,\rho}(\theta))$-generic over $V[g]$. The map $\pi_{0,\rho}\colon H_\theta\to M_\rho$ admits a canonical extension $\pi\colon V\to N$, where N is transitive and $\pi(H_\theta)=M_\rho$. Of course, h is also $\mathrm{Col}(\omega,\pi_{0,\rho}(\theta))$-generic over N. Let $x\in\mathbb{R}\cap N[h]$ code $\pi(\mathcal{M})$ in a natural way. The existence of a model which certifies $\langle\langle\rangle,\langle\rangle,\langle\rangle\rangle$ with respect to $\pi(\mathcal{M})$ is then easily seen to be a $\Sigma^1_1(x)$ statement which holds true in $V[g,h]$, as being witnessed by $V[g]$. By absoluteness, this statement is then also true in $N[h]$. That is, inside $N^{\mathrm{Col}(\omega,\pi_{0,\rho}(\theta))}$ there is a model which certifies $\langle\langle\rangle,\langle\rangle,\langle\rangle\rangle$ with respect to $\pi(\mathcal{M})$. By elementarity, in $V^{\mathrm{Col}(\omega,\theta)}$ there is therefore a model which certifies $\langle\langle\rangle,\langle\rangle,\langle\rangle\rangle$ with respect to $\mathcal{M}$. $\square$

We will now prove some lemmata which will make sure that the generic filter indeed produces a generic iteration leading to $\langle H_\theta;\in,I\rangle$. If $p\in\mathbb{P}$, then from now on we shall often just say that $\mathfrak{A}$ *certifies* p to express that $\mathfrak{A}$ is a model which certifies p with respect to $\mathcal{M}$.

Lemma 6. *Let $p \in \mathbb{P}$, let u be finite such that $\mathrm{dom}(p) \subseteq u \subseteq \omega_1$. There is $p' \leqslant p$ such that $u \subseteq \mathrm{dom}(p')$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\theta)}$ certify p. We may define p' such that $\mathrm{dom}(p') = u$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa_i^{p'} = \kappa_i^{\mathfrak{A}}$ for $i \in u$, $\pi_i^{p'} = \pi_i^p$ for $i \in \mathrm{dom}(p)$, $\pi_i^{p'} = \emptyset$ for $i \in \mathrm{dom}(p') \setminus \mathrm{dom}(p)$, and $\tau_i^{p'} = \tau_i^p$ for $i \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies p', and of course $p' \leqslant p$. □

Lemma 7. *Let $p \in \mathbb{P}$, $i \in \mathrm{dom}(p)$ and $\xi < \theta$. There is a $p' \leqslant p$ and an $\alpha \in \mathrm{dom}(\pi_i^{p'})$ such that $\xi < \pi_i^{p'}(\alpha)$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\theta)}$ certify p. Let α be such that $\pi_{i,\omega_1}^{\mathfrak{A}}(\alpha) > \xi$. (Such an α exists, as the iteration map $\pi_{i,\omega_1}^{\mathfrak{A}}$ is cofinal.) We may define p' such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa_j^{p'} = \kappa_j^p$ for $j \in \mathrm{dom}(p)$, $\pi_j^{p'} = \pi_j^p$ for $j \in \mathrm{dom}(p) \setminus \{i\}$, $\pi_i^{p'} = \pi_i^p \cup \left\{\langle \alpha, \pi_{i,\omega_1}^{\mathfrak{A}}(\alpha)\rangle\right\}$, and $\tau_j^{p'} = \tau_j^p$ for $j \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies p', and of course $p' \leqslant p$. □

Lemma 8. *Let $p \in \mathbb{P}$, $i \in \mathrm{dom}(p)$, $\xi < \zeta$ and $\zeta \in \mathrm{dom}(\pi_i^p)$. There is a $p' \leqslant p$ such that $\xi \in \mathrm{dom}(\pi_i^{p'})$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\theta)}$ certify p. We may define p' such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa_j^{p'} = \kappa_j^p$ for $j \in \mathrm{dom}(p)$, $\pi_j^{p'} = \pi_j^p$ for $j \in \mathrm{dom}(p) \setminus \{i\}$, $\pi_i^{p'} = \pi_i^p \cup \left\{\langle \xi, \pi_{i,\omega_1}^{\mathfrak{A}}(\xi)\rangle\right\}$, and $\tau_j^{p'} = \tau_j^p$ for $j \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies p', and of course $p' \leqslant p$. □

Lemma 9. *Let $p \in \mathbb{P}$ and $\xi \in H_\theta$. There is a $p' \leqslant p$ such that $\xi \in \mathrm{ran}(\pi_i^{p'})$ for some $i \in \mathrm{dom}(p')$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\theta)}$ certify p. Let $i < \omega_1$, $i \notin \mathrm{dom}(p)$, and $\bar{\xi}$ be such that $\pi_{i,\omega_1}^{\mathfrak{A}}(\bar{\xi}) = \xi$. We may define p' such that $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{i\}$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa_j^{p'} = \kappa_j^{\mathfrak{A}}$ for $j \in \mathrm{dom}(p')$, $\kappa_i^{p'} = \kappa_i^{\mathfrak{A}}$, $\pi_j^{p'} = \pi_j^p$ for $j \in \mathrm{dom}(p) \setminus \{i\}$, $\pi_i^{p'} = \left\{\langle \bar{\xi}, \xi\rangle\right\}$, and $\tau_j^{p'} = \tau_j^p$ for $j \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies p', and of course $p' \leqslant p$. □

Lemma 10. *Let $p \in \mathbb{P}$, $i \in \mathrm{dom}(p)$, $j \in \mathrm{dom}(p)$, $i < j$, $\xi \in \mathrm{ran}(\pi_i^p)$. There is a $p' \leqslant p$ such that $\xi \in \mathrm{ran}(\pi_j^{p'})$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\theta)}$ certify p. Let $\bar{\xi}$ be such that $\pi^{\mathfrak{A}}_{j,\omega_1}(\bar{\xi}) = \xi$. We may define p' such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa^{p'}_k = \kappa^{\mathfrak{A}}_k$ for $k \in \mathrm{dom}(p)$, $\pi^{p'}_k = \pi^p_k$ for $k \in \mathrm{dom}(p) \setminus \{j\}$, $\pi^{p'}_j = \pi^p_j \cup \{\langle \bar{\xi}, \xi \rangle\}$, and $\tau^{p'}_k = \tau^p_k$ for $k \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies p', and of course $p' \leqslant p$. □

Lemma 11. *Let* $p \in \mathbb{P}$, $i, i+1 \in \mathrm{dom}(p)$. *Let* $\xi \in \mathrm{ran}(\pi^p_{i+1})$. *There is some* $p' \leqslant p$ *such that* ξ *is definable over* $\mathcal{M}$ *from parameters in* $\mathrm{ran}(\pi^{p'}_i) \cup \{\kappa^p_i\}$.

Proof. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\theta)}$ certify p. Since $M^{\mathfrak{A}}_{i+1} = \mathrm{Ult}(M^{\mathfrak{A}}_i, G^{\mathfrak{A}}_i)$ there is an $f \colon \kappa^p_i = \omega_1^{M^{\mathfrak{A}}_i} \to M^{\mathfrak{A}}_i$, $f \in M^{\mathfrak{A}}_i$ such that $(\pi^p_{i+1})^{-1}(\xi) = \pi^{\mathfrak{A}}_{i,i+1}(f)(\kappa^p_i)$, i.e., $\xi = \pi^{\mathfrak{A}}_{i,\omega_1}(f)(\kappa^p_i)$. Due to the presence of $<$ in $\mathcal{M}$, the function $\pi^{\mathfrak{A}}_{i,\omega_1}(f)$ is definable over $\mathcal{M}$ in some ordinal parameter $\lambda < \theta$. Let $\bar{\lambda}$ be such that $\lambda = \pi^{\mathfrak{A}}_{i,\omega_1}(\bar{\lambda})$. We may define p' such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa^{p'}_j = \kappa^{\mathfrak{A}}_j$ for $j \in \mathrm{dom}(p')$, $\pi^{p'}_j = \pi^p_j$ for $j \in \mathrm{dom}(p) \setminus \{i\}$,

$$\pi^{p'}_i = \pi^p_i \cup \{\langle \bar{\lambda}, \lambda \rangle\},$$

and $\tau^{p'}_i = \tau^p_i$ for $i \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies p', and of course $p' \leqslant p$. □

Lemma 12. *Let* $p \in \mathbb{P}$, *and let* $\lambda \in \mathrm{dom}(p)$ *be a limit ordinal. If* $\xi \in \mathrm{ran}(\pi^p_\lambda)$, *then there is some* $p' \leq p$ *and some* $i < \lambda$ *with* $i \in \mathrm{dom}(p')$ *such that* $\xi \in \mathrm{ran}(\pi^{p'}_i)$.

Proof. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\theta)}$ certify p. Because $\mathrm{ran}(\pi^{\mathfrak{A}}_{\lambda,\omega_1}) = \bigcup_{i<\lambda} \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$, there is some $i < \lambda$ such that $\xi \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$. Let us without loss of generality assume that $i \in \mathrm{dom}(p)$. Let $\bar{\xi}$ be such that $\pi^{\mathfrak{A}}_{i,\omega_1}(\bar{\xi}) = \xi$. We may then define p' such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa^{p'}_j = \kappa^p_j$ for $j \in \mathrm{dom}(p)$, $\pi^{p'}_j = \pi^p_j$ for $j \in \mathrm{dom}(p) \setminus \{i\}$, $\pi^{p'}_i = \pi^p_i \cup \{\langle\langle \bar{\xi}, \xi \rangle\}$, and $\tau^{p'}_i = \tau^p_i$ for $i \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies p', and of course $p' \leqslant p$. □

Lemma 13. *Let* $p \in \mathbb{P}$, $i \in \mathrm{dom}(p)$ *and let* ξ *be definable over* $\mathcal{M}$ *from parameters in* $\mathrm{ran}(\pi^p_i)$. *There is a* $p' \leqslant p$ *such that* $\xi \in \mathrm{ran}(\pi^{p'}_i)$.

Proof. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\theta)}$ certify p. We must have that $\xi \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$, as $\mathfrak{A}$ certifies p (cf. condition (b)). Let $\pi^{\mathfrak{A}}_{i,\omega_1}(\bar{\xi}) = \xi$. We may define p' such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa^{p'}_j = \kappa^p_j$ for $j \in \mathrm{dom}(p)$, $\pi^{p'}_j = \pi^p_j$

for $j \in \mathrm{dom}(p) \setminus \{i\}$, $\pi_i^{p'} = \pi_i^p \cup \{\langle\bar{\xi}, \xi\rangle\}$, and $\tau_j^{p'} = \tau_j^p$ for $j \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies p', and of course $p' \leqslant p$. □

Lemma 14. *Let $p \in \mathbb{P}$, let $i \in \mathrm{dom}(p)$, and suppose that $D \in H_\theta$ is definable over $\mathcal{M}$ from parameters in $\mathrm{ran}(\pi_i^p)$. Suppose also that*

$$\mathcal{M} \models \text{“}D \text{ is dense in the partial order } \langle I^+, \leq_I\rangle.\text{”}$$

Then there is some $p' \leq p$ and some $X \in D$ which is definable over $\mathcal{M}$ from parameters in $\mathrm{ran}(\pi_i^{p'})$ such that $\kappa_i^p \in X$.

Proof. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\theta)}$ certify p. Let $\bar{D} \in M_i^{\mathfrak{A}}$ be such that $\pi_{i,\omega_1}^{\mathfrak{A}}(\bar{D}) = D$. As $G_i^{\mathfrak{A}}$ is $\langle (I_i^{\mathfrak{A}})^+, \leq_{I_i^{\mathfrak{A}}}\rangle$-generic over $M_i^{\mathfrak{A}}$, $\bar{D} \cap G_i^{\mathfrak{A}} \neq \emptyset$. There is thus some $\bar{X} \in \bar{D}$ such that $\kappa_i^p = \kappa_i^{\mathfrak{A}} \in \pi_{i,i+1}^{\mathfrak{A}}(\bar{X}) \subset \pi_{i,\omega_1}^{\mathfrak{A}}(\bar{X})$. Let $X = \pi_{i,\omega_1}^{\mathfrak{A}}(\bar{X})$. Then $X \in D$ and $\kappa_i^p \in X$. Due to the presence of $<$ in $\mathcal{M}$, there is some $\lambda < \theta$ such that X is definable over $\mathcal{M}$ from the parameter λ. Let $\bar{\lambda}$ be such that $\lambda = \pi_{i,\omega_1}^{\mathfrak{A}}(\bar{\lambda})$. We may define p' such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa_j^{p'} = \kappa_j^{\mathfrak{A}}$ for $j \in \mathrm{dom}(p')$, $\pi_j^{p'} = \pi_j^p$ for $j \in \mathrm{dom}(p) \setminus \{i\}$,

$$\pi_i^{p'} = \pi_i^p \cup \{\langle\bar{\lambda}, \lambda\rangle\},$$

and $\tau_i^{p'} = \tau_i^p$ for $i \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies p', and of course $p' \leqslant p$. □

Now let G be $\mathbb{P}$-generic over V. Set

$$\kappa_i = \kappa_i^p \text{ for some (all) } p \in G \text{ with } i \in \mathrm{dom}(p),$$

$$\pi_i = \bigcup\{\pi_i^p ; p \in G \wedge i \in \mathrm{dom}(p)\}, \text{ and}$$

$$\beta_i = \mathrm{dom}(\pi_i).$$

By Lemmas 6, 7, and 8, $\pi_i \colon \beta_i \to \theta$ is a well-defined cofinal order preserving map, and by Lemma 9, $\theta = \bigcup\{\mathrm{ran}(\pi_i); i < \omega_1\}$. For $i < \omega_1$, let X_i be the smallest $X \prec \mathcal{M}$ such that $\mathrm{ran}(\pi_i) \subseteq X$. By Lemma 13, $\mathrm{ran}(\pi_i) = X_i \cap \theta$. Let $\tilde{\pi}_i \colon M_i \cong X_i \prec \mathcal{M}$ be the uncollapsing map, so that $\tilde{\pi}_i \supset \pi_i$. For $i \leq j \leq \omega_1$, let $\tilde{\pi}_{i,j} = \tilde{\pi}_j^{-1} \circ \tilde{\pi}_i$. We then have that $\tilde{\pi}_{i,j} \colon M_i \to M_j$ is then well-defined by Lemma 10. For $i \leq \omega_1$, let $I_i = \tilde{\pi}_i^{-1}(I)$ and $\kappa_i = \tilde{\pi}_i^{-1}(\omega_1)$, and for $i < \omega_1$, let

$$G_i = \{X \in \mathcal{P}(\kappa_i) \cap M_i \ ; \ \kappa_i \in \tilde{\pi}_{i,i+1}(X)\}.$$

Using Lemmas 11, 12, and 14, we then have the following.

Lemma 15. *$\langle\langle M_i, \tilde{\pi}_{i,j}, I_i, \kappa_i\ ;\ i \leqslant j \leqslant \omega_1^V\rangle, \langle G_i\ ;\ i < \omega_1\rangle\rangle$ is a generic iteration of M_0 such that if $i < \omega_1$, then M_i is countable, and $M_{\omega_1} = \langle H_\theta; \in, I\rangle$.*

Let us now discuss the third component of a condition $p \in \mathbb{P}$.

Lemma 16. *Suppose that $\mathfrak{A}$ is a model. Let $p \in \mathbb{P}$ and $i \in \mathrm{dom}(p)$. Let $x \in H_\rho$ be such that τ_i^p is the complete $\mathcal{H}$-type of x over H_θ, i.e., having φ range over $\mathcal{H}$-formulae with free variables $u, \vec{v}$,*

$$\tau_i^p = \{\langle\ulcorner\varphi\urcorner, \vec{z}\rangle\ ;\ \vec{z} \in H_\theta \wedge \mathcal{H} \models \varphi[x, \vec{z}]\}.$$

Then the following are equivalent.

i. $\mathfrak{A}$ certifies p with respect to $\mathcal{M}$.

ii. $\theta + 1 \subset \mathrm{wfp}(\mathfrak{A})$, $H_\theta \in \mathfrak{A}$, $\mathfrak{A} \models \mathrm{ZFC}^-$, for all $S \in I^+$, $\mathfrak{A} \models$ "S is stationary," and inside $\mathfrak{A}$, there is a generic iteration

$$\langle\langle M_i^{\mathfrak{A}}, \pi_{i,j}^{\mathfrak{A}}, I_i^{\mathfrak{A}}, \kappa_i^{\mathfrak{A}}; i \leqslant j \leqslant \omega_1\rangle, \langle G_i^{\mathfrak{A}}; i < \omega_1\rangle\rangle$$

such that if $i < \omega_1$, then $M_i^{\mathfrak{A}}$ is countable, $M_{\omega_1}^{\mathfrak{A}} = \langle H_\theta; \in, I\rangle$, if $i \in \mathrm{dom}(p)$, then $\kappa_i^p = \kappa_i^{\mathfrak{A}}$ and $\pi_i^p \subseteq \pi_{i,\omega_1}^{\mathfrak{A}}$, and if $i \in \mathrm{dom}_-(p)$, then one of the following equivalent conditions holds.

(a)
$$\mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}}) \cup \{x\}) \cap H_\theta = \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}}).$$

(b) The map $\pi_{i,\omega_1}^{\mathfrak{A}} \colon M_i \to \mathcal{M}$ extends to some elementary map $\tilde{\pi} \colon H \to \mathcal{H}$ with $\tilde{\pi}(M_i) = \langle H_\theta; \in, I\rangle$, $\tilde{\pi} \restriction M_i = \pi_{i,\omega_1}^{\mathfrak{A}}$, and $x \in \mathrm{ran}(\tilde{\pi})$.

(c) $\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}}) \prec \langle H_\theta; \in, I, <, \tau_i^p\rangle$.

Proof. i. $\Rightarrow$ ii.(a): Let $y \in \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}}) \cup \{x\}) \cap H_\theta$. Then y is definable over $\mathcal{H}$ from parameters $\vec{z}$, x in $\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}}) \cup \{x\}$. For some $n < \omega$, we then have that y is unique with $(n, y^\frown\vec{z}) \in \tau_i^p$. As $\mathfrak{A}$ certifies p (cf. condition vi.(e) in Definition 4), we then get that in fact $y \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$.

ii.(a) $\Rightarrow$ ii.(b): Let $\tilde{\pi} \colon H \cong \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}}) \cup \{x\}) \prec \mathcal{H}$, where H is transitive. It is obvious that this map works.

ii.(b) $\Rightarrow$ ii.(a): As $x \in \mathrm{ran}(\tilde{\pi})$ and $\tilde{\pi} \supset \pi_{i,\omega_1}^{\mathfrak{A}}$, $\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}}) \subset \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}}) \cup \{x\}) \cap H_\theta \subset \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\tilde{\pi})) \cap H_\theta = \mathrm{ran}(\tilde{\pi}) \cap H_\theta = \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$.

ii.(a) $\Rightarrow$ ii.(c): We need to show that if $\vec{z} \in \text{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$ and φ is a formula (of the language associated with $\langle H_\theta; \in, I, <, \tau^p_i \rangle$) such that

$$\langle H_\theta; \in, I, <, \tau^p_i \rangle \models \exists v \varphi(v, \vec{z}), \tag{1}$$

then there is some $u \in \text{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$ with

$$\langle H_\theta; \in, I, <, \tau^p_i \rangle \models \varphi(u, \vec{z}).$$

There is some recursive $\ulcorner \psi \urcorner \mapsto \ulcorner \psi^* \urcorner$ (assigning to each formula of the language associated with $\langle H_\theta; \in, I, <, \tau^p_i \rangle$ a formula of the language associated with $\langle H_\rho; \in, H_\theta, I, <, x \rangle$) such that for all $\vec{w} \in H_\theta$,

$$\langle H_\theta; \in, I, <, \tau^p_i \rangle \models \psi(\vec{w})$$

iff

$$\langle H_\rho; \in, H_\theta, I, <, x \rangle \models \psi^*(\vec{w}).$$

Hence if (1) holds, then there is some $u \in H_\theta$ such that

$$\langle H_\rho; \in, H_\theta, I, <, x \rangle \models \varphi^*(u, \vec{z}).$$

There is then also some such $u \in H_\theta$ which is in $\text{Hull}^{\mathcal{H}}(\text{ran}(\pi^{\mathfrak{A}}_{i,\omega_1}) \cup \{x\})$, so that $u \in \text{ran}(\pi_{i,\omega_1})^{\mathfrak{A}}$ by ii.(a). But then

$$\langle H_\theta; \in, I, <, \tau^p_i \rangle \models \varphi(u, \vec{z}),$$

where $u \in \text{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$.

ii.(c) $\Rightarrow$ i.: Let $n < \omega$ and $\vec{z} \in \text{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$. Suppose there to be some $y \in H_\theta$ such that $(n, y^\frown \vec{z}) \in \tau^p_i$. Then

$$\langle H_\theta; \in, I, <, \tau^p_i \rangle \models \exists y (n, y^\frown \vec{z}) \in \tau^p_i,$$

so that there is some $y \in \text{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$ with

$$\langle H_\theta; \in, I, <, \tau^p_i \rangle \models (n, y^\frown \vec{z}) \in \tau^p_i,$$

as needed for condition vi.(e) in Definition 4. $\square$

It is easy to see that if $X \in I$ and $X \in \text{ran}(\tilde{\pi}_{i,\omega_1})$, where $i < \omega_1$, then $\{\kappa_j ; i \leq j < \omega_1\} \subset \omega_1 \setminus X$. This means that no set in I will be stationary in $V^{\mathbb{P}}$.

Lemma 17. *If $S \in I^+$, then S is stationary in $V^{\mathbb{P}}$.*

Proof. Let $S \in I^+$, and let $p \in \mathbb{P}$ and $\dot{C}$ be such that $p \Vdash \dot{C}$ is club in $\check{\omega_1}$. We need to see that there is some $p' \leqslant p$ and some $\alpha < \omega_1$ such that $p' \Vdash \check{\alpha} \in \dot{C} \cap \check{S}$.

Let

$$R = \{(p, \delta); p \in \mathbb{P}, \delta < \omega_1, \text{ and } p \Vdash_{\mathbb{P}} \check{\delta} \in \dot{C}\}.$$

Notice that $p, R, \leq_{\mathbb{P}} \in H_\rho$. Let τ the the complete $\mathcal{H}$-type of $\langle p, R, \leq_{\mathbb{P}} \rangle$ over H_θ. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\theta)}$ certify p with respect to $\mathcal{M}$. Recall that $H_\theta \in \mathfrak{A}$ and $\omega_1^{\mathfrak{A}} = \omega_1^V$. We have that $\langle \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1}); i < \omega_1 \rangle$ is a continuous tower of countable substructures of H_θ with $\bigcup \{\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1}); i < \omega_1\} = H_\theta$. Since S is stationary in $\mathfrak{A}$, we may therefore pick an $\alpha < \omega_1$ such that

i. $\kappa^{\mathfrak{A}}_\alpha = \alpha$ and $\mathrm{dom}(p) \subseteq \alpha$,

ii. $\mathrm{ran}(\pi^{\mathfrak{A}}_{\alpha,\omega_1}) \prec \langle H_\theta; \in, I, <, \tau \rangle$, and

iii. $\alpha \in S$.

We may define p' such that $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{\alpha\}$, $\mathrm{dom}_-(p') = \mathrm{dom}(p)_- \cup \{\alpha\}$, $\kappa^{p'}_i = \kappa^p_i$ for all $i \in \mathrm{dom}(p)$, $\kappa^{p'}_\alpha = \alpha$, $\pi^{p'}_i = \pi^p_i$ for all $i \in \mathrm{dom}(p)$, $\pi^{p'}_\alpha = \emptyset$, $\tau^{p'}_i = \tau^p_i$ for all $i \in \mathrm{dom}_-(p)$, and $\tau^{p'}_\alpha = \tau$. Using Lemma 16, we see that $\mathfrak{A}$ still certifies p' by the above choice of α. Also, notice that if $i \in \mathrm{dom}_-(p)$, then τ^p_i is (trivially) definable over $\mathcal{H}$ from the parameter p, so that because τ is the complete $\mathcal{H}$-type of $\langle p, R, \leq_{\mathbb{P}} \rangle$ over H_θ, we get that there is an $n < \omega$ such that

$$\tau^p_i = \{(m, \vec{z}) \ ; \ (n, m^\frown \vec{z}) \in \tau\}.$$

We thus have $p' \in \mathbb{P}$, and of course $p' \leqslant p$.

We claim that $p' \Vdash \check{\alpha} \in \dot{C} \cap \check{S}$. Suppose not. Then p' does not force $\dot{C} \cap \check{\alpha}$ to be unbounded in $\check{\alpha}$. Pick $q \leqslant p'$ and $\xi < \alpha$ such that

$$q \Vdash \sup(\dot{C} \cap \check{\alpha}) = \check{\xi}. \tag{2}$$

Let the model $\mathfrak{B}$ certify q with respect to $\mathcal{M}$. By Lemma 16,

$$\mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}} \rangle\}) \cap H_\theta = \mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}). \tag{3}$$

Let us now set

$$q' = \langle \langle \kappa^q_i; i \in \mathrm{dom}(q) \upharpoonright \alpha \rangle, \langle \pi^q_i; i \in \mathrm{dom}(q) \upharpoonright \alpha \rangle, \langle \tau^q_i; i \in \mathrm{dom}_-(q) \upharpoonright \alpha \rangle \rangle.$$

Of course, $q \leqslant q' \leqslant p$. If $i \in \mathrm{dom}_-(q') = \mathrm{dom}_-(q) \upharpoonright \alpha$, then there is some $n < \omega$ and some $\vec{u} \in \mathrm{ran}(\pi^q_\alpha)$ such that

$$\tau^{q'}_i = \{(m, \vec{z}) \ ; \ (n, \vec{u}^\frown m^\frown \vec{z}) \in \tau^q_\alpha = \tau\}.$$

By the choice of τ, we must then have that

$$\tau^{q'}_i \in \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\})$$

for every $i \in \mathrm{dom}_-(q')$, because if

$$\tau = \{\langle \ulcorner \varphi \urcorner, \vec{z}\rangle; \vec{z} \in H_\theta \wedge \mathcal{H} \models \varphi[\langle p, R, \leq_{\mathbb{P}}\rangle, \vec{z}]\},$$

then

$$\tau^{q'}_i = \tau^q_i = \{\langle m, \vec{z}\rangle; \vec{z} \in H_\theta \wedge \mathcal{H} \models \varphi[\langle p, R, \leq_{\mathbb{P}}\rangle, \vec{u}^\frown m^\frown \vec{z}]\}.$$

This implies that in fact

$$q' \in \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\}). \tag{4}$$

Because $q' \Vdash_{\mathbb{P}}$ "$\dot{C}$ is club in $\check{\omega}_1$," there is some $\gamma > \xi$ and some $q'' \leq_{\mathbb{P}} q'$ such that $q'' \Vdash_{\mathbb{P}} \check{\gamma} \in \dot{C}$, i.e., $(q'', \gamma) \in R$, and therefore by (4)

$$\mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\}) \models \exists \gamma > \xi \ \exists q'' \leq_{\mathbb{P}} q' \ (q'', \gamma) \in R,$$

which means that there is some $q'' \leqslant q'$ with

$$q'' \in \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\}) \tag{5}$$

such that

$$q'' \Vdash_{\mathbb{P}} \sup(\dot{C} \cap \check{\alpha}) > \check{\xi}.$$

In particular, $\mathrm{dom}(q'') \subseteq \alpha$. We must now have that

$$q'' \text{ and } q \text{ are incompatible.}$$

We derive a contradiction by constructing some $q^* \leqslant q'', q$.

Let

$$\tilde{\pi} \colon H \cong \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\}) \prec \mathcal{H},$$

where H is transitive. By (3), $M^{\mathfrak{B}}_\alpha = \tilde{\pi}^{-1}(\langle H_\theta; \in, I\rangle) \in H$ and $\tilde{\pi} \upharpoonright M^{\mathfrak{B}}_\alpha = \pi^{\mathfrak{B}}_{\alpha,\omega_1}$. In $V^{\mathrm{Col}(\omega,\theta)}$, there is a model $\mathfrak{C}$ which certifies q''. In $\mathcal{H}^{\mathrm{Col}(\omega,\theta)}$, there is hence some generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that $M_{\omega_1} = \langle H_\theta; \in, I\rangle$ and for all $i \in \mathrm{dom}(q'')$, $\kappa_i^{q''} = \kappa_i$ and $\pi_i^{q''} \subseteq \pi_{i,\omega_1}$. By the elementarity of $\tilde{\pi}$, there is hence in $H^{\mathrm{Col}(\omega,\tilde{\pi}^{-1}(\theta))} \subseteq V^{\mathrm{Col}(\omega,\theta)}$ some generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \alpha\rangle, \langle G_i; i < \alpha\rangle\rangle$$

such that $M_\alpha = \tilde{\pi}^{-1}(\langle H_\theta; \in, I\rangle) = M_\alpha^{\mathfrak{B}}$ and for all $i \in \mathrm{dom}(q'')$, $\kappa_i^{q''} = \kappa_i$ and $\tilde{\pi}^{-1}(\pi_i^{q''}) \subseteq \pi_{i,\alpha}$, i.e., $\pi_i^{q''} \subseteq \tilde{\pi} \circ \pi_{i,\alpha} = \pi^{\mathfrak{B}}_{\alpha,\omega} \circ \pi_{i,\alpha}$. Because $M_\alpha^{\mathfrak{B}}$ is countable in $\mathfrak{B}$, $\theta + 1 \subset \mathrm{wfp}(\mathfrak{B})$, and $\mathfrak{B} \in V^{\mathrm{Col}(\omega,\theta)}$, there is therefore by absoluteness some generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \alpha\rangle, \langle G_i; i < \alpha\rangle\rangle \in \mathfrak{B}$$

such that $M_\alpha = M_\alpha^{\mathfrak{B}}$ and for all $i \in \mathrm{dom}(q'')$, $\kappa_i^{q''} = \kappa_i$ and $\pi_i^{q''} \subseteq \pi^{\mathfrak{B}}_{\alpha,\omega_1} \circ \pi_{i,\alpha}$. Let

$$\langle\langle M_i^*, \pi_{i,j}^*, I_i^*, \kappa_i^*; i \leqslant j \leqslant \omega_1\rangle, \langle G_i^*; i < \omega_1\rangle\rangle \in \mathfrak{B} \tag{6}$$

be defined as follows. If $i \leq j \leq \alpha$, then we set $M_i^* = M_i$, $\pi_{i,j}^* = \pi_{i,j}$, $I_i^* = I_i$, $\kappa_i^* = \kappa_i$, and if $i < \alpha$, then we set $G_i^* = G_i$. If $\alpha \leq i \leq j \leq \omega_1$, then we set $M_i^* = M_i^{\mathfrak{B}}$ (there is no conflict for $i = \alpha$, as $M_\alpha^{\mathfrak{B}} = M_\alpha$), $\pi_{i,j}^* = \pi_{i,j}^{\mathfrak{B}}$, $I_i^* = I_i^{\mathfrak{B}}$, $\kappa_i^* = \kappa_i$, and if $\alpha \leq i < \omega_1$, then we set $G_i^* = G_i^{\mathfrak{B}}$. Finally, if $i \leq \alpha \leq j$, then we set $\pi_{i,j}^* = \pi^{\mathfrak{B}}_{\alpha,j} \circ \pi_{i,\alpha}$. The existence of the generic iteration (6) inside $\mathfrak{B}$ clearly shows that $\mathfrak{B}$ in fact certifies q''. However, as $\mathrm{dom}(q'') \supseteq \mathrm{dom}(q) \upharpoonright \alpha$, the very same generic iteration (6) shows that $\mathfrak{B}$ certifies q.

Let us now define $q^* \in \mathbb{P}$ as follows. Let $\mathrm{dom}(q^*) = \mathrm{dom}(q) \cup \mathrm{dom}(q'')$ and $\mathrm{dom}_-(q^*) = \mathrm{dom}(q)_- \cup \mathrm{dom}_-(q'')$. (Neither $\mathrm{dom}(q)$ and $\mathrm{dom}(q'')$ nor $\mathrm{dom}(q)_-$ and $\mathrm{dom}_-(q'')$ need to be disjoint, but $\mathrm{dom}(q) \cap \alpha \subseteq \mathrm{dom}(q'')$ and $\mathrm{dom}(q)_- \cap \alpha \subseteq \mathrm{dom}_-(q'')$.) For $i \in \mathrm{dom}(q^*)$ set $\kappa_i^{q^*} = \kappa_i^*$. For $i \in \mathrm{dom}_-(q'')$ set $\tau_i^{q^*} = \tau_i^{q''}$, and for $i \in \mathrm{dom}_-(q)$, set $\tau_i^{q^*} = \tau_i^q$. Also, for $i \in \mathrm{dom}(q'')$ set $\pi_i^{q^*} = \pi_i^{q''}$. Finally, for $i \in \mathrm{dom}(q) \setminus \alpha$, we need some adjustment in order to actually get a condition. By (5), there is some finite $\vec{u} \subseteq \mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1})$ such that

$$q'' \in \mathrm{Hull}^{\mathcal{H}}(\{\vec{u}, \langle p, R, \leq_{\mathbb{P}}\rangle\}).$$

We then also have some $n < \omega$ such that for every $i \in \mathrm{dom}_-(q'')$,

$$\tau_i^{q''} = \tau_i^{q^*} = \{(m, \vec{z})\ ;\ (n, \vec{u}^\frown m^\frown \vec{z}) \in \tau_\alpha^{q^*} = \tau_\alpha^{p'} = \tau\}.$$

We may assume without loss of generality that $\pi^*_{i,\omega_1}{}''\,\mathrm{dom}(\pi_i^{q^*}) \subseteq \vec{u}$ for $i \in \mathrm{dom}(q'') \subseteq \alpha$. For $j \in \mathrm{dom}(q^*)$, $j \geqslant \alpha$, we then set

$$\pi_j^{q^*} = \pi^*_{j,\omega_1} \upharpoonright ((\pi^*_{j,\omega_1})^{-1}(\vec{u}) \cup \mathrm{dom}(\pi_j^{q''})).$$

It is now straightforward to see that $q^* \in \mathbb{P}$. Notice that if $i \in \mathrm{dom}_-(q^*) \cap \alpha = \mathrm{dom}_-(q'')$ and $j \in \mathrm{dom}_-(q^*) \setminus \alpha = \mathrm{dom}_-(q) \setminus \alpha$, and if

$$\tau_\alpha^{q^*} = \tau_\alpha^q = \{(m, \vec{z}); (k, \vec{v}^\frown m^\frown \vec{z}) \in \tau_j^{q^*} = \tau_j^q\},$$

where $\vec{v} \in \mathrm{ran}(\pi_j^{q^*}) = \mathrm{ran}(\pi_j^q)$, then

$$\tau_i^{q^*} = \tau_i^{q''} = \{(m, \vec{z}); (n, \vec{u}^\frown m^\frown \vec{z} \in \tau_\alpha^{q^*}\} = \{(m, \vec{z}); (k, \vec{v}^\frown n^\frown \vec{u}^\frown m^\frown \vec{z} \in \tau_j^{q^*}\}$$

and $\vec{v}$, $\vec{u} \subseteq \mathrm{ran}(\pi_j^{q^*})$. Of course, $q^* \leqslant q, q''$. We have reached a contradiction. □

This finishes the proof of Theorem 3.

A straightforward adaptation yields the following result.

Theorem 18. *Let I be a precipitous ideal on ω_1, and let $\theta > \omega_1$ be a regular cardinal. Suppose that $H_\theta^\#$ exists. There is a poset $\mathbb{P}$, preserving the stationarity of all sets in I^+, such that if G is $\mathbb{P}$-generic over V, then in $V[G]$ there is a generic iteration*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that if $i < \omega_1$, then M_i is countable and $M_{\omega_1} = \langle H_\theta^\#; \in, I\rangle$. In particular, M_0 is generically $\omega_1 + 1$ iterable. If $I = \mathsf{NS}_{\omega_1}$, then $\mathbb{P}$ is stationary set preserving.

Proof. Let $\rho > 2^{2^\theta}$, and let $\mathbb{P} = (\mathrm{Col}(\rho, \rho) \times \mathrm{Col}(\theta^+, \theta^+)) * \mathbb{P}(I, \theta^+)$, where $\mathbb{P}(I, \theta^+)$ is as in Theorem 3. Let

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

be a generic iteration which is added by forcing with $\mathbb{P}$. Setting $N_i = \pi_{i,\omega_1}^{-1}(H_\theta)$, we will have that $\pi_{i,\omega_1}^{-1}(H_\theta^\#) = N_i^\#$. The iterability of M_0 follows from Lemma 2. Notice that $\langle N_0^\#; \in, I_0\rangle$ is generically $\omega_1 + 1$ iterable iff $\langle L[N_0]; \in, I_0\rangle$ is generically $\omega_1 + 1$ iterable. □

Lemma 19 (Woodin)**.** *Let M be a countable transitive model of ZFC^* + "ω_1 exists," and let $I \subseteq \mathcal{P}(\omega_1^M)$ be such that $\langle M; \in, I\rangle \models$ "I is a uniform and normal ideal on ω_1^M." Let $\alpha < \omega_1$, and suppose $\langle M; \in, I\rangle$ to be generically $\alpha + 1$ iterable. Let z_0 be a real which codes $\langle M; \in, I\rangle$, let z_1 be a real which codes α, and let $z = z_0 \oplus z_1$. Let*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \alpha\rangle, \langle G_i; i < \alpha\rangle\rangle$$

be a generic iteration of $\langle M;\in,I\rangle$ *of length* $\alpha+1$*. Then* $M_\alpha\cap\mathrm{OR}<\omega_1^{L[z]}$.

Proof. The proof is taken from [Woo99, p. 56f.]. Let $A\subset\mathbb{R}$ be defined by $x\in A$ iff x codes a countable ordinal ξ (which we write as $\xi=||x||$) such that for some generic iteration

$$\langle\langle M_i',\pi_{i,j}',I_i,\kappa_i';i\leqslant j\leqslant\alpha\rangle,\langle G_i';i<\alpha\rangle\rangle$$

of $\langle M;\in,I\rangle$ of length $\alpha+1$, $\xi\subseteq M_\alpha'$. The set A is $\Sigma_1^1(z)$, so that by the Boundedness Lemma (cf. [Jec03, Corollary 25.14]),

$$\sup\{\xi;\exists x\in A\xi=||x||\}<\omega_1^{L[z]}.$$

In particular, $M_\alpha\cap\mathrm{OR}<\omega_1^{L[z]}$. □

Lemma 20. *Suppose* I *to be a precipitous ideal on* ω_1*. Let* $\theta\geq\omega_2$ *be regular, and suppose that* $H_\theta^\#$ *exists. Let* $\mathbb{P}=\mathbb{P}'(I,\theta)$ *be as in Theorem 18, and let* G *be* $\mathbb{P}$*-generic over* V*. In* $V[G]$*, let*

$$\langle\langle M_i,\pi_{i,j},I_i,\kappa_i;i\leqslant j\leqslant\omega_1\rangle,\langle G_i;i<\omega_1\rangle\rangle\in V[G]$$

be a generic iteration such that if $i<\omega_1$*, then* M_i *is countable and* $M_{\omega_1}=\langle H_\theta^\#;\in,I\rangle$*. Let* $z\in\mathbb{R}\cap V[G]$ *code* $\langle\pi_{0,\omega_1}^{-1}(H_\theta);\in,I_0\rangle$*. Then* $\theta<\omega_1^{+L[z]}$*. In particular,* $V[G]\vDash\theta<\underset{\sim}{\delta}_2^1$*.*

Proof. For a canonical choice of z, $z^\#$ exists in $V[G]$ and $z^\#$ codes $\langle M_0;\in,I_0\rangle$. It therefore suffices to prove $\theta<\omega_1^{+L[z]}$. Suppose that $\omega_1^{+L[z]}\leq\theta$. Let us work in $V[G]$ to derive a contradiction. Let $X\prec H_\Omega$ be countable (where Ω is regular and large enough) such that $z^\#$ and

$$\langle\langle M_i,\pi_{i,j},I_i,\kappa_i;i\leqslant j\leqslant\omega_1\rangle,\langle G_i;i<\omega_1\rangle\rangle$$

are both elements of X, and let $\sigma\colon N\cong X\prec H_\Omega$, where N is tranitive. Let $\alpha=X\cap\omega_1=\omega_1^N$. Since $z^\#\in X$, we have that

$$\mathcal{P}(\alpha)\cap L[z]\subseteq\mathcal{P}(\alpha)\cap N,$$

so that $\sigma^{-1}(\omega_1^{L[z]})=\alpha^{+L[z]}$. Also,

$$\sigma^{-1}(\langle\langle M_i,\pi_{i,j},I_i,\kappa_i;i\leqslant j\leqslant\omega_1\rangle,\langle G_i;i<\omega_1\rangle\rangle)=$$

$$\langle\langle M_i,\pi_{i,j},I_i,\kappa_i;i\leqslant j\leqslant\alpha\rangle,\langle G_i;i<\alpha\rangle\rangle,$$

so that $\sigma^{-1}(\theta)=M_\alpha\cap\mathrm{OR}$. Let $g\in V[G]$ be $\mathrm{Col}(\omega,\alpha)$-generic over N. Then $M_\alpha\cap\mathrm{OR}\geq\alpha^{+L[z]}=\omega_1^{L[z\oplus g]}$. This contradicts Lemma 19. □

Recall that Bounded Martin's Maximum, BMM, may be formulated as follows. If $\mathbb{Q} \in V$ is a stationary set preserving forcing, then

$$H_{\omega_2}^V \prec_{\Sigma_1} H_{\omega_2}^{V^{\mathbb{Q}}}.$$

It was shown in [Sch04] that BMM implies that V is closed under sharps. Of course, having a precipitous ideal on ω_1 also yields that the reals are closed under sharps.

Corollary 21. *Suppose that* BMM *holds and* NS_{ω_1} *is precipitous. Then* $u_2 = \omega_2$.

Proof. Let $\alpha < \omega_2$. Let $\varphi \equiv \exists z \in \mathbb{R}(\alpha < \omega_1^{+L[z]})$. The statement φ is Σ_1 over H_{ω_2} in the parameters ω_1, α, and φ holds in $V^{\mathbb{P}}$, where $\mathbb{P} = \mathbb{P}'(NS_{\omega_1}, \omega_2)$. Therefore, φ must hold in V. As α was arbitrary, we have shown that $u_2^V = \omega_2$. □

Recall that the Bounded Semiproper Forcing Axiom, BSPFA, may be formulated as follows. If $\mathbb{Q} \in V$ is a semiproper forcing, then

$$H_{\omega_2}^V \prec_{\Sigma_1} H_{\omega_2}^{V^{\mathbb{Q}}}.$$

For a formulation of the Reflection Principle RP cf. [Jec03, p.688].

Corollary 22. *Suppose* BSPFA *and* RP *both hold. Then* $u_2 = \omega_2$.

Proof. The Reflection Principle RP implies that all stationary set preserving forcings are semiproper, and it implies that NS_{ω_1} is precipitous (cf. [Jec03, p.688]). The rest of the proof is the same as that of the previous corollary. □

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