19.08.2010 9:00  John Steel

Q: Conv (PFA) \Rightarrow Conv (Supercompact)

Structure so far:

\[ \text{Strong hypotheses} \Rightarrow \text{Co-Model} \Rightarrow \text{Absolutely definable scales} \]

\[ \text{Descriptive inner model theory} \]

**Derived model theorem (Steel's paper)**

**Def** Homogeneity systems, $S^x$, $\text{Hom}_x$

**Def** $n$-absolute complementing trees, $n$-abs UB

**Def** $n$-weakly homogeneous set

**Facts**

1. $\text{Hom}_x \Rightarrow$ Hom $x^+$ sets are determined

(1) If $\delta$ is Woodin, then:

- $\delta$ is $S^+$-weakly homogeneous $\Rightarrow \text{TA: } \delta$ is $<\delta$-homogeneous

(2) If $\lambda$ is a limit of Woodin cardinals then $\text{Hom}_\lambda$ is closed under $\exists^\delta$ and $\forall$, and also under $\forall^\delta$ and $\exists^\delta$.

- Exist $\exists y < \lambda: \text{Hom}_y = \text{Hom}_\lambda$

(3) Let $\delta$ be Woodin then

- $\delta$ is $S^+$-UB $\Rightarrow$ $\delta$ is $<\delta$-homogeneous

- $\lambda$ is a limit of Woodin cardinals then

  $\text{Hom}_\lambda$ is UB $\Rightarrow$ $\text{Hom}_\lambda$ is Whom $\lambda$
Tree production lemma. Let \( \mathcal{S} \) be Woodin, \( \mathcal{V}(\mathfrak{w}) \) be a formula, \( \mathfrak{a} \) be a set. Suppose

\( \mathfrak{a} = \mathcal{S} - \text{generic} \) and \( \mathfrak{a} \) that is \( \mathcal{S} - \text{generic} \\ on \mathcal{V}(\mathfrak{a}) \) and \( \mathfrak{a} \in [\mathbb{R}]^{\mathfrak{a}}; \\ \mathcal{V}(\mathfrak{a}) \models \varphi([\mathfrak{a}], \mathfrak{a}) \implies \mathcal{V}(\mathfrak{a}, \mathcal{G}) \models \varphi([\mathfrak{a}], \mathfrak{a}) \)

(1) (Stationary tree conjecture)

For \( \mathcal{G} \) that is \( \mathcal{Q}_{\mathfrak{a}} - \text{generic} \) and \( \sigma: V \rightarrow \mathcal{M} = \text{Ult}(V, \mathcal{G}) \) and \( \mathfrak{a} \in [\mathbb{R}]^{\mathfrak{a}}; \\ \mathcal{V}(\mathfrak{a}) \models \varphi([\mathfrak{a}], \mathfrak{a}) \) off \( \mathcal{M} \models \varphi([\mathfrak{a}], \mathfrak{a}) \),

then there are \( \delta \) absolute complements \( T \) of \( U \) such

\[ p(T) = \{ \mathfrak{a} \mid \varphi([\mathfrak{a}], \mathfrak{a}) \} \] in all \( \mathcal{V}(\mathfrak{a}) \) when \( q \) is \( \mathcal{S} - \text{generic} \). 

Corollary. Every \( \mathcal{G} \) of \( \mathfrak{S} \) has a \( \text{Hom}_{\mathfrak{S}} \) scale

for \( \mathfrak{a} \) a limit of Woodins.

Derived model theorem. Let \( \mathfrak{x} \) be a limit of Woodins. Let \( \mathcal{G} \) be \( \mathcal{V} - \text{generic} / \text{Col}(\mathfrak{w}, \mathfrak{x}) \). Put

\[ \mathbb{R}^x = \bigcup \mathbb{R}_{\mathcal{G}^+} \]

where \( \mathcal{G} \) is \( \mathcal{S} - \text{generic} \) and \( \mathcal{G}^+ \) is \( \mathcal{S} - \text{generic} \) absolute complementing

\[ \mathcal{V}(\mathfrak{a}, \mathcal{G}) \models \varphi([\mathfrak{a}], \mathfrak{a}) \implies \mathcal{V}(\mathfrak{a}) \models \varphi([\mathfrak{a}], \mathfrak{a}) \].

\[ A_G = \{ A \in \mathbb{R}^x \mid A \in \mathcal{V}(\mathbb{R}^x) \} \text{ and } L(A, \mathbb{R}^x) \models \mathcal{L}^{\mathcal{G}^+} \}

Then

(1) \( L(A, \mathbb{R}^x) \models \mathcal{L}^{\mathcal{G}^+} \)

(2) \( \text{Hom}_G = \{ A \in \mathbb{R}^x \mid A \text{ is \mathcal{S} - generic complementing} \}

\[ L(A, \mathbb{R}^x) \}

(3) \( \Sigma^2_1 \) reflects to \( \mathcal{S} - \text{generic} \), complementing

\[ L(A, \mathbb{R}^x) \models \mathcal{M}_y \models V \text{ for } y \text{ the collection of } \mathcal{S} - \text{generic complementing} \text{ scales} \]
Def. For $A \leq \mathcal{P}(\mathbb{R}) = \mathcal{P}$,

$$M_A = \{x \in \mathbb{R}^+ \mid x \text{ is countable and } \exists \mathcal{F} \subseteq \mathcal{P} \text{ s.t.}
\begin{align*}
(x_i, \mathcal{F}_i) &\leq (m, \mathcal{E}) \text{ for all } i,
\end{align*}
$$

Use when $A \leq \mathcal{P} = L_{\kappa}(\mathcal{P}(\mathbb{R})) \text{ and } \kappa(\mathbb{R}) \leq \Delta$ for $\mathcal{E} = \{E \subseteq \mathbb{R}^+ \mid f : \mathbb{R}^+ \rightarrow E\}

\begin{align*}
M_A = \{x \in \mathbb{R}^+ \mid f : \mathbb{R}^+ \rightarrow E^+ \}
\end{align*}

Proof. Set $HC^* = HC(U_{\mathcal{P}})$ for $A \leq \text{Hom}_{x_0}^x$,

$$A^* = p[T] \cap \mathcal{P} \text{ for all } x \text{ and } \forall y < x \text{ and } T^*
$$

So, $A^* = p[T]$. (Exercise: independent of $T$)

Or $\text{Hom}^x = \{A^* \mid A \in \mathcal{U}_{\mathcal{P}} \text{ and } x < \mathbb{R} \text{ generic} \}

Lemma \quad (HC, \epsilon, A)^V \prec (HC^*, \epsilon, A^*)

Con. Every $\text{Hom}^x$-set has a $\text{Hom}^x$-scale

Thus: if $\text{Hom}^x \leq A^*$ then $U(\mathbb{R}^+)^x \leq \text{Hom}^x \leq \mathcal{U}(\mathbb{R}^+)^x
$\n
Here is the use of the assumption

Lemma \quad $\text{Hom}^x \leq A^*$.

Let $A \leq \text{Hom}_{x_0}^x$. We may $L(A^*) \leq \mathcal{P}$.

Show: if $\exists B \in L(A^*) \leq \mathcal{F}_{\mathcal{P}}[B, A^*]$

Then, if $B \in \text{Hom}_{x_0}^x$,

$$(HC, \epsilon, A, B)^V \prec \mathcal{F}_{\mathcal{P}}[B, A]$$

So, $(HC^*, \epsilon, A^*, B^*) \in (C^*, A^*)$.

So, if $\Sigma^* \Sigma^* \text{ reflects to Satelin co-Satelin in } L(A^*, \mathcal{P})$,

$$(M_2, \mathcal{F}(\mathbb{R})) \leq \Sigma^* \leq L(A^*, \mathcal{P})$$

\begin{align*}
\text{L0. Easy because } \mathcal{P} \text{ is regular in } L(A^*, \mathcal{P}^*).
\end{align*}
Proof. Take \( q = \emptyset \). Let \( y_0 \) be least s.t.
\[
L_{y_0}(A^*_1, R^*$) \models ZF^{+ \omega} + GB (HC, E, A^*_1 R) \upharpoonright q
\]
Pick \( B \ni OD_{\omega_0} \) some \( \omega \in R^* \) and the least sequence of ordinals possible is
\[
x \in B \text{ off } L_{y_0}(A^*_1, R^*) \models \omega \setminus \{ x, 2, 0, A^*_1 \}
\]
Let \( A^*_0 = p \langle T \rangle \) for some absolute constructible \( (T, V) \) in \( V \).
\[
\eta_1(x; T, 0) \equiv \forall x \in R \text{ and } \forall y \text{ least s.t. } \eta_1(x; T, 0) \equiv \eta_{y_1}(x; T, 0)
\]
Now \( B \models \bigwedge_{y \in R^*} p \langle T, 0 \rangle \).

(a) \( \eta \) is stationary hence correct.

Lemma. Let \( x \) be a limit of Woodin's.

(b) Let \( j : V \rightarrow M = \text{Lb}(V, H) \) where
\( H = \mathfrak{I}_{<\lambda} \)-generic, \( j \leq x \) Woodin. Then
\[
\eta_1(x; V, H, \lambda) \equiv \eta_{\text{Hom}_x}(V, H, \lambda)
\]
as a \( \leq_{\text{W}} \)-limit segment. \( \lambda = \text{R}^* \).

Now take break from the proof of the DIT.

Recall. Let \( F \in \mathbf{P}^2(\mathbb{R}) \) be \( \omega \)-parametrized, closed under \( \forall^R \), \( \exists^0 \text{ recursion substitution} \), and \( \text{PW}^0(\mathbb{R}) \). Assume also \( \exists^0 \mathbb{R} \subseteq \mathbb{R} \).

(We say \( \mathbb{R} \) is "inductive like". Example:
\[
\mathbb{R} = \mathbb{Z}_{\omega} + \mathbb{R}_{1,2} \text{ where } 2^1(12) \equiv \text{KP}.
\)
$m_{\mu} = (\mu, e, A)$ unless

if \( \varphi : U \to \mu \) is a \( \Gamma \)-morphism on \( U = \text{universal \( \Gamma \)-set} \)

\( A(x, \beta) \) iff \( x = u \) and \( \varphi(x) = \beta \)

\( \nu = \Sigma_{i=1}^{\infty} m_{\mu} \), also \( M_{\delta} = (\bigcup_{\delta} (1), e, A) \).

\( \delta_{\nu} = \sup \text{ of } p_{\nu} \text{'s in } \Delta \).

Assume scale \( (\mu) \). Let

\( \tau = T_{\mu} \) when \( \nu \) is a \( \Gamma \)-scale on universal \( \Gamma \)-set

(\( \tau \in (\mu) \).

We put for \( x, y \in \mathbb{R} \)

\( x \in C_{\nu}(y) \) iff \( \exists x < \delta_{\nu}(x) \) \( x \in \delta_{\nu} \) \( \text{from } \nu \text{ on } (\bigcup_{\delta} (1), e, A) \).

Note: "\( x \in C_{\nu}(y) \)" is \( \Sigma_{1}^{\mu} \) so \( x < \mu \).

Assume \( M_{\rho} \) is AD. Then by \( C_{\nu}(y) \) is countable.

Lemma \( C_{\nu}(y) = \mathbb{R} \cap L [T_{\mu}, y] \).

Let \( \mathbb{R} \cap \langle T_{\mu}, x \geq 1 \rangle < \delta_{\nu} \) be \( \Sigma_{1}^{\mu} \)

\( \mathbb{R} \cap \langle T_{\mu}, x \geq 1 \rangle < \delta_{\nu} \) is \( \Sigma_{1}^{\mu} \).

\( \mathbb{R} \cap \langle T_{\mu}, x \geq 1 \rangle < \delta_{\nu} \) by Hausdorff-Solovay.

Let \( < y \) be the canonical w.o. on \( C_{\nu}(y) \) \( \rightarrow \Sigma_{1}^{\mu} \)

\( R(x, y, t) \equiv \langle x < y, t \rangle \) in \( \Delta \).
$I(z,y) \iff z = (w,u)$ when $u \in \omega$ and $w$, $z$ is the real in $A_p(y)$.

$I \in \Sigma^1_1$, hence $\Pi$.

Now assume $AD^+$. Suppose also not all sets of reals are Suslin. (Eq $V=L(A,\mathbb{R})$)

Let $\kappa$ be largest Suslin cardinal.

$S(w) = \{ p \in \mathbb{T} \mid w \prec p \}$ the class of $w$-Suslin trees.

$S(w) = \Sigma^1_{\kappa}$ for $\kappa$-lightface inductive like with Scale($\Pi$).

Let $T = T^\kappa$ where $\kappa$ is a $\Pi$-scale on the univalent $\kappa$-set.

Let $T_0: \text{left}(T, \mathbb{D})$ where $\mathbb{D}$ is the Martin measure.

**Theorem:** $V = L(T_0, \mathbb{R})$

**Proof:** Force our $U$ with "pointed Sacks forcing".

**Conditions:** Trees on $w$ which are

- perfect

- pointed: $(\forall x \in [\mathbb{T}]) (T \leq_T x)$

$\{ (x)_T \mid x \in [\mathbb{T}] \} = \{ \mathbb{D} \mid [\mathbb{T}] \approx \mathbb{D} \}$

Martin: if $\mathbb{A}_1 \equiv_T \mathbb{A}_2$ then there is a perfect pointed tree $T$ s.t. $[\mathbb{T}] \leq \mathbb{A}$.

$P_{sacks} = \{ T \mid T \text{ perfect pointed } \}$.
Let \( q \) be \( V \)-generic \( P_{\text{sack}} \). For \( \mathbf{A} \in \mathbb{R} \), \( \mathbf{A} \in V \\
A \in U_{x_2} \Leftrightarrow (\exists t \in g) (c t) v \subseteq A \\
Martin \Rightarrow U_{x_2} \text{ is an UF over } P(\mathbb{R}) \cap V \\
Consider \\
\prod_{x_2 \in \mathbb{R}} L_{[t_1 x]} / U_{x_2} \\
We have to show that \\
x \in \text{Cod}. \\
If \( f : \omega_2 \rightarrow \omega_1 \), \( f \in V \) and \( T \in P_{\text{sack}} \) then \\
\( \exists \mathbb{U} \preceq T \Leftrightarrow \forall x, y \in [V] \\
x \Rightarrow y \Rightarrow f(x) = f(y) \\
So \( f \) is a function on \( \{ d \in \mathbb{D} \cup \mathbb{U} \} \). \\
to Ud in \( \text{Wei}(V, \text{Martin}) \). So \\
\([f]_U = [g]_{\text{Martin}} \\
So \ [c_T]_U = T^* \\
Then \ A_{1+} \Rightarrow \text{Wei}(V, \text{martin}) \text{ is WF. To come.} \\
Fact: \text{ Wei } \mathbf{A} \in V. \text{ Then for a } T \text{-cone of } x, \ A \in L_{[t_1 x]} \in L_{[t_2 x]}. \\
Need here \( x \) = largest Suslin.
Let $A^* = \{ e_A \} u$. Then $A = A^* \cap IR^V$.
So $A^* \in L(T^*, x_B)$.
So $A \in L(T^*, IR^V)$.
But the latter of any $x_B$, so $A \in L(T^*, IR^V)$.

In the situation above:

Let $\exists z \in 0$ iff $\exists \{ x \} \in A^*$. Then
$\exists z \in C_T(x_B) = IR, L(T^*, x_B)$ and
$A = \{ \{ x, y \} \mid \exists z \in z \in C_T(x_B) \}$

Working in $L(T^*, IR)$, say
$(\rho, \beta)$ codes $A \in IR$ (relative to $T^*$) iff
$\rho \beta$ is a $\beta$-th real $z = z^T^* x_B$ in

$\exists \rho \beta \in C_T(x_B)$ as determined by $T^*$.

Then  \(\exists \rho \beta \in C_T(x_B)\) iff $(x_0, y, x) \in \rho^f(T^*)$
and $x \in T^*$ and $A = \{ x \} \in \{ x \} \in \rho^f(T^*)$

In particular: take $x = x_B$.

(2B+DC)

**Theorem (Woodin) Let $M, N \subseteq AD^+$ and $IR \subseteq M \cap N$. Suppose $\mathcal{P}(IR)^M \neq \mathcal{P}(IR)^N$ and $\mathcal{P}(IR)^M \neq \mathcal{P}(IR)^N$.

Then if $\Gamma = \mathcal{P}(IR)^N \cap M \cap N$ $L(\Gamma, IR) = M$. Let $\Gamma$ be Sushin.
Work through in $\text{ZF} + \text{DC} + \text{AD}$.

We describe the structure of the Suslin cardinals and scaled pointclasses under $\text{AD}$.

**Def.** $\kappa$-Suslin sets, $\kappa^+$

**Def.** Suslin cardinals.

**Def.** Semi-scale.

**Fact.** A $\in \mathcal{P}(\mathbb{R})$ for $\omega < \kappa$ and iff A admits a $\kappa$-scale.

**Def.** Scales.

**Lemma.** TRATE

1. A is $\kappa$-Suslin
2. A admits a $\kappa$-scale
3. A admits a $\kappa$-scale
4. A admits a very good $\kappa$-scale.

**Lemma.** Let $\kappa$ be a Suslin cardinal.

1. There is a short, increasing sequence of $\kappa$-Suslin sets
2. If $\kappa \not\subseteq \kappa$ then $\max(\kappa) < \kappa$.

**Lemma.** For every $\kappa$, $S(\kappa)$ is closed under $\mathbb{P}^R$, $\mathbb{R}$, and $\mathbb{V}$.

**Theorem.** (Kechris) If $\kappa$ is a Suslin cardinal then each $A_\kappa$ $S(\kappa)$ is non-selfdual.

**Def.** $\kappa$-norm, $\kappa$-scale

**Def.** $\text{pwo}(\kappa)$, scale ($\kappa$)

**Pointclasses**

A pointclass $\Gamma$ is a collection $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ closed under $\subseteq$.

**Lemma.** Wedge's Lemma, Wedge degrees

**Theorem.** (Martin-Montag) $\mathfrak{w}$ is well-founded.
\[ o(\Gamma) = \sup \{ o(A) \mid A \in \Gamma \} \]

Def. Self-dual Wadge degrees.

Non-selfdual pointclasses correspond to non-selfdual Wadge degrees. Can occur in 4 different ways.

Thm. (Steel, van Wesep)

Def. Sep(\Gamma)

Theorem. For non-selfdual \( \Gamma \): exactly one of \( \Gamma, \neg \Gamma \) holds.

Fact. For any non-selfdual \( \Gamma \): \( \text{pwo}(\Gamma) \Rightarrow \text{sep}(\Gamma) \)

Need \( \Gamma \) closed under \( \forall \Gamma \)

Def. A non-selfdual \( \Gamma \) is a "long" pointclass iff

\[ \Gamma \text{ is closed under } \exists^R \text{ or } \forall^R \]

\[ \Sigma^1_2, \Delta^0_2 \text{ enumerate long pointclasses closed under } \exists^R \text{ or } \forall^R \]

Projects - like hierarchies: Steel: Closure properties of pointclasses (Cabal)

Def. \( S(\Gamma) = \sup \) of lengths of all pws in \( \Gamma \).

Fact. If \( \Gamma \) is selfdual then \( o(\Gamma) = \sup S(\Gamma) \)

Tied under \( \forall^R \).
Def (Steel) For a pointclass $\Pi$:

$$\Lambda(\Pi) = \bigcup \{ \Lambda(B) \mid B \text{ is selfdual closed under } \exists^B, \land \}$$

So $\Lambda(\Pi)$ is selfdual, closed under $\exists^B, \land$ and

$$\lambda = \omega(\Lambda(\Pi)) \cup \text{ a limit ordinal.}$$

**Type 1** $\text{cf}(\lambda) = \omega$ if self-dual and $\exists^B = \cup_\omega \Lambda$ Build projective hierarchy over it.

**Type 2** $\text{cf}(\lambda) > \omega$, $\text{cf}(B) = \lambda$ if non-selfdual. Assume they are not closed under quantifiers.

Let $\Sigma^B_{\leq 1}$ be the side with the separatin property and $\Pi^B_{\geq 1}$ the other side. Build $\Sigma^B_{\leq 1}, \Pi^B_{\geq 1}$.

(Type 3 means $\Sigma^B_{\geq 1}$ closed under $\land$, Type 2 not.)

**Type 4** $\text{cf}(\lambda) > \omega$ and both $B, R - B$ closed under quantifiers. Let $\tilde{\Gamma} = \tilde{\Gamma}(\lambda)$ be the side with the separation property. Let $\Sigma^B_{\leq 1} = \tilde{\Gamma} \cup \tilde{\Pi}$, $\Pi^B_{\geq 1} = \tilde{\Gamma} \cup \tilde{\Pi}$

Apply quantifiers to build $\Sigma^B_{\leq 1}, \Pi^B_{\geq 1}$.

The classes $\Pi^B_{\geq 1}$ are called Steel pointclasses.

The $\Sigma^B_{\leq 1}, \Pi^B_{\geq 1}$ $n \geq -1$ enumerate all Levy pointclasses.
Prewellordering property

Type 1 

Let $A = \bigcup_{n \in \Delta} A_n$. Let
$$c_\Delta(x) = \min \{ y \mid (x \in A_y) \}. \quad \text{Then } x <^* y \text{ iff } (\exists b)(x \in A_b \land y \subseteq A_b)$$

Similarly for $\leq^*$. Propagate by periodicity.

Type 2 

Then $\Gamma = \Pi_1^\delta$ has the pwo. By AD there is some $\delta \geq \omega_1$. $\Delta$ is not closed under $\rho$-unions.

Let $\rho$ be least possible. Then $\rho = c_\Delta(x)$.
1. $\rho \geq c_\Delta(x)$ by Coding Lemma.
2. $\rho \leq c_\Delta(x)$ by compactness with cofinalities.

Following Steele:

$$\Pi^\delta = \{ (A_n)_{n\in \Delta} \in (\Delta^\delta)^{\Delta \alpha} \mid \text{is } \Sigma^\delta_1 \text{-bounded} \}$$

Claim: $\Pi = \Pi^\delta$.

Type 4 We have $\Pi = \Pi^\delta = \exists^\delta \Gamma = \bigcup \Delta$. Note: $\rho = \lambda$.

$A = B \cap (12 - 1)$ are bounded unions.

Useful Fact

Lemma (Martin) Let $\Gamma$ be a non-selfdual p-class closed under $\forall \forall^\delta$, $\forall$ and assume prov(\Gamma). Then $\Delta$ is closed under $\exists \Delta(\Delta)$ unions and intersections.

Well-ordered unions

Theorem Suppose $\Gamma$ is non-selfdual, prov(\Gamma) and $\forall^\delta \Gamma \subseteq \Gamma$.

Then $\Gamma$ is closed under well-ordered unions.
Case 1. \( A \) closed under \( \land, \lor, \text{but not } \forall^w \). Let
\[
S_1 = \sup \{ k \mid k < \varepsilon \in \mathbb{E} \text{ is wellfounded} \}
\]
\[
S_2 = \sup \{ k \mid k < \varepsilon \in \mathbb{E}^w \text{ is wellfounded} \}
\]
So \( S_1 < S_2 \). We have
\[
p = \text{least } \varepsilon \text{ s.t. } U A_\varepsilon \neq \emptyset \text{ for all } A_\varepsilon \in \mathbb{E}.
\]
Thus \( p \) is regular.
Easily \( S_1 < S_2 \) (not closed under \( \forall^w \)). We must have
\[
U p \geq \mathbb{E}^w \Rightarrow \forall \varepsilon, \exists \varepsilon_1 \text{ s.t. } \Delta_1 = \Delta(\mathbb{E}^w) \text{ not closed under } \forall^w.
\]
Then \( U p A_\varepsilon \geq \mathbb{E}^w \Rightarrow \forall \varepsilon \). \( p \)

So \( d_1 < d_2 < p \).
We have \( \forall \varepsilon, \exists \varepsilon_1 \text{ s.t. } \Delta_1 = \Delta(\mathbb{E}^w) \text{ not closed under } \forall^w \). Write
\[
\varepsilon = \bigcup A_\varepsilon \text{ for each } A_\varepsilon \in \mathbb{E}.
\]
For \( x \in dom(\varepsilon) \) let \( \delta(x) \) = the eventual rank of \( x \in A_\varepsilon \).
This is an order-preserving map of \( x \) into \( S_1 \). \( \& \)
Let \( \langle x \rangle \) be a limit of Woodin cardinals \( \gamma \). Then it is not necessarily true that \( j^*(\text{Hom}_x) = \text{Hom}_x \).

If \( x \in R^{\exists} \) holds in \( V \), then \( j^* \circ (\varepsilon^2_{x})_{\text{Hom}_x} \) holds in \( V \). Let \( m \in x \) such that \( \text{Hom}_x \) is a homomorphism from \( \text{Hom}_x \) to \( \text{Hom}_x \) but not a \( \Sigma_1 \)-homomorphism.

This is true in \( M_{\omega_1} \). Let \( \pi \) be a homomorphism from \( M_{\omega_1} \) to \( M_{\omega_1} \) for some \( M_{\omega_1} \) with \( j : V \rightarrow M \) as above.

\[ x \in (\exists y) \cdot \Sigma_1 \]
\[ M \models \forall x \cdot (\exists y) \cdot \Sigma_1 \]

Let \( B \) witness \( x \in (\exists y) \cdot (\varepsilon^2_{x})_{\text{Hom}_x} \).

Since \( (HC, \varepsilon, B)^{V[G]} \models (HC^*, \varepsilon, B^*) \),

we get \( x \in 0^\varepsilon \cdot \text{Hom}^* \), (this is definable in a symmetric way) so \( x \varepsilon V \). 

**Example**

Exercise 9.4: \( \langle x \rangle \) is \( \lambda \)-generic. If

\[ \forall [G] \in \mathcal{F} \cdot \text{Hom}_x \cdot \langle HC, \varepsilon, B \rangle \models [G] \cdot [x] \]

then \( [G] \models \exists \mathcal{F} \cdot \text{Hom}_x \cdot \langle HC, \varepsilon, B \rangle \models [G] \cdot [x] \)
Theorem. If $M, N \models AD^+$ and $P(\mathbb{R})^M \not\subseteq N, P(\mathbb{R})^N \not\subseteq M$ and $\omega^M = \mathbb{R} = \omega^N$. Then letting

$$\Gamma_0 = P(\mathbb{R})^M \cap P(\mathbb{R})^N$$

$L(\Gamma_0, \mathbb{R})$ is all sets are Suslin.

Proof. If not, let

$n = \text{the largest Suslin card of } L(\Gamma_0, \mathbb{R})$

Let $S(n) = \mathfrak{M}$ for some $\mathfrak{M}$ which is inductive-like; also Scale ($\mathfrak{M}$).

Let $U$ be a universal $\Gamma$-set. Let

$$T = T_\mathfrak{M}$$

for a scale of $\mathfrak{M}$ on $U$.

Let $A, B$ be sets of reals that witness the divergence; i.e.,

$A \uparrow_n = \Theta^{L(\mathfrak{M}, \mathbb{R})}$ in $M$, $A \not\subseteq N$

$B \uparrow_n = \Theta^{L(\mathfrak{M}, \mathbb{R})}$ in $N$, $B \not\subseteq M$

Then

$L(A, \mathbb{R}) \not\subseteq L(\mathfrak{M}, \mathbb{R})$ is not Suslin.

$L(B, \mathbb{R}) = \mathbb{R}$

after $n$

Why: $AD^+ \Rightarrow$ The next Suslin cardinal $\mathfrak{m}$ is $\omega$-cofinal

But if $(A_n \cup \omega) \in \text{Wadge-\neg\:cofinal }$ in $\Gamma$, then

$(A_n \cup \omega) \in \text{MnN }$ ( $A_n \subseteq A$ in $\mathfrak{M}$ and $(\omega, \mathbb{R}^\omega) \in \mathfrak{M}$)

Let $T^*_A = (\Pi \Gamma / \text{Martin}) L(A, \mathbb{R})$

$T^*_B = (\Pi \Gamma / \text{Martin}) L(B, \mathbb{R})$

Then $A$ has a code relative to $T^*_A$

$B \uparrow_n T^*_B$

We used: on a Turing cone: of $x$:

$A \subseteq L[T_1x] \subseteq L[T_1x]$
\[ \text{Def. } \mathbb{E}(T) = \{ A \mid A \in L(T, \mathbb{R}) \} \]

Recall: \((p, \beta) \) codes \( A \) relative to \( T_A^* \) iff
\[ p \text{-sacks } \gamma \in L(A, \mathbb{R}) \text{ for } \gamma = \beta \text{-th real on } \mathbb{R}^* \text{ we have:} \]
\[ \text{if } \gamma \vartriangleleft \gamma' \text{ then } \gamma(0) = 0 \text{ iff } \gamma \vartriangleleft \gamma' \in A \]

We showed that there is a \((p, \beta)\)-coding \( A \) relative to \( T_A^* \).
Now recall:
\[ x \vartriangleleft y \text{ s.t. } (x, y) \in p \subseteq T_A^* . \]
So
\[ y \vartriangleleft x \text{ s.t. } (x, y) \in p \subseteq T_A^* . \]

Can also get \( y \) s.t.
\((p, y)\) codes \( B \) relative to \( T_B^* \) in \( L(B, \mathbb{R}) \).
We may assume \( \beta \leq \delta \). (The same \( p \)!)
\[ p \text{-sacks } \gamma \in T_B^* \text{ has a } \beta\text{-th real} \]
\[ L(B, \mathbb{R}) \]

Claim: \( L(B, \mathbb{R}) \models x, A \ni \exists y \in p \ni \]
\[ q \text{-sacks } \gamma \vartriangleleft x \text{ if } \gamma(0) = 0 \text{ and } \gamma \vartriangleleft y \ni x \]

so \( A \in L(B, \mathbb{R}) \), contradicts \( \text{KH} \).

Why: otherwise we could \( \text{read } \gamma \), \( y \in p \) s.t.
\[ q \text{-sacks } \gamma \vartriangleleft x \text{ if } \gamma(0) = 0 \]

and
\[ q \text{-sacks } \gamma \vartriangleleft y \text{ if } \gamma(0) = 1 \]

Let \( \pi: H \rightarrow V \) when \( H \) countable transitive
with every \( y \) relevant in \( \text{rng}(\pi) \).
\[ \pi(S^*) = T_A^* , \pi(U^*) = T_B^* \]
Then \( S^* \rightarrow T \)

\[ U^* \rightarrow T \]

Since Martin's measure is countably complete.
Here we are using full DC.

Let \( G \) be \( \kappa \)-accessible generic \( \mathbb{H} \) with \( q \in V \). So \( q \in HC \).

\[ x_q \in \mathbb{R} = \mathbb{V}^M = \mathbb{V}^N \]

Then \( <_{x_q} \subseteq <_{U^*} \) where \( \beta = \alpha^*(\beta) \).

Why: let \( I(w, z <_{x_q} y) \) in \( L(\mathbb{A}, \mathbb{V})^+ [x_q] \).

Then \( I(w, z <_{x_q} y) \) holds in \( V \), and \( S^* \rightarrow T \).
so \( <_{x_q} \) and \( <_{U^*} \) are compatible.

Now go back to:
- \( \lambda \) a limit of Woodin
- \( G \) is \( Col(\omega, \lambda) \) - generic \( / V \)
- \( L^\mathbb{V}_G \), \( Han^G \),
- \( \mathbb{A}^G \subseteq \{ A \in L^\mathbb{V}_G | A \in V(\mathbb{R}^*) \} \) and \( L(\mathbb{R}^+, \mathbb{A}) \models AP^+ \)

**Lemma**: Let \( A, B \in A_G \). Then \( A \leq_w B \) if and only if \( B \leq_w 1^{\mathbb{R}^+} - A \)

**Proof**:\nWina \( A \in Han^* \). (But for all \( w \) we know \( \text{Hom}^* \notin L(\mathbb{A}, 1^{\mathbb{R}^+}) \))

Let \( \Gamma \) be the largest Suslin in \( L(\mathbb{A}^*, 1^{\mathbb{R}^*}) \)

\( \Gamma \) inductive like, \( Scale(\Gamma) \), \( \Gamma = S(\mathbb{K}) \)

\( U \) universal \( \Gamma \)-set, \( U = \{ s \} \), \( s \in w \times x \)

Let \( \varphi_0 : U \rightarrow w \) be a \( \Gamma \)-name on \( U \)
Let \( p(T_0) = \{ x \in \mathscr{P}_y \mid x \subseteq \mathcal{G}_y \} \) for an \( n \times k \)
and \( T_1 \) on \( w \times n \) for \( y \in U \).
\[
p(T_1, y) = \{ x \mid x \subseteq \mathcal{G}_y \}
\]
where \( T_0, T_1 \in V \), \( \text{in } V(\mathbb{R}^k) \) so in some \( V(\mathbb{R}^k) \).

Claim: \( p(T_0)^V \) is a point \( p(C) \) and each unit
of \( p(T_0)^V \) is \( \text{Hom}_{<\lambda} \).

\((T_1, y, T_0)\) give \( \lambda \) absolute complements.

Claim: \( U \notin \text{Hom}^* \).

\[\text{Pt: If } b, \text{ have } C \in \text{Hom}^* \text{ coding a scale on } -U.\]
\[\text{So } L(C, \mathbb{R}^*) \models \text{AD}^+. \text{ Since } U \text{ is the largest}\]

Subline of \( L(A, \mathbb{R}^*) \) exists: \( C \subseteq L \langle A, \mathbb{R}^* \rangle \).

On the other hand: if \( A \in L(C, \mathbb{R}^*) \) then \( A \subseteq C \),
so \( A \in \text{Hom}^* \) as \( C \).

So \( L(A, \mathbb{R}^*) \) and \( L(C, \mathbb{R}^*) \) differ past \( U \),
but \( L(A, \mathbb{R}^*) \) has no more subline cardinals,
so contradiction - if we had \( DC \) in \( V(\mathbb{R}^*) \).

To fix this,

\[ M_\alpha \]

\[ j : V \rightarrow M_\alpha \]

\[ M_\alpha = \text{lim} \text{Ult}(V, H(\lambda)) \]

is sufficiently well-founded.

\[ C = (C)^*, C \in V, j(C) = C. \text{ (As } C \in \text{Hom}_{<\lambda}) \]

But \( A \in M_\alpha \).

Let \((p, \beta)\) code \( A \) relative to \( S^* = \bigcap S/\text{Martin} \)
\( S^* \rightarrow j(S^*) \).

Then \( \left< S^*_x \right> \) is an initial segment of \( \left< j(S)_y \right> \) for \( x, y \).

Thus, \( p \) packs generic over "everything".


\[(p, \beta) \text{ codes } A \text{ relative to } j(S^x) \text{ in } M_{\omega}. \text{ So } A \in M_{\omega} \in \mathcal{R}.\]

Now apply "no divergent models" in \(M_{\omega}\).

Thus \(U \in \text{Hom}_*\).

\[\text{Exercise: Every Hom}_* \text{ set is } \leq_\omega \text{ some proper DS of } p[\mathcal{T}_0]^V.\]

Now as \(U \in \text{Hom}_*\), the tree product lemma fails for \(q(\eta, \xi) \equiv \times e \in p[\mathcal{S}]\).

This means that the stationary tower correctness fails.

We have \(M \leq_{\text{ult}} (U, \mathcal{M})\) true in \(V_{\aleph_0}\),

\[i^V : \mathcal{V} \begin{array}{c} \longrightarrow \ \kappa \ \longrightarrow \ M_{\omega} \in \text{successor of } \mathcal{P}_{\Omega} \end{array} \sim \mathcal{V} \leq_\omega \mathcal{V} \subset \mathcal{V} .\]

Now:

\[\text{Using } T_1 \text{ get: } p[\mathcal{T}_0] \cap (\mathbb{R}^*)^2 \text{ a proper DS of } p[\mathcal{T}_0] \cap (\mathbb{R}^*)^2 \text{ as } e \in p[\mathcal{T}_0] \mathcal{S}.\]

Use \(T_0 \to j(T_0)\) such that

\[T_0 \leq j(T_0) \text{ no new points below } \text{fold}(p[\mathcal{T}_0]).\]

Then \(U \in M_{\omega}\) and \(U \in j(\text{Hom})\).

Also \(A \in M_{\omega}\) using \(S^x_\alpha \to j(S^x_\alpha)\) as before.

So \(A \in j(\text{Hom}_{\omega})\) using divergent models \(T_1 \in M_{\omega}\).

Similarly: \(B \in j(\text{Hom}_{\omega})\) for the same \(j\).
But then $B \notin \text{Hom}^*$ as our $B \notin \text{Hom}_c^*(U_B)$

A similar argument to that above shows: $B \in \text{M}_x$.

$U_B = \text{universal set at largest Suslin of } L(B, R^*)$

$U_B \subseteq \text{Hom}_c^*(U_B)$ so $B \in \text{Hom}_c^*(U_B)$

So $A \subseteq B$ or $B \subseteq \text{IR} - A$ via some $\sigma \in \text{IR}^*$. □

Lemma: Either $\text{Hom}_c^*$ is closed under $\# \circ 0$

$(\exists A \in \text{Hom}_c^*) L(A, IR^*) = L(A_G, IR^*)$

Pf: Exercise. □

So $\text{wma} : \text{Hom}_c^*$ is closed under $\#$.

Corollary: $\text{Hom}_c^*$ is closed under $\#$.

Pf: Exercise, "not quite trivial".

Case 1: $\text{Hom}_c^* = A_G$ (the case in the old DMT paper)

Case 2. Otherwise for $A \in A_G - \text{Hom}_c^*$ we have

$\text{Hom}_c^* \subseteq L(A, IR^*)$. □ Claim 1

Let $n = $ the largest Suslin of $L(A, IR^*)$

Then $n = \# \text{Hom}_c^* \#$

Let $\mathcal{F} = S(n)$, $\mathcal{F}$ inductive like, $T$ tree on $\omega \times \mathcal{F}$

$\mathcal{T} = \text{universal } \mathcal{T}$-

Let $T = (t^*) \in L(A, IR^*)$ for any/all $A$ with $1A \downarrow_0 \omega$

$T^* = \bigoplus T_d^*$
\( T^*, T \in V[\mathbb{R}] \text{ some } A \in \mathcal{H}[\mathbb{R}^*] \)

\[
i \rightarrow M \quad x \in \mathbf{p}(\mathcal{M}) - \mathbf{p}(\mathcal{T})
\]

\[
\downarrow \quad j : V[\mathbb{R}] \\
\downarrow \quad M_\infty \quad \mathbb{R}^{M_\infty} = \mathbb{R}^* \text{ and } M_\infty \text{ well-founded}
\]

Then using \( T^* \rightarrow j(T^*) \):

\[ A_G \leq M_\infty , \text{ No divergence models in } M_\infty , \]

\[ A_G \leq j(\text{How } \leq) \]

\begin{itemize}
  \item Remark: \( M_\infty \leq \mathcal{E}_1 \mathcal{H}[\mathbb{R}] \text{ in } L(A_G, \mathbb{R}^*) \)
  \item This is an easy by-product: In each case,
  \item it is how we showed \( L(A_G, \mathbb{R}^*) = A \mathcal{D}^+ \)
  \item \( M_\mathcal{H}[\mathbb{R}] \leq \mathcal{E}_1 V \)
  \item But to prove: \( M_\gamma \leq \mathcal{E}_1 V \text{ in } L(A_G, \mathbb{R}^*) \)
  \item takes more work.
\end{itemize}
Well-ordered unions

We are proving:

**Theorem** \( \Gamma \) is non-self-dual, \( \text{pwo}( \Gamma ) \) and \( \exists^R \Gamma \in \Pi \).

Then \( \Gamma \) is closed under well-ordered unions.

We did Case 1 \( \Gamma \) closed under \( \forall^\infty \), \( \forall^\omega \) but not \( \forall^R \).

**Case 2** \( \Gamma \) closed under \( \forall^R \). So \( \Gamma \) is closed under \( \forall^\infty \), \( \forall^\omega \).

By \( \text{pwo}( \Gamma ) \): \( \Gamma \) not closed under well-ordered unions.

We have \( \varphi_1 \) for \( \Gamma \) and \( \varphi_2 \) for \( \bar{\Gamma} \).

We show \( \varphi_1 < \varphi_2 \). Notice \( \forall^R \exists^R \bar{\Gamma} \). We have \( A \in \varphi_1 \Gamma \).

\( A \in \varphi_2 \Delta \). Write \( A \) as \( \Xi^f \)-bounded union. Play game:

I \( \forall^\infty \) \( \exists \Delta \) universal \( \Gamma \)-set. I plays \( x \in A \).

II \( \forall^\omega \) \( \exists \Delta \) \( y \in y \upharpoonright \alpha \) and

\( x \in A_\alpha = \bigcup_{\omega \leq \alpha} A_\beta \) for some \( \varphi_1 \alpha \geq \varphi_2 (\alpha) \).

Standard. I cannot win. Let \( \alpha \) be a \( \Lambda S \) for \( \bar{\Gamma} \).

Define a \( \leq \) on \( A \):

\( x < y \) if \( x \in A_\alpha \neq (\delta(y))_\alpha \in A_\alpha \).

then \( \leq \) is a well-founded relation of length \( \varphi_1 \alpha \).

Since \( \varphi_1 \alpha \) is regular. By coding lemma: \( \varphi_1 \alpha > \varphi_2 \alpha \).

Likewise show \( \varphi_1 \alpha > \varphi_1 \alpha \). Contradiction.

**Case 3** \( \Gamma \) not closed under \( \forall^\infty \), \( \forall^\omega \).

Actually: We only use "not closed under \( \forall^R \)."

**Proof** Let \( \varphi \in \forall^R \Gamma \). As before let \( S \) be a minimal \( \Xi^f \)-set of \( A \).

Note: \( \Gamma \) closed under \( \forall^\infty \), \( \forall^\omega \).
Play the game and define $\prec$ as before. By coding lemma:
$\gamma_0 \Gamma \leq \Sigma_1^r$ and $\gamma_0 \Gamma = \Sigma_1^r$.

Theorem (Chang): Let $\Gamma$ be non-self-dual closed under $\forall^R$ and $\forall$. Then $\varphi$ is strictly increasing or decreasing sequence of length $(\Sigma_1^r)^*$. Here
$\delta(\Gamma) = \text{map of all lengths of } \Delta \Gamma \in \Pi_1 - \text{proofs}.
$

Proof: Let $\{ A_\alpha \}_{\alpha < \beta}^+ \in \text{a strictly increasing sequence of } \Pi_1 - \text{sets}.
\quad \text{Wrt:} \quad \bigcup_{\alpha < \beta} A_\alpha \in A_\beta. \quad \text{Since proofs } \text{of } \varphi(A_\beta) = \text{proofs } \varphi(\text{all } A_\alpha).

Let $\psi$ be the map corresponding to $\{ A_\alpha \} _{\alpha \in \alpha^+}$

The associated proofs in $\varphi(A_\beta)$. Let $\Upsilon$ be universal for $\Gamma$.

Define:
$C(\psi_1, \psi_2) \iff \exists \xi < \beta^+ \quad \text{such that } A_\xi = A_\beta \quad \forall \xi \in A_\alpha, \quad \forall (\psi_1, \psi_2) = \alpha \quad \forall (\psi_1, \psi_2) = \beta$.

Let $S \subseteq C$ be a choice set by Coding lemma, $S \subseteq \varphi(A_\beta)$.

Now: $\text{by set in } \Delta$ or a $\delta$-union of $\Delta$-sets, $S$ is a $\delta$-union of $\Delta$-sets in $\varphi(A_\beta) \in \Gamma$. So $S = \bigcup_{\psi \in S} \psi$.

Define:
$\psi(\psi_1, \psi_2) \leq_\beta (\eta_1, \eta_2) \iff \text{both on } \psi \in S \land \eta_2, \eta_2 \in (\eta_1, \eta_2)$
$\quad \iff \psi \beta \leq \eta_1 \in U_{\alpha_2}$
$\quad \iff \psi \beta \leq \eta_2 \leq \beta \in U_{\gamma_1}$

So $\leq_\beta \in \Gamma$. This gives an injection map $S^+ \rightarrow S \times S$ by $\alpha \rightarrow (\beta, \alpha)$. For $x \beta$ let $\beta$ be the least $\delta^+ : (x, y_1, y_2) \in S$. Then let $S = \{(x, y_1, y_2) \leq \beta$.

Case 2: $\Gamma$ closed under $\exists^R$

Proof: As before, just $X \in \exists^R(A_\beta)$.
Q: Can one remove the assumption on the closure under $\mathcal{V}$ in Chang's theorem $^2$.

Remark There are 2 cases where $\mathcal{V}$.

Well need: (black box)

Theorem (Steel, Woodin)
$\text{AD} \Rightarrow$ Sushlin cardinals are closed and below their $\text{upp}$
$\text{AD}^+ \Rightarrow$ The Sushlin cardinals are closed

Theorem (Martin, Steel) $\#(\text{AD})$ if $\text{lt} \leq \text{sup}$ of Sushlins
and $T$ is a tree on $\omega_1 \times \omega_1$ then $\mathcal{T}$ is weakly homogeneous

Type 1 Case $\kappa$ is a limit Sushlin cardinal.

First consider $\text{cf}(\kappa) = \omega$. Let $\lambda = S(\kappa)$. So $\lambda$ is closed
under $\mathfrak{P}(\kappa)$, $\mathfrak{P}(\kappa)$, and $\mathcal{T}$: let $\lambda = \theta(\lambda)$.

Claim $\lambda \leq \kappa$.

Proof If $\lambda > \kappa$ then $\kappa$ is a $\lambda$-pwo of length $\mathcal{T}$. By (a)
$S(\kappa) \subseteq \lambda$, $\mathcal{T}$. So $\lambda \leq \kappa$. We shall show that
there are $\lambda$-pwo's of length $\mathcal{T} < \kappa$.

Let $\langle \mathcal{B}_\alpha \rangle_{\alpha < \kappa}$ be an increasing $\mathcal{T}$.
Find $\mathcal{E} \in \mathcal{L}$ with $\mathfrak{P}(\mathcal{E})$, $\mathfrak{P}(\mathcal{E})$.

By closure properties of $\mathfrak{S}(\kappa)$: $S(\kappa) \subseteq \Sigma^k_1$.

By Coding lemma: $S(\kappa) \subseteq \Sigma^k_1$. 
We show: Scale (Σ₀^₇).

Now given A ∈ Σ₀^₇: Write A = ∪ Aₙ, where Aₙ ∈ S(κₙ). Define a scale in the standard way.

Then propagate by finiteness.

By Kunen–Martin: all Aₙ real here are ≤ κ⁺.

By Coding Lemma, κ⁺ = sup of lengths of Σ₂⁻¹-ω₁ of relations.

So: κ⁺ is regular.

→ Such argument on the behavior for Σⁿ₂.

**Type 2,3 Case**  cf(n) > ω.

Let A = S(cₙ). Recall: Π₁⁻¹ not closed under Σ₄^¹₂.

As before λ = n.

Σ₀^₇ is closed under well-ordered unions, so

S(λ) = ∪ₙ S(κₙ) ≤ Σ₀^₇.

So either S(ω) = Σ₀^₇ or S(ω) = Σ₀⁻¹. We show the latter does not occur.

Let Γ = the collection of all A = p[T] where T is homogeneous on λ.

Have: S(ω) = Σ₀^₇ as every tree on ω₁₅ is weakly homogeneous. So Γ ≠ ∅.

Claim: ∀p ∈ S(ω).

Cor: S(ω) = Σ₀⁻¹

Proof: Let A(ω) = (B₁) B(ω) where B(ω) = B = p[T] for some homogeneous T on ω₁₅ × ω₁₅.

Play the game: Fix x ∈ A.

x ∈ A

y ⊨ T

It has a w.s.p. against a closed game for T.

By the usual determinacy argument for homogeneously

A₂₅
So \( S_0 = \mathcal{E}_0 \).

\[ \text{Con} \, \mathcal{R}_1 = \mathcal{E}_1 \text{ IT is homogeneous } \]

Now prove \( \text{Scale} (S_0) \).

Claim \( \text{Scale} (\mathcal{E}_0^+) \) vs. \( \text{Scale} (S_0) \)

\[ \text{Pf.} \quad \text{Basically same as before.} \]

Now show \( \text{Scale} (\mathcal{R}_1) \).

Let \( \mathcal{R} \) be stb.

\[ \mathcal{R} = \Sigma_1^1 \text{-bounded p-union of } \Delta \text{-sets} \]

Let \( \mathcal{A} \in \mathcal{R} \) and \( U \) be universal \( \mathcal{E}_1 \).

As before: \( C \) is \( \mathcal{E}_1 \) some Ton \( w \times \mathcal{R} \), \( V = \mathcal{E}_1 \).

Define \( V \) on \( (\mathcal{R})^2 \times N \) by

\[ (s,t,u,v) \in S \quad \Rightarrow \quad (s,t,u) \in S \quad \text{ and } \quad (t,v) \in V \]

Clearly: \( \mathcal{A} \epsilon \mathcal{E}_1 [V] \). Modify as follows: get \( V' \) on \( w \times \mathcal{R} \)

s.t. for \( (s,t,u,v) \in V' \), \( d_0 > d_1 > d_2 > 0 \).

The corresponding scales is in

For instance: Show \( \mathcal{E}_0 = U \otimes p \), where

\[ (x,y) \in \mathcal{E}_0 \quad \Rightarrow \quad \exists \beta < \xi \left[ (x,y) \in A \otimes \varphi_{\xi}(x) = \beta \right] \]

\[ \forall (y \epsilon A \otimes \varphi_{\xi}(y) \leq \beta) \]

where \( f: \beta \rightarrow \mathcal{E}_0 \) is cofinal. Show this is a

\( \Sigma_1^1 \)-bounded union.
21.7. 9:30  John Steel

Remarks

1. \( AD^+ + V = L(\mathcal{P}(\aleph_1)) \) \( \Rightarrow \)
   \[ M \models AD + M \models L(\mathcal{P}(\aleph_1)) + V = L(\mathcal{P}(\aleph_1)) \]
   on proof of DMT involved sharing
   \[ L(\mathcal{V}_0, \mathcal{V}^+ \mathcal{V})] \models \text{truth}
   \]

2. \( AD^+ \Rightarrow (a) \) Scale \( (\mathbb{Z}, 1) \)
   (b) \[ M_{\Delta_1} < \mathbb{Z}, M_y < M_{\Delta_1} \mathcal{P}(\aleph_1) < \mathbb{Z} \]

Ref: The Derived model Then paper.

3. \( AD^+ \Rightarrow \) the class of Suslin is closed
   All sets on \( \omega^* \)-Borel
   Ordinal determinacy
   (Kethered on website)

Thus let \( \lambda \) be a limit of \( \lambda \) in \( \lambda \) that \( \delta \) does not satisfy
then \( \mathcal{A}_\delta = \text{Ham}^+ \). So

\[ L(\mathcal{A}_\delta, \mathcal{V}) \] All sets of reals are Suslin \( \gamma = \mathcal{P}(\delta) \)

Pf: On we have a largest Suslin on \( L(\mathcal{A}_\delta, \mathcal{V}) \)

\[ \nu = \mathcal{P}^{\mathcal{A}_\delta} \] Let \( T \) be \( \omega^* \), \( \mathcal{T} \) is a \( \omega^* \) relation

on \( \mathcal{V} \) of rank \( \geq \nu, \) \( \text{TH} \in L(\mathcal{A}_\delta, \mathcal{V}) \),

Let \( \mathcal{T} \in \mathcal{V}(\mathcal{V}) \), \( \mathcal{T} \in \mathcal{V}(\mathcal{V}) \).

Let \( \mathcal{U} \) be a site \( \mu \)-generic,

\[ \mu < \nu < \lambda \]

\[ \mathcal{V}(\mathcal{U}) \models \mu \text{ is } \lambda \text{-strong} \]

Let \( \gamma \) be \( \mathcal{V}(\mathcal{U}) \)-generic for \( \mathcal{U} \) \( \omega_1 + 2 \) \( \) and

\[ \mathcal{V}(\mathcal{U}) \models \gamma \models \text{Ham}^+ \quad \gamma < \lambda \]

Let \( j: \mathcal{V}(\mathcal{U}) \to \mathcal{M}(\gamma) \) witnesses that \( \gamma \) is \( \gamma \)-strong

where \( \gamma = \text{the } 3^\text{rd} \text{ Woodin } > \gamma \).
Then \( \mathcal{U}(\mathbb{R},\mathcal{G}) \not\subseteq j(\mathcal{U}) \) is \( \mathcal{A}^+ \)-absolutely complemented. So
\[
\mathcal{U}(\mathbb{R}) \ni \mathcal{U} \ni \mathcal{A} \ni \mathcal{A} \text{ a.e. pair } \Rightarrow \\
\mathcal{U}(\mathbb{R}) \ni \mathcal{U} \ni \mathcal{A} \ni \mathcal{A} \text{ a.e. pair } \\
p(\mathcal{U}) = p(j(\mathcal{U})).
\]
Now:
(1) \( p(\mathcal{U}) \cap \mathcal{A}^* \) is a \( \mathcal{A}^* \) relation.
(2) \( p(\mathcal{U}) \cap \mathcal{A}^* \) is a \( \mathcal{A}^* \) relation of rank \( \geq \).
Contradiction.

Exercise: If \( \mathcal{A} \) is a limit of Woodin and \( \mathcal{E} \subseteq \mathcal{A} \) s.t. \( \mathcal{E} \subseteq \mathcal{A} \) strong then
\[
L(\mathcal{A}, \mathbb{R}) \subseteq \mathcal{E}
\]
Equivalently: \( \mathcal{E} \supseteq \mathcal{A} \)

Notation: \( \text{D}(\mathcal{M}, \mathcal{A}) \) for \( \mathcal{M} \models \text{ZFC} + \mathcal{A} \) a limit of Woodin

"is" \( L(\mathcal{A}, \mathbb{R}) \) for \( \text{Col}(\omega_1, \mathcal{A}) \)-generic.

Exercise (a) \( \text{D}(\text{Add}(\omega_1, \mathcal{A}), \mathcal{A}) \subseteq \mathcal{V} = L(\mathbb{R}) \)
(b) \( \text{D}(\text{Add}(\omega_1, \mathcal{A}), \mathcal{A}) \subseteq \mathcal{V} = L(\mathbb{R}) \)
(c) \( \text{D}(\text{Add}(\omega_1, \mathcal{A}), \mathcal{A}) \subseteq L(\mathbb{R}) \cap \mathcal{V} \)

Converse direction

Given \( \text{M} \models \text{AD}^+ \), realize it as a derived model.
LARGEST SUSLIN CARDINAL CASE

Suppose in \( V \) we have \( AD^+ \) and there is a largest Suslin \( T \).
Let \( T \) on \( x \) be witnesses \( \tau = \text{the largest Suslin} \).

\( T^* \) as before, so \( V = L(T^*, R) \) \((2) \text{ fact})\).
Every \( A \in \mathbb{R} \) is "countably captured" on \( T \).
\( An L(T, x) \subseteq L(T, x) \) on a cone of \( x \).

Claim. For a cone of reals \( Z \):
\[
\exists \{x \in \mathbb{R} \mid x \in OD(T, x)\}
\]
\[
\exists \{x \in \mathbb{R} \}
\]
\( Z \) if not, let
\[
\min \{x \text{ largest OD}(T, x) - \text{real not in } L(T, x)\}.
\]
\( A(x, y, z) \iff f(x)(y) = z \)

Have \( A \wedge L(T, x) \in L(T, x) \) on a cone of \( Z \).
\( A(x, y, z) \iff f(x) \in L(T, x) \).

let \( x_0 \) be base of \( Z \) a cone \( Z \) from above. Claim holds.

If \( A \) is countable transfinite with \( x_0 \) \( A \)
\( b(x, y, z) \in L(T, x) \) \( \iff b \in OD(T, x) \) and member of \( A \).

If: Exercise; reduce this to the case of reals.

Let \( T \) be the Prikry tree forcing corresponding to
the Martin measure on \( D \).

Conditions: \( (s, F) \) when \( s \) is a finite sequence
of Turing degrees and
\[
F: \omega^\omega \to \omega(\omega)
\]
\( F(s) \) is of measure 1
\[
T_F = \{ x \in \omega^\omega \mid \text{there exists some } u \in \omega^\omega \text{ such that } u \in F(u x) \} \}
\]
then let
\[(s, F) \leq (t, G)\]
if \(s \in T_G\) and \(t \in S\) and \(F(v) \leq G(v)\) for \(s \leq v\).

If \(\exists a \gamma \) is \(\beta\)-generic then
\[\bigcup (s, F) = G \in \mathcal{P}_\kappa\] s.t.
\[\forall d \in D \exists \alpha \leq s(\kappa)\]

Basic property: let \(\{y_i\}_{i \in \omega_1}\) be a family of sentences in the forcing language. Let \((s, F) \in P\)
then \(\exists G \in \mathcal{P}_\kappa\) \(s, G) \leq (s, F)\) and for all \(i\):
\[(s, G) \not\models y_i\) (Proof without DC - [Trang's paper])

(CAP)

Theorem (Woodin) If \((R, S)\) are sets of ordinals then
\[
\text{on a } \mathcal{P}_\kappa\text{-cone of } x \quad L[HOD]_{\mathcal{P}_\kappa} \models [s, R]\]

Let a be well-founded, admissible, well-ordered, and satisfying \(\mathcal{P}_\kappa\) a.e. Let \(x \in \mathcal{P}_\kappa\) be s.t. a is coded by some \(y \leq x\). Let
\[
\delta^x = \omega_2 \quad L[T, x]\]
\[
\omega = \text{HOD} [s, \omega] \quad \delta^x + 1 \quad (\forall x \in \mathcal{P}_\kappa, \text{HOD equipped with its canonical w.o.})

Note: \(\omega^x \) depends only on \([x]\)
so \(\delta^d, \omega^d\) for \(d \in D\) makes sense.
Claim let $x_0$ be as in the previous claim, let
$a$ be countable transitive with $x_0 \in a$. Then for
a cone of $a : P(a) \cap Q^a = P(a) \cap L[T^a, a]$. 

Now take a Pickley $V$.
\[
<\text{di}	ext{line}> = \bigcup \{s \mid 1 \in \text{GF}(s, F) \text{ eg } Y \}
\]
assume each $<\text{di}	ext{line}>$ meets certain measure one
sets in Martin $x \ldots x$ Martin.

Let $Q_0 = Q^x_0$
\[
\omega_1 = \omega_{Q_0}
\]

\[
\omega_i = \omega \omega_i
\]

Then $P(\omega_i) = P(\omega_i) \omega_i \omega_{Q_i} \omega_{\omega_i}$ (by restricting to
measures one sets

set $\omega_i = \omega_i \omega_i$

Consider $L[T^x, \omega_i]$

Exercise For any $i$, $P(\omega_i) \cap L[T^x, \omega_i] = P(\omega_i) \omega_i \omega_{Q_i} \omega_{\omega_i}$

Claim $\omega = \omega_i \omega_i$ for some $G = \text{Col}(\omega, \omega_{\text{sup} \omega_i}) \omega_{\text{sup} \omega_i}$

Proof $\omega_i \cap \omega_i \cap (L[T^x, \omega]) \exists x : x \in \omega_i \Rightarrow \exists \omega_i \omega_{\omega_i}$

$\omega_{Q_i}$ by Vopenka.

... $\vdots$ for points of $\omega_0 < \text{sup} \omega_i$...

$\omega = L[T^x, \omega_i] \subseteq L(A, \omega_i)$
However, we actually have that, as otherwise we get a # for \( L[\mathbb{R}^\omega, \mathbb{R}^\omega] \) by forcing.

**Proof of:** \( M \models \mathcal{P}(\mathbb{R}) \neq V \)

**Cases:**

1. There is a largest Suslin \((\Rightarrow \Theta \text{ is regular})\) (Exercise (Take hulls))
2. No largest Suslin cardinal + cf(\( \Theta \)) = \( \omega \)
3. No largest Suslin cardinal + cf(\( \Theta \)) \( > \omega \).

**Case 2 Strategy:** Pick one force to get a model \( N \) s.t. \( D(N, \mathbb{R}^\omega) = V \). Let \( <\xi, \xi, < \omega \rangle \) be s.t. \( \mathcal{P}(\mathcal{R}) / \Theta \).

**Recall:** \( (ZF + DC + AX) \) let \( X \) be a set. The Solovay sequence is defined by

\[
<\theta^X_x, \xi < \xi_x > \text{ is defined by:}
\]

\[
\theta^x_0 = \sup \{ a \mid \exists \pi : \mathcal{R} \overset{a}{\rightarrow} d \text{ s.t. } \pi \text{ is OD}_x \}
\]

\[
\theta^x_x = \sup \{ a' \mid \exists \pi : \mathcal{R} \overset{a'}{\rightarrow} d' \text{ s.t. } \pi \text{ is OD}_x \text{ for some/all } A \text{ with } |A|_\mathbb{R} = 2 \}
\]

\( \sup \) limits for a limit.

**Remark (AX):** For \( \Theta^X_x < \Theta^x_x \):

\[
\Theta^x_{x+1} = \sup \{ \xi (\exists \pi : \mathcal{R} \overset{a}{\rightarrow} d) \text{ s.t. } \pi \text{ is OD}_x \}
\]

**Claim:**

Now let \( d \) be as above.

**Notation:** \( \kappa^X = \sup \delta^X (A) \) where \( |A|_\mathbb{R} = \Theta^X_x \).
For each $i$ let $\mu_i$ be the supercompact measure on $\mathcal{P}(\mathcal{P}(\kappa_i))$. $\mu_i$ is unique, hence OD. Let

$$X_i = \{ x \in \mathcal{P}(\mathcal{P}(\kappa_i)) \mid \text{HOD}\sigma_{\mu_i} = \text{AD} + \exists \mathcal{A} \Delta \} \quad \text{and} \quad \sigma_i = \text{HOD}\sigma_{\mu_i} \uparrow \text{HOD}\sigma_{\mu_i} \uparrow \text{HOD}\sigma_{\mu_i} \quad \text{where}\}

$$

2. $\sigma_i$ collapses $\mathcal{P}(\kappa_i)$ to $\text{HOD}\sigma_{\mu_i}$

3. $\sigma_i$ is a successor on the Solovay tree.

Claim $\mu_i(\kappa_i) = 1$ for all $i$.

Fix $i$. Let $M = \prod\text{HOD}\sigma_{\mu_i} / \mu_i$, where

the ultrapower is taken in $\text{HOD}\sigma_{\mu_i}$.

The ultrapower is well-founded since $\text{HOD}\sigma_{\mu_i}$ is well-founded.

Let $\pi_i : R \to \sigma_i$ in $\text{HOD}\sigma_{\mu_i}$ (this exists)

Let $\sigma_0 = [\pi_i]_{\mu_i}$. Then we have $\text{To}_i$:

$$M \models ? \sigma_0 \iff \sigma_0 \in \text{HOD}\sigma_{\mu_i} \uparrow \sigma_0$$

This follows from normality of $\mu_i$. 

Note:
- $\sigma^n$ collapses $\mathcal{P}(\mathbb{R})$
- $\mathbb{R} \cap M = \mathbb{R}$
- $\mathcal{P}(\mathbb{R}) \cap M = \{ A \subseteq \mathbb{R} \mid \forall \omega < \theta_{2, \omega+1} \forall Y \subseteq \mathcal{P}(\mathbb{R}) \cap \text{HOD}_{\theta_{2, \omega+1}} Y \}$

Lemma Above: $\text{HOD}_{\omega \cup \omega}$, played angle of $L(T, \mathbb{R})$

Let $T_0 = \{ \langle \sigma_0, \ldots, \sigma_n \rangle \mid \sigma_i \in X_i \text{ for } i \leq n \}$

$T = \{ \langle \sigma_0, \ldots, \sigma_n \rangle \in T_0 \mid$

1. $\mathcal{P}(\mathbb{R}) \cup \text{HOD}_{\theta} = \mathcal{P}(\mathbb{R}) \cup \text{HOD}_{\theta}$
2. $\sigma_i \in \sigma_j$ whenever $i < j$
3. $\sigma_k \in \text{HOD}_{\omega \cup \omega}$ for all $k \leq i$
4. $\sigma_k$ is countable in $\text{HOD}_{\sigma_0 \cup \sigma_k}$ for all $k < i$
5. Let $\theta_i = \theta \cup \text{HOD}_{\omega \cup \omega}$, then

$\text{HOD}_{\sigma_i} \supseteq \{ \theta_i : \text{Woodin} \}$
6. $\mathcal{P}(\mathbb{R}) \cup \text{HOD}_{\sigma_0 \cup \sigma_k} = \mathcal{P}(\mathbb{R}) \cup \text{HOD}_{\sigma_k}$

Want: Let $s = \langle \sigma_0, \ldots, \sigma_n \rangle \in T$. Then

Let $\sigma \in \mathcal{P}_\omega \mathcal{S} \subseteq \mathcal{T}$. Let $H = \text{HOD}_{\mathcal{S} \subseteq \mathcal{T} \setminus \mathcal{S}}$

Then $H = \text{HOD}_{\mathcal{S} \subseteq \mathcal{T} \setminus \mathcal{S}}$

Proof:

$H \subseteq \text{HOD}_{\sigma_0 \cup \sigma_1}$ so RHS makes sense.

Case 2: $\theta_1$ is not Woodin.

Let $\theta = \langle \sigma_0, \ldots, \sigma_n \rangle \in T$. Then $\forall \sigma \in \mathcal{S} \subseteq \mathcal{T} \setminus \mathcal{S}$. Then}

\[ \text{HOD}_{\sigma_0 \cup \sigma_1} \subseteq \mathcal{S} \subseteq \mathcal{T} \setminus \mathcal{S} \]
Proof. Enough to check the case (5).

Let \( t = \mathcal{E}^{\mathcal{C}} \mathcal{O} \) and \( H = \text{HOD}^+ \).

We know: \( H = \text{HOD}^+ \).

**Theorem (Kollnich-Wooldridge)** Assume \( \text{ZF} + \text{DC} + \text{AD} \).

Let \( x, y \) be sets. Then \( \text{HOD}^+_x \not\subseteq \text{HOD}^+_{y} \).

We want to show:

\[ H = \Theta^+ \text{ is Woodin when } \Theta^+ = \Theta^+ \text{ HOD}^+ \]

**Proof.** Work in \( \text{HOD}^+ \). So \( H = \text{HOD}^+ \).

**Note:** \( \Theta = \Theta^{\mathcal{L}(\mathcal{H}, \mathcal{H})} \) some \( \mathcal{L} \) of \( \mathcal{H} \).

This is because \( \text{HOD}^+ \) has largest Suslin, \( \Theta \) is successor (hence regular) and \( \text{DC} \).

More detailed:

\[ \forall \mu \in H \text{ OD}^+ \subseteq \text{AD}^+ + \text{AD}^+_R \text{ so } \forall \mu \in H \text{ OD}^+ \text{ has a largest Suslin } \Theta \text{ HOD}^+ \text{ is a successor on the Solovay sequence } \exists \gamma \text{ (a) regular on } \text{HOD}^+ \text{ . Also DC holds in } \text{HOD}^+ \text{.} \]

Now apply the above theorem to \( \Theta = \Theta^{\mathcal{L}(\mathcal{H}, \mathcal{H})} \) in the \( \text{HOD}^+ \).

Get: \( \text{HOD}^+_H = H \) and \( H = \Theta^+ \text{ is Woodin.} \)

To see that \( \forall \mu \in H \text{ OD}^+ \subseteq \text{AD}^+ + \text{AD}^+_R \text{ so } \forall \mu \in H \text{ OD}^+ \text{ has a largest Suslin } \Theta \text{ HOD}^+ \text{ is a successor on the Solovay sequence } \exists \gamma \text{ (a) regular on } \text{HOD}^+ \text{ . Also DC holds in } \text{HOD}^+ \text{.} \]

Now apply the above theorem to \( \Theta = \Theta^{\mathcal{L}(\mathcal{H}, \mathcal{H})} \) in the \( \text{HOD}^+ \).

Get: \( \text{HOD}^+_H = H \) and \( H = \Theta^+ \text{ is Woodin.} \)

**To see that** \( \forall \mu \in H \text{ OD}^+ \subseteq \text{AD}^+ + \text{AD}^+_R \text{ so } \forall \mu \in H \text{ OD}^+ \text{ has a largest Suslin } \Theta \text{ HOD}^+ \text{ is a successor on the Solovay sequence } \exists \gamma \text{ (a) regular on } \text{HOD}^+ \text{ . Also DC holds in } \text{HOD}^+ \text{.} \]

**Test need:** \( \forall \mu \in H \text{ OD}^+ \subseteq \text{AD}^+ + \text{AD}^+_R \text{ so } \forall \mu \in H \text{ OD}^+ \text{ has a largest Suslin } \Theta \text{ HOD}^+ \text{ is a successor on the Solovay sequence } \exists \gamma \text{ (a) regular on } \text{HOD}^+ \text{ . Also DC holds in } \text{HOD}^+ \text{.} \)

If not: Assume \( \forall \mu \in H \text{ OD}^+ \subseteq \text{AD}^+ + \text{AD}^+_R \text{ so } \forall \mu \in H \text{ OD}^+ \text{ has a largest Suslin } \Theta \text{ HOD}^+ \text{ is a successor on the Solovay sequence } \exists \gamma \text{ (a) regular on } \text{HOD}^+ \text{ . Also DC holds in } \text{HOD}^+ \text{.} \)

Take \( A_0 \) to be the least such. By normality
of rank, and the fact that $\Theta_n$ is fixed countable ordinal:

$$\forall \alpha \in \text{Ord} \quad A^\alpha = A^\Theta$$

but $A^\Theta$ OD from $S$ because $\Theta_n$ is.
We look at \( H \uparrow \mathbb{L}([x,E]) \).

\( M_1 \) is minimal proper class \( \mathbb{L}([E]) \)-model such that \( \mathbb{L}([E]) \) \( E \) Woodin cardinal and that is fully iterable.

Minimal = \( M_1 \) without last \( \mathbb{L} \) ladder.

Def: \( M \) is \( M_1 \)-like iff \( M \) is a proper class \( \mathbb{L}([E]) \)-model and \( M \models E \) exactly one Woodin cardinal.

\( S^M \) the unique Woodin of \( M \).

Deal mostly with \( S^M \prec \omega_1 \).

Note: \( M = \mathbb{L}[M_1, S^M] \cong \mathbb{L} \) when \( \mathcal{C} \not\subset \mathbb{E} \).

Question: Suppose \( M_1 (S^M \in \mathbb{L}(x), \dot{S}^M \prec \omega_1, \mathcal{C}) \). How much does \( \mathcal{C} \) know about the unique iteration strategy for \( M_1 \)?

Def (Uniqueness) Let \( J \) be a normal IT on \( M \).

Recall \( F(J) = \exp \{ \mathbb{L}(E^J \circ \delta, E^J) \} \).

\( M(J) = \bigcup M_0 \restriction \mathbb{L}(E^J, \delta_c, \delta_c) \).

Def: \( J \) is maximal iff \( \mathbb{L}(M(J)) = F(J) \) is Woodin.

Otherwise \( J \) is short (\( J \not\subset \mathcal{C} \)).
Then: if \( T \) is short then \( b = \Sigma(T) \) for \( \Sigma \)
any strategy for \( M_1 \) then \( M_1^T \) is well-founded.

Let \( 0 \leq \alpha \) be least \( \beta \) s.t. \( \Sigma_{\alpha+1}(M(T)) \not\subseteq \Delta(T) \) is not Woodin.

Then either:
- \( \Sigma_{\alpha+1}(M(T)) \not\subseteq \Delta(T) \) and
- \( \exists \beta \leq \alpha \) such that there is some set \( A \) (a function?) definable over \( \Sigma_{\alpha+1}(M(T)) \) that witnesses the failure of Woodinness.

Define \( \omega(T) = \Sigma_{\alpha+1}(M(T)) \). Then:

\[ b = \text{the unique cofinal } \xi \text{ s.t. } \omega(T) \leq M_{\xi+1} \]

(Minimality of \( \omega(T) \)). Then we can find \( b \in L(\forall x) \)
by absoluteness.

Now assume \( T \) is maximal. Then we may not find
\( b \in L(\forall x) \). Why: \( (L(\forall x)) \) does not admit \( \Delta \) generic iteration which makes \( \times \) generic. (Do it outside.) Then
\( \omega(T) \) can be recovered in \( L(\forall x) \). We can recover \( T \) if \( T \) is maximal.

Then:

\[ L(M(T(\forall x))) \not\subseteq \Delta(T(\forall x)) \text{ is Woodin.} \]

So for \( b = \Sigma(T(\forall x)) \) we have \( b^M = \Sigma(T(\forall x)) \).

So \( X \) is \( \text{BST} \) \( \forall x \) generic, then \( N = L(M(T(\forall x))) \).

But \( \delta^N \) is a cardinal in the generic extension.

But then \( \delta^N \) is \( \text{ST} \) \( \forall x \) \( \delta^N \).

The genericity situation cannot take \( \Sigma \).

Then \( M_{\xi+1} \not\subseteq N \).
Summary:

- So \( \text{Ch}(T) = \omega_1 \) if \( T \) has no branch in \( L(\mathcal{V}) \).
- \( x \) is generic for \( B \) in \( L(\mathcal{M}(\mathcal{V})) \).

What we have: We know the final model in \( L(\mathcal{V}) \):
\[ M^T_\omega = L(M(\mathcal{V})) \]

This works in general: if \( T \) is maximal in \( M \)
and \( b = \mathcal{E}(T) \) then \( L(M(\mathcal{V})) = M^T_\omega \) (and \( b \) does not drop).

Theorem: Let \( N \) be \( M_1 \)-like, \( T \) a tree on \( N \),
\( T \) is maximal and \( b, c \) be coinitial w.f. branches
of \( T \). Then \( b = c \).

Lemma: Let \( N \) be \( M_1 \)-like and \( \gamma \) be a proper
class of ordinals. Then
\[ \text{Hull}^N(\gamma) \cap \delta^N \text{ is coinitial in } \delta^N \]

Proof: Let \( \gamma = \sup(\text{Hull}^N(\gamma) \cap \delta^N) < \delta^N \).

Exercise: \( \text{Hull}^N(\gamma \cup T) \cap \delta^N = \gamma \), because \( \delta^N \) is singular.

Let \( M = \text{collapse of } \text{Hull}^N(\gamma) \setminus \{ \gamma \} \in \mathcal{N} \).

Then \( g^M = \delta^N \). So \( L[N(\delta^N)] \models M \models \delta^M \) not Vadin, 12.

Proof of the theorem:

\[ N \overset{b}{\longrightarrow} L(M(\mathcal{V})) \text{, since } \varphi(T) \text{ exists.} \]

with \( b(\delta^N) = c(\delta^N) = \delta(T) \)
Let $\mathbb{N} = \{(x, y) : \neg \exists z \in x \land \neg \exists z \in y \}$. If $\mathbb{N}$ is a proper class.

But $\text{Hull}^{L_{(\mathbb{H}^N)})}(\mathbb{N}) \in \text{arg} \{i_b \cap \text{arg} (i_c) \}.$

This implies $L = C$.

**Lemma.** Take any two cofinal non-chopping branches $b, c$ of a normal ultrapower tree $\mathbb{H}$, and let $\mathbb{N}$.

Let $\xi = i_b(\gamma) = i_c(\gamma)$ with $\gamma < \mathbb{S}(\mathbb{N})$.

Then $\gamma = \gamma$ and $\xi_b \mathbb{S}(\gamma + 1) = \xi_c(\gamma + 1)$.

**Proof.** Exercise: notice that from the point where the branches start to diverge we have overlaps of the extenders, and the ordinals above the initial p.t. are not in the intersection of the ranges. Use this to analyze the situation.

This will give the equality $\text{Ext} =$ .

**Def.** For $\mathbb{N}$ that is $M_\omega$-like and so on, let

\[ \gamma^N S^N = (\text{Hull}^{M_\omega \times (\downarrow S^N)}) \mathbb{S} = S^N \cup S^N \mathbb{S} = S^N \mathbb{S} \mathbb{S}. \]

Let $H^N_S = \text{Hull}^{M_\omega \times (\downarrow S^N)}(\gamma^N \cup S^N).

So $H^N_S \cap S^N = \gamma^N_S$.

**Lemma.** If $\mathbb{N}$ is a maximal iteration on $\mathbb{N}$ and $b, c$ are cofinal non-chopping, $\mathbb{N}$ is $M_\omega$-like and $i_b(S) = i_c(S) = S$ then $i_b \cap H^N_S = i_c \cap H^N_S$.

Since $i_b \cap \gamma^N_S = i_c \cap \gamma^N_S$ by the last lemma and $i_b(\gamma) = i_c(\gamma) = S$ by hyp.,

Since $\text{max}(S) > \gamma^N_S$ and this means $\text{wtp}^{M_\omega}(\mathbb{H})$,

\[ \forall \mathbb{N} \in \text{arg} \{i_b \cap \text{arg} (i_c) \}. \]
Theorem. Assume $M_1$ exist and is fully iterable. Let $M_1^f \subseteq x$. Let $\mathcal{G}$ be Col($\omega_1, \omega_1$) $\omega$-generic over $L[x]$ where $\kappa$ is least inaccessible of $L(x)$.

Then

$$\text{HOD}(x, \mathcal{G}) = L(M_\kappa, \mathcal{M}_\kappa)$$

where

$M_\kappa$ is an iterate of $M$

$\mathcal{M}_\kappa \models \{ \delta \mid \delta$ on $M_\kappa$ and $\delta \in L_{\kappa+}(M_\kappa) \}$

Proof. $M_\kappa$ is direct limit of all $\xi$-iterates of $M_1$

via a finite stack of $\xi$ of normal trees

with $\xi \subseteq H.C.L(x, \mathcal{G})$ under the composition maps.

---

Working in $L(x, \mathcal{G})$

$I = \{(N, \xi) \mid N$ is $M_1$-like, $\kappa^+ \notin \text{Ord}_{\xi+}$

and $N$ is strongly $\xi$-iterable $\}$

Here: $N$ is $\xi$-iterable iff for any finite stack

$< T_0, \ldots, T_n, U >$ s.t. $T_0$ is on $N$, $T_i$ is maximal

for all $i$ or has a last model $M_i^f$ with no drops in the branch $[0, \omega]_{T_i}$ and

$T_{i+1}$ is on $L(M(T_i))$ if $T_i$ is maximal

and last model of $T_i$ otherwise.

Also: $U$ is a tree on $L(M(T_n))$ of $T_n$ maximal

and last model of $T_n$ if $T_n$ is short.
Then (1) If $U$ is short then $U$ has a cut $b$ s.t.
$Q(M(U)) \leq M^b$
(2) If $U$ is maximal then in $L[x_0, G] \text{Col}(\omega, \text{max}(s))$
there are branches $b_0, \ldots, b_n$ such that
and cofinal in $S_0, \ldots, S_n$, respectively.
$s.t. \ i_{b_k}(s) = 5$ for all $k \leq n+1$.

Rem. For any $s \in \omega$ there is a $\Sigma^1_1$-iterate of $M_i$
call of $N$, in $H \subseteq [x_0, G]$ s.t. $L[x_0, G] \models N$ is iterable.

Say that $N$ is strongly iterable iff $N$ is iterable
and for a stack $(T_0, \ldots, T_n)$ as above on $N$ and
$b_0, \ldots, b_n$, $c_0, \ldots, c_n$ in $L[x_0, G] \text{Col}(\omega, \text{max}(s))$

sequences witnessing $\tau$-iterability, we have
$i_{b_0} \cdots i_{b_n} \upharpoonright H^N = i_{c_0} \cdots i_{c_n} \upharpoonright H^N$

$\Sigma^1_1$ is an $\tau$-iterable strategy on $N$ as $N$ is a
$\Sigma^1_1$-iterate of $M_i$.

Exercise. Hint: $\Sigma^1_1$ has the Dodd-Jensen property.

Say $N(s, i) \models (M_i, t)$ for $(N(s), (M_i, t)) \in I$
iff there is a good stack in the last model
$M$ and $s \leq t$.

$(N(s))$ indexes $H^N_s$ in the direct limit system
$\pi_{(N, s), (M, t)} : H^N_s \to H^M_t$ and the common value

of all maps witnessing $\tau$-iterability of $N$

$\pi_{(N(s))(M, t)} = \pi_{(M, t)} \circ \pi_{(N(s))(R, U)}$
by strong
let \( \mathcal{F} \) be this direct limit system.

\[ M_\omega = \text{direct limit of } \mathcal{F} \]

Claim: \( \mathcal{F} \) is directed.

Proof:

\[ (N_\omega) \]

\[ (M_\omega) \longrightarrow (R_{\text{out}}) \eta \]

Notice: we can have maximal trees in this compareion, so may not have cofinal branches. If the tree is maximal, we are done.

\[ \mathcal{E}_{M_1} \in \mathcal{E} \]

\[ M_1 \longrightarrow (P_{\text{out}}) \]

\( R \) is strongly surjective. Notice \( R \in \text{HC} \).

as the outcome of the iteration \( \omega \) may be uncountable in \( L^x \).

Let \( M_\omega^+ = \) the direct limit of all \( \mathcal{E}_{M_1} \) clusters of \( M_\omega \) that are in \( \text{HC} \).

Then there is an embedding \( \pi: M_\omega \rightarrow M_\omega^+ \).

\( \pi \) is defined by a straightforward translation between the two direct limit systems.

If \( z \in M_\omega \), then \( z = (N_\omega) \in (P_{\text{out}}) \) with \( z \in H_\omega^+ \).

\[ (N_\omega) \rightarrow (R_\omega) \eta \]

\[ (P_\omega) \rightarrow (R_{\text{out}}) \eta \]

\[ \mathcal{E}_{M_1} \leftarrow \mathcal{E}_{M_1} \]

\[ M_\omega \rightarrow M_\omega^+ \]

Note: \( M_\omega \subseteq \mathcal{E}_{M_1} \).

We have \( \text{On} \leq M_\omega \) since \( \mathcal{U}(N_\omega) \geq (P_{\text{out}}) \geq (N_\omega) \).

Let \( S_\omega = S_{M_\omega} \), then one can show: \( S_{\omega} = S_{M_\omega}^+ \) and \( \eta \mathcal{M}(S_{\omega}^+) = \mathcal{A} \). So \( M_{\omega} = M_{\omega}^+ \). (\( \pi \neq \text{id} \) seems possible.)
Then $m_0 \equiv 0 \mod L_{[x,G]}$. Let

\[ m = \exists m \in I \text{ trees on } M_y \text{ that belong to } M_{\alpha} \]

where $n_0 = \text{the first inaccessible } > n_0$.

Claim: $m \in \text{PD} \cup L_{x,G}$
Under $AD^+ + \mathcal{V} = L(P(\mathcal{R}))$

\[ \mathcal{W}_S \cap P(\mathcal{R}) \subseteq \mathcal{V} \]

**Cases:**

1. $\mathcal{V}$ regular (3 longest Suslin and not in the list case; not sure if $\mathcal{V}$ and regular model)
2. No longest Suslin, $\mathcal{V} = \omega$
3. $\mathcal{V} = \omega$

We defined

\[ T = \{ \sigma = \langle \sigma_0, \ldots, \sigma_n \rangle \mid \sigma_i \in \mathcal{X}_i \text{ and} \]

\[ 1. \ \sigma_i(\mathcal{R}) = \sigma_i(\mathcal{K}) \cup \delta \sigma_i \]

\[ 2. \ \forall i \leq n : \sigma_i \subseteq \delta \sigma_i \text{ and} \]

\[ 3. \ \sigma_i \text{ countable in } HOD_{\sigma_i} \text{ for all } i < \omega \]

\[ 4. \ \text{Let } \Theta_i = \Theta^{HOD_{\sigma_i}}. \text{ Then} \]

\[ P(\Theta_i) \cap HOD_{\sigma_i(\mathcal{R})} = \Theta_1 \cap HOD_{\sigma_i(\mathcal{R})} \text{ and} \]

\[ HOD_{\sigma_i(\mathcal{R})} = \Theta_i \text{ is well-founded.} \]

**Lemma (Essentially Vopěnka):** Let $\mathcal{S} = \langle \sigma_0, \ldots, \sigma_n \rangle \in T$.

Then $\exists$ partial ordering $\mathcal{P}$ s.t.

1. $HOD_{\mathcal{S}} = \mathcal{P}$ is a c.b.a. of size $\Theta_n$
2. Let $\mathcal{K} = \sup \delta_n(A)$ where $A \in HOD_{\sigma_0, \ldots, \sigma_n}$ and

$\langle \mathcal{K}, \mathcal{W} \rangle$ is the predecessor of $\Theta_n$ on the Solovay sequence.