of \( \text{HOD}_\text{σνΩ} \).

Then there is a filter \( G_A \) on \( P \) s.t.
* \( G_A \) is \( \text{HOD}_S \)-generic
* \( \text{HOD}_{σνΩ} = \text{HOD}_S [ G_A ] \)

Proof. Let \( H = \text{HOD}_S \). Know: \( H = \text{HOD}_{σνΩ} \).

Working in \( \text{HOD}_{σνΩ} \).

Let \( P \) be the Vopěnká algebra for adding a subset of \( σνΩ \) to \( \text{HOD}_S \).

\( ( P, ≤ ) \models \forall c \in P ( c ∈ \text{OD}_H μ ) \).

\( G_A = \{ c ∈ P | A ∈ π(c) \} \)

(1), (2) are then standard facts about Vopěnká algebra. \( \square \)

Define

\( P \) = the set of all pairs \( ( s, F ) \) s.t.

- \( s ∈ T \) and \( F : T → V \)
- \( F(∅) = \mathcal{P}_ω (κ_0) \)
- \( \forall < s_0...s_n > ∈ T : μ_{n+1}(F(< s_0...s_n >)) = 1 \)

Ordering

\( < s_0, F_0 > ≤ < s_1, F_1 > \) iff

- \( s_0 ≥ s_1 \)
- \( F_0 ≤ F_1 \)
- \( ∀ c ∈ \text{dom}(s_0) - \text{dom}(s_1) : s_c(c) ∈ F_1(σ_{f(c)}) \)

Lemma (Prikry property) \( P \) has the Prikry property, i.e.

if \( z \) is a countable set of terms \( ( s_0, F_0 ) ∈ P \) and \( z \)

is a formula then there is some \( t \) s.t. \( ( s_0, t ) ∈ P \) and

\( ( s_0, H ) \models \forall t \exists t \) for all \( t ∈ z \).
Proof: Exercise: there is a DC-free proof of the lemma.

Now let \( G \) be \( \not\emptyset \)-quency \( V \). Let
\[
S_G = \{ \sigma \in (\mathcal{Q}F) \mid \exists \sigma', \sigma'' \in \sigma \, \text{s.t.} \, \sigma' \leq \sigma'' \} = \langle \sigma, 1 \leq \sigma \rangle
\]
We will use the Prikry property to show:

\[
\text{Lemma } (\forall \xi < \omega) \quad \mathcal{P}(\xi) \cap \text{HOD}^V_{S_G(\xi+1)} = \mathcal{P}(\xi) \cap \text{HOD}^V_{S_G(\xi+1)}
\]

Rem. If lemma holds: then
\[
\text{HOD}^V_{S_G(\xi+1)} = \exists \Gamma \text{ infinitely many Wadding cardinal}
\]
This is because \( (\forall \xi < \omega) \, \text{HOD}^V_{S_G(\xi+1)} = \emptyset \) is Wadding.

Proof of the lemma

\( \leq \) Holds because we use \( V \) as a predicate

\( \geq \) If not: There are:

- Formula \( \phi(x_1, x_2, x_3) \),
- \( \exists \sigma \in \mathcal{O} \)
- \( \sigma > \xi \)
  
\( \sigma \cdot (S_G \cap F) \in G \)
  
\( \sigma \cdot (S_G \cap F) \in G \)

By Prikry property: there are densely many conditions of the form \( (S_G \cap F, H) \) that decide the statement

\[
(\forall \xi) (S_G(\xi+1) \in \text{HOD}^V_{S_G(\xi+1)})
\]

so \( \sigma \cdot (S_G \cap F) \in G \). This mean:

\[
(\forall \xi) (S_G(\xi+1) \in \text{HOD}^V_{S_G(\xi+1)})
\]

But then this set is in \( \text{HOD}^V_{S_G(\xi+1)} \) by our above arrangements. Contradiction.
Let \( N = \text{HOD}^{(\mathcal{U}, \mathcal{V})} \), where \( G \) is \( \mathcal{R} \)-generic \( (\mathcal{V}, \mathcal{U}) \), and \( s_0 = (\mathcal{V}, \mathcal{U}) \). So \( N \models \mathcal{E} \text{C} \) and \( \omega^N_1 = \sup\theta_i \).

**Lemma.** \( \mathcal{V} \) is a derived model of \( N \). More precisely:

There is a \( \mathcal{G} \)-generated \( (\mathcal{W}, \mathcal{U}) \)-generic \( / N \) filter \( \mathcal{K} \) s.t.

\( \mathcal{V} = L^V(\text{Ham}^{\mathcal{K}}_K, \mathcal{U}_K) \).

**Proof.** Let \( N_i = \text{HOD}^{(\mathcal{V}, \mathcal{U})} \), \( \theta_i = \text{HOD}^{(\mathcal{V}, \mathcal{U})} \).

We know:

\( \theta(\theta_i) \cap N_i = \theta(\theta_i) \cap N_i = \theta(\theta_i) \cap N \) for \( j \geq i \).

In \( \mathcal{V} \) there is a filter \( \mathcal{K} \) that is \( \mathcal{G} \)-generated \( / N \).

\( \mathcal{V} = \mathcal{L}(\text{Ham}^{K}) \). This is because each \( x \in \mathcal{V} \) s.t. \( s_0(x) \) then \( x \) can be absorded by a Vopěnka algebra of size \( \omega_1^N \), namely a Vopěnka algebra for \( N \).

Now to see that \( \mathcal{G}(\mathcal{V}) = \text{Ham}^{K} \). Enough to see \( \mathcal{G}(\mathcal{V}) \subseteq \text{Ham}^{K} \) otherwise we get a sharp for \( \mathcal{V} \) in the generic extension of \( \mathcal{V} \).

Let \( B = \mathcal{G}(\mathcal{V}) \). \( B \) is Suslin co-Suslin.

Markin's theorem \( (\mathcal{A}D + \mathcal{D} \mathcal{K}) \) \( B \) is homogeneously Suslin. Then we can code the homogeneity system \( \mathcal{A}D + \mathcal{D} \mathcal{K} \) by the \( \omega_1 \)-adical a countable sequence of ordinala that is bounded below \( \theta \).

(\text{Measures are OD by Kunen.}) So we can get trees \( T_i \) s.t. \( p(r) = B = \mathcal{U}(p(T)) \) and \( T_i \) are OD from that sequence. Now since \( f \) is bounded below \( \theta \), there is \( i \) s.t. \( s_0(i) \geq f \) and \( s_0(i) \cap \Theta \cap \mathcal{N} \).

There is a \( \mathcal{G} \)-generic over \( \mathcal{N}_{i+1} \) s.t. the collapse of \( f \) is in \( \mathcal{N}_{i+1} \). Since the corresponding collapse \( \mathcal{T} \in \mathcal{E} \mathcal{N}_{i+1} \).

We have \( f \in \mathcal{N}_{i} \). So for all \( j \geq i \) \( \mathcal{N}_{i} \cap \mathcal{E} \mathcal{N}_{j} \) can decode \( f \) to recover the trees \( T \) and \( U_i \). So

\( p(CT_{\mathcal{N}} \cap \mathcal{E} \mathcal{N}_{i+1} = B \cap \mathcal{U}(p(T)) \cap \mathcal{N}_{i+1} \).

This shows that \( \mathcal{H}_{\mathcal{V}} \).
Next goal: let \( \varphi \) be a \( \Sigma_1 \)-formula

let \( \psi \) be a \( \Sigma_1 \)-formula and \( \psi \in \varphi(\mathbb{R}) \). WTS: \( \varphi(\mathbb{R}) \) \( \subseteq \varphi(\mathbb{R}) \).

**Lemma** There is \( A \in \text{Hom}_{\omega_1}^\mathbb{R} \) s.t. \( L(A, \mathbb{R}^A) \models \varphi(\mathbb{R}) \).

**Proof** Let \( \gamma \) be least s.t. \( L_\gamma(\mathbb{R}(\mathbb{R})) \models \varphi(\mathbb{R}) \) and there is a sequence \( \langle \alpha_i \mid i < \omega_1 \rangle \) s.t. \( \Theta = \sup \Theta_i \) and \( \langle \alpha_i \mid i < \omega_1 \rangle \) is definable in \( L_\gamma(\mathbb{R}(\mathbb{R})) \) from a set of reals and no ordinal parameters. Let \( j: (\mathbb{N}, \epsilon) \to (\mathbb{M}, \in) \) be a stationary tower map induced by a \( R_{\omega_1}^\mathbb{N} \)-generic \( \mathcal{G}/\mathcal{N} \). We have:

1. \( \omega(j) = \omega_1^\mathbb{N} \) and \( j(\omega_1^\mathbb{N}) = \omega_1^\mathbb{M} \)
2. \( L(\mathbb{M}, \epsilon) = \mathbb{R} \)
3. \( j(\text{Hom}_{\omega_1}^\mathbb{N}) \supseteq \mathbb{R}(\mathbb{R}) \)
4. \( j(A) = A^* \) for all \( A \in \text{Hom}_{\omega_1}^\mathbb{N} \)
5. \( \gamma \in \text{wp}(\mathbb{M}, \epsilon) \)

**Case 1.** Suppose \( j(\text{Hom}_{\omega_1}^\mathbb{N}) \not\supseteq \mathbb{R}(\mathbb{R}) \). So there is some \( A \in j(\text{Hom}_{\omega_1}^\mathbb{N}) \) \( \not\in \mathbb{R}(\mathbb{R}) \). Since \( \mathcal{M}, \epsilon) \models (L_\gamma(A, \mathbb{R}(\mathbb{R})) \models \varphi(\mathbb{R}) \) (because \( \mathcal{M}, \epsilon) \models \varphi(\mathbb{R}) \)) by elementaryness of \( j \) we have \( A \in \text{Hom}_{\omega_1}^\mathbb{N} \) s.t. \( L(A, \mathbb{R}(\mathbb{R})) \models \varphi(\mathbb{R}) \).

**Case 2.** \( j(\text{Hom}_{\omega_1}^\mathbb{N}) \supseteq \mathbb{R}(\mathbb{R}) \).

We can pick \( \gamma \) s.t. \( \gamma \not\in j(\text{Hom}_{\omega_1}^\mathbb{N}) \). Then \( L_\gamma(\mathbb{R}(\mathbb{R})) \not\models \varphi(j) \).

Hence there is some sequence \( \langle \alpha_i \mid i < \omega_1 \rangle \) s.t. \( \Theta = \sup \Theta_i \). Why? We know such a sequence is definable in some \( B \supseteq \mathbb{R}(\mathbb{R}) \) without ordinals.

Now \( B = C^* \) for some \( C \in \text{NC}_{\varphi}(\varphi) \) where \( \varphi \) is \( \omega_1^\mathbb{R} \)-generic over \( \mathbb{N} \). By replacing \( \varphi \) by \( \varphi(\mathcal{G}) \) if necessary, we can assume \( \mathcal{G} \subseteq \mathbb{N} \) and \( C^* = B \). So \( B = C^* = j(C) \models \varphi(j) \)

hence \( \langle \alpha_i \mid i < \omega_1 \rangle \) \( \models \varphi(j) \). Say \( j(\langle \alpha_i \mid i < \omega_1 \rangle) = \langle \alpha_i \mid i < \omega_1 \rangle \).

From \( \langle \beta_i \mid i < \omega_1 \rangle \) we choose a sequence \( \langle B_i \mid i < \omega_1 \rangle \) cofinal in \( \text{Hom}_{\omega_1}^\mathbb{N} \). This is a contradiction.
as we can code \( \text{Def}(\text{Hom}_{\text{w}_1}) \) by a \( D \in \text{Hom}_{\text{w}_1} \),
\( B_i \in D \) all \( i \). But \( \langle B_i : i \in \text{Ord} \rangle \) is cofinal in \( \text{Hom}_{\text{w}_1} \).

Now since \( L(A, \text{Ord}) \models \text{ZF(Ord)} \) and \( j \rightarrow L(A^j, \text{Ord}) \models \text{ZF(Ord)} \),
here \( A^j \in \text{Ord}^2 \).

So \( M_{\text{Ord}} \models \text{ZF(Ord)} \).

**CASE 3** No largest Suslin cardinal \( \omega + \Theta \) singular.

Since \( \text{cf}(\Theta) > \omega \) we have DC by Solovay.

Since every regular \( < \Theta \) is measurable, let \( \mu \) be a measure on \( \text{rng}(\text{cf}(\Theta)) \) cofinal increasing. For each \( \alpha \in \Theta \), \( \text{cf}(\alpha) = \omega \) let

\[ I_\alpha = \{ A \in \Theta_\alpha \mid \text{sup}(A) < \Theta_\alpha \} \]

\[ \Rightarrow \{ \neg \text{HOD}^{I_\alpha}_{\text{Ord}} = \text{AD}^+ + \text{AD}_\text{R}^+ \}
\[ \Theta_\alpha = \text{\neg HOD}^{I_\alpha}_{\text{Ord}} \]

\[ \forall X \in \text{HOD}^{-1}_{I_\alpha} : \text{HOD}^{-1}_{I_\alpha} \cap \text{Ord} = \text{limit of Woodin in HOD}^{-1}_{I_\alpha} \]

Our \( N \) will be a ZFC model set.

\( \omega^{\aleph_1} = \text{limit of limits of Woodins in } N \)

Let \( \mu \) be a supercompact measure on \( \text{P}_{\text{w}_1}(I_\alpha) \).

**Lemma** For each \( \alpha \) s.t. \( \text{cf}(\alpha) = \omega \), \( \Theta_\alpha \) there is

\[ \sigma \in \text{P}_{\text{w}_1}(I_\alpha) \text{ s.t.} \]

\[ \text{HOD}_{\text{P}_{\text{w}_1}(I_\alpha)} = \text{AD}^+ + \text{AD}_\text{R}^+ \]

\[ \sigma \text{ has transitive collapse } = \{ A \in \Theta \mid \text{sup} A < \Theta \} \]

as computed in \( \text{HOD}_{\text{P}_{\text{w}_1}(I_\alpha)} \).
Define

\( T_0 = \{ \tau \subseteq \Lambda \delta \mid \text{all } \tau_i \leq \mu : \}

\begin{align*}
&\bullet \sum_{\omega \in \tau_i} \left( \text{cf}(\delta_i) = \omega \land \Theta_i \subseteq \Theta \right) \\
&\bullet \Theta_i \subseteq \sup \{ \delta_i \mid \gamma \in \delta_i \} \\
&\bullet \delta_i \in \mathcal{P}_\omega (I_{\delta_i}) \\
&\bullet \text{HOD}_{\delta_i} (\omega_{\delta_i}) = \text{AD}^+ + \text{AD}_{\omega_{\delta_i}}^\omega \\
&\bullet \delta_i \text{ collapses to } \{ \alpha \subseteq \Theta \mid \sup (\alpha) < \Theta \} \text{ in } \text{HOD}_{\delta_i} (\omega_{\delta_i})
\end{align*}

\( T = \{ \text{the set of all } s = \langle \delta_0, \ldots, \delta_n \rangle \text{ s.t.} \}

\begin{align*}
&\bullet s \subseteq T_0 \\
&\bullet \mathcal{P}(\mathcal{R}) \text{ HOD}_s = \mathcal{P}(\mathcal{R}) \text{ HOD} \\
&\bullet (\forall c \leq \mu)
\begin{align*}
&\bullet \alpha_i < \alpha_{i+1} \\
&\bullet \delta_i \subseteq \delta_i^+ \text{ and } \delta_i \in \text{HOD}_{\delta_i} (\omega_{\delta_i}) \text{ for all } \lambda \leq \delta_i \\
&\bullet \text{HOD} (\omega_{\delta_i}) \text{ countable in } \text{HOD}_{\delta_i} (\omega_{\delta_i}) \text{ for all } \lambda < \delta_i \\
&\bullet \mathcal{P}(\Theta_i) \otimes \text{HOD}_{\delta_i} (\omega_{\delta_i}) = \mathcal{P}(\Theta_i) \otimes \text{HOD}_s \\
&\text{where } \Theta_i = \Theta \cap \text{HOD}_{\delta_i} (\omega_{\delta_i})
\end{align*}
\end{align*}

Now define Birkhoff forcing:

\( \mathcal{P} = \{ \text{the set of all pairs } \langle s, F \rangle \text{ such that set}, \)

\( F : T \to V \text{ and } \)

\( (\forall t \in T) \ t^* \langle s \rangle \in T \text{ for all } s \in F(t) \text{ and } \)

\( \forall t. \forall \gamma. \exists F \subseteq \omega_{\delta_i} \text{ such that } \gamma \in F(t) \)

Ordering:

\( \langle s_0, F_0 \rangle \leq \langle s_1, F_1 \rangle \) \iff \( s_0 \supseteq s_1 \) and

\( \langle \forall i \in \text{dom}(s_0) \setminus \text{dom}(s_1), s_0(i) \in F_1 (s_1(i)) \rangle \)

\( F_0 \subseteq F_1 \)
The largest Suslin cardinal.

Assume there is a largest Suslin cardinal \( \kappa \).

**Claim.** \( \kappa \) is a regular limit cardinal.
- \( \kappa = S(\kappa) \quad \text{and} \quad \text{Scale}(\kappa) \)
- \( S(\kappa) \) is closed under quantifiers.

The Envelope

Let \( \Gamma \) be a pointclass and \( \kappa \in \text{On} \). We define \( \text{\textunderbar{\Gamma}, \kappa}-\text{envelope} \) as follows.

**Definition (Martin)** Let \( \Delta = (A_x : x \in R) \) each \( A_x \subseteq R \).
Then \( \text{\textunderbar{\Delta}} \) is the set of all \( A \in \theta(\Gamma) \) such that for all countable \( S \subseteq R \) there is a \( z < \kappa \) s.t. \( S \cap A = S \cap A_z \).

We let
\[
\Lambda(\Gamma, \kappa) = \{ \text{\textunderbar{\Delta}} | A \subseteq \Gamma \text{ and } \text{card}(A) \leq \kappa \}
\]

**Lemma.** Let \( \Gamma \) be nonselfdual, closed under \( \forall R \) and \( \rho \alpha \text{ (if } \Delta \text{ not closed under } \exists \text{ assume scale } (3^R \Delta \text{ with monus } \kappa \text{ of either}) \). Then
\[
\Lambda(\Delta, \kappa) = \Lambda(\Gamma, \kappa) = \Lambda(3^R \Gamma, \kappa) \text{ where } \kappa = S(\Delta).
\]

**Lemma.** Assume assumptions of the previous lemma. Then there is a single \( \Delta = (A_x : x \in R) \) with each \( A_x \subseteq \Delta \) s.t. every set in \( \Lambda(\Gamma, \kappa) \) is Wadge reducible to a set in \( \text{\textunderbar{\Delta}} \).
**Corollary** Under same hypotheses: \( \Lambda(G, n) \) is closed under \( \wedge, \vee, \neg \).

Why: \( \Lambda \) because \( \Lambda(G, n) = \Lambda(\Delta, n) \).

**Lemma** Suppose \( \Gamma \) is nonselfdual closed under \( \forall \), \( \forall^* \), and \( \neg \cdot \neg \) (\( \Gamma \)). Let \( n = o(\Delta) \). Then \( \Lambda(G, n) \) is closed under \( \forall^*, \forall^* \).

**Coding measures**

Let \( m \), \( n = o(\Delta) \) be as above. Fix an \( \forall^* \) norm \( (W, \| \cdot \|) \) of length \( k \) (with each \( W_i \in \Delta \)).

Let \( U \) be a universal \( \forall^* \) set. For \( x \in W \) let \( B_x = \{ f < n | (\exists x \in W) (\forall (n) = f) \} \). By the Coding Lemma every subset of \( n \) is of the form \( B_x \). For a measure \( \mu \)

on \( n \):

\[
C_\mu = \{ z | \mu(B_z) = 1 \}
\]

**Lemma** \( \Gamma \) as above. For \( A \in \Delta(G, n) \) iff there is a measure \( \mu \) on \( n \) s.t. \( A \leq \mu C_\mu \).

**Upper bound for the next semiscale**

**Theorem** \( \Gamma \) nonselfdual, closed under \( \forall^* \) and \( \neg \cdot \neg \) (\( \Gamma \)). Assume every \( \forall^* \) set admits a \( \forall^* \) scale with norm \( n \leq \Sigma(\Delta) \). Assume also that there is a Suslin cardinal greater than \( n \). Then every set in \( \forall^* \) admits a semiscale \( m \leq n \).
Remark. It is not clear if we can get a scale whose
members are in \( \mathcal{A}(\mathbb{R}, \kappa) \):

**Question.** Can we find a homogeneous tree \( T \) on
\( \mathbb{R} \times \kappa \) a countable family \( \mathcal{A}_\kappa \) of \( \mu_\kappa \) of measure \( \kappa \).

\( \forall x \left( \text{there is a } \nu \text{ s.t. } \mathcal{A}_\kappa \subseteq \mathcal{A}_\nu \right) \Rightarrow \left[ \begin{array}{c}
\left( \mathcal{A}_\nu \right)_{\mu_\kappa} = \text{leftmost branch of scale}
\\
\left( \mathcal{A}_\nu \right)_{\mu_\kappa}^\prime
\end{array} \right] 
\)

Lower bound for the next scale

**Lemma.** If non-selfdual, closed under \( \mathcal{A}(\mathbb{R}) \) and \( \text{proj}(\mathbb{R}) \)
let \( A \) be \( \mathcal{A}(\mathbb{R}) \)-complete. Then \( A \) does not admit
a scale all of whose members are \( \kappa \)-reducible to some \( B \in \mathcal{A}(\mathbb{R}, \kappa) \).

Proof. Idea. This is the "largest countable \( \mathcal{A} \)" argument.

Remark. A semiscale can be converted to a scale within
the next projective class

**Lemma.** Suppose \( \mathcal{A} \) is non-selfdual, closed under
quantifiers and scale \( \mathcal{A}(\mathbb{R}) \) (and \( \kappa \leq \Theta(A) \) is not
the largest Suslin cardinal). Then every set in \( \mathcal{A}(\mathbb{R}, \kappa) \)
is \( \kappa \)-Suslin.

Assume \( \mathcal{A}(\mathbb{R}, \kappa) \) is closed under quantifiers, \( \kappa \leq \Theta(A) \)
is not the largest Suslin cardinal and \( \mathcal{A} = \mathcal{A}(\mathbb{R}, \kappa) \).

Let \( \mu = \Theta(A) \). So cf(\( \mu \)) = \( \omega \).

Let \( \mathcal{E}_0 = \bigcup_{\gamma < \mu} S(\kappa, \gamma) \).

Recall \( \text{proj}(\mathcal{E}_0), \text{proj}(\mathcal{R}) \)

eca.
Lemma \( S(x) = E_2 \).

Lemma \( S_\tau = S_1(A) \cap \mathcal{X}^+ \) and \( \mathcal{X}^+ \) is regular.

Lemma Let \( B \in \mathcal{A}_1, \rho < \sigma \) and \( B = \{ B_\beta \mid \beta < \rho \} \) be s.t. \( B_\beta \subseteq \omega \) \( B \) for each \( \beta \). Then \( B \in \mathcal{A}_1 \).
Continuation of the lecture in the morning.

IP has the Priery property:
Let \( G \) be \( V \)-generic for \( P \), \( s_G = U \{ s \mid (s,F) \in G \} \),
\( N = \text{HOD}^{(vca_7,v)} \models ZFC + \omega_I = \text{limit of limit of Woodins} \).
Let \( s_0 = \langle s_0 | i \in \omega \rangle \).

**Lemma (a) (VCA,\( \zeta \))** \( \mathcal{P}(\omega_I) \cap \text{HOD}^V_{s_G(\zeta + 1)} = \mathcal{P}(\omega_I) \cap \text{HOD}^V_{s_G(\zeta + 1)} \)
where \( \zeta = \text{HOD}^V_{\kappa_I(\omega_I)} \)

(b) Upenope holds. Assume \( A \subseteq \omega_I \) bounded.

Fix \( \theta_I < \博弈 \) in \( \text{HOD}^V_{s_G(\omega_I)} \) and
\( \text{HOD}^V_{s_G(\omega_I), A} = \text{HOD}^V_{s_G(\omega_I), [G_A]} \).

(c) \( \theta_I \) is a limit of Woodins in \( \text{HOD}^{(vca_7,v)}_{s_0,1} \)

To show (a): Note that \( \text{HOD}^V_{s_G(\omega_I), A} \models A \Delta \rho \) so \( \text{HOD}^V_{s_G(\omega_I), A} \models \theta_I \) is a limit of Woodins

Fix \( G \) \( V \)-generic \( / V \) and \( s_G = \langle s_0 | i \in \omega \rangle \).

(2) \( s_0 \in \text{HOD}^{(vca_7,v)} \models ZFC + \omega_I = \text{limit of limit of Woodins} \)
and \( V = D(\text{HOD}^{(vca_7,v)} \mathcal{V}, \omega_I) \)

**Def (Woodin)** Assume \( J \) is a limit of Woodins.

\( \text{Hom}^{<\omega} \) is weakly closed if the following holds.

1) If \( \theta_I < J \) is Woodin and \( G \subseteq \theta_I \) is generic \( / V \)

Let \( j: \text{Ult}(V(G)) \rightarrow \text{Ult}(V(G)) \) be the generic map, then \( j(\text{Hom}^{<\omega} J) = \text{Hom}^{<\omega} J \)
2) \( \forall \delta \) holds in VTR for any \( \delta \) that is \(<\delta\text{-generic} >\).

**Lemma**  Exactly one of the following holds:

1. \( \exists x \in \mathbb{R} \text{ s.t. } A \in \text{Hom}_{\mathcal{N}^{<\omega_1}} \text{ s.t. } L(A, \mathcal{R}^{\mathcal{V}}) \models \varphi^{\mathcal{V}} \)

   (we are assuming \( V \models \varphi^{\mathcal{V}} \text{ when } \varphi \text{ is } \Sigma_1 \))

2. \( \text{Hom}^{<\omega_1}_v \text{ is weakly sealed} \).

**Proof** Assume \( V \models \varphi^{\mathcal{V}} \text{ where } \varphi \text{ is } \Sigma_1 \). Let \( A \) be large enough s.t. \( L(A, \mathcal{R}^{\mathcal{V}}) \models \varphi^{\mathcal{V}} \). For \( x \in \mathbb{R}^{\mathcal{V}} \) let

\( j_x : (\mathcal{N}^{<\omega_1}, \in) \rightarrow (M_x, \mathcal{E}_x) \) induced by a \( \varphi^{\mathcal{V}} \text{-generic} \) s.t.

1. \( \mathcal{U}(j_x) = \omega^\mathcal{V} \text{ and } j_x(\omega^\mathcal{V}) = \omega^*_1 \)

2. \( \mathcal{V}(M_x, \mathcal{E}_x) = \mathcal{V}^{\mathcal{V}} \)

3. \( \text{Hom}^{<\omega_1}_v \subseteq j_x \left( \text{Hom}_{\mathcal{N}^{<\omega_1}} \right) \)

4. \( \forall A \in \text{Hom}^{<\omega_1}_v \exists j_x(A) = A^x \)

5. For every successor Woodin cardinal \( \kappa < \omega_1^\mathcal{V} \) in \( \mathcal{N}^{<\omega_1} \) there is an \( \mathcal{N}^{<\omega_1} \text{-generic} \) \( H \in \mathcal{P}^{\mathcal{V}} \text{ inducing} \)

\( j_H : \mathcal{N}^{\mathcal{V}} \rightarrow \text{Ult}(\mathcal{N}^{\mathcal{V}}, H) \) and

\( k_H : \text{Ult}(\mathcal{N}^{\mathcal{V}}, H) \rightarrow (M_x, \mathcal{E}_x) \) so that

\( j_x = k_H \circ j_H \)

**Case 1** \( P(\mathbb{R}^{\mathcal{V}}) \subseteq j_x \left( \text{Hom}^{\mathcal{N}^{<\omega_1}} \right) \) for some \( x \in \mathbb{R}^{\mathcal{V}} \).

Already done.

**Case 2** \( P(\mathbb{R}^{\mathcal{V}}) = j_x \left( \text{Hom}^{\mathcal{N}^{<\omega_1}} \right) \) all \( x \in \mathbb{R}^{\mathcal{V}} \).

We have:

\( j_x \left( \text{Hom}_{\mathcal{N}^{<\omega_1}} \right) = \text{Hom}_{\mathcal{N}^{<\omega_1}} \text{ (takes a little)} \)

(Note: We don't get weakly sealed this way as \( \mathcal{E}_x \text{ are not weakly homogeneous} \)

(2) Holds by varying the embedding \( j_x \) to include any given condition.
This gives (1) in the statement of the Main Lemma. 

Now: if (2) holds then 

\[
\text{Lemma } \text{Hom}^N_{\omega_1} = L(\text{Hom}^N_{\omega_1}) \cap \text{PCR}
\]

Assuming this lemma: Then \( L(\text{Hom}^N_{\omega_1}) \) is a counterexample to the theorem on the sense that \( L(\text{Hom}^N_{\omega_1}) \models \text{AD}^+ \cup \text{C}^\dagger \text{[R]} \) but for no \( A \in \text{Hom}^N_{\omega_1} \) 
\[
L(A, \text{[R]}) \models \text{C}^\dagger \text{[R]} \text{. } & \text{in } L(\text{Hom}^N_{\omega_1}) \text{, } \omega_1 \text{ does not exist .}
\]

By repeating this we get an infinite descending sequence of ordinals.

**Proof of the lemma**

**Sublemma** \( \text{if } \text{Pr} \text{ in } \text{V}^N_{\omega_1}, \text{ G O P generic in } N \text{ then} \)

\( \text{in } N[G] \) there is an elementary embedding \( \mathcal{E} \) \( \text{so that } \)

\( \mathcal{E} : L(Hom^N_{\omega_1}) \rightarrow L(Hom^N_{\omega_1}[G]) \)

s.t.

\( \mathcal{E}(Hom^N_{\omega_1}) = Hom^N_{\omega_1}[G] \)

Assuming the sublemma, we prove now the lemma:

If the lemma fails, let \( \xi \) be least s.t.

\( \text{Hom}^N_{\omega_1} \nsubseteq L(\text{Hom}^N_{\omega_1}) \cap \text{PCR} \)

Take \( A \) is definable without ordinal parameters s.t. \( \forall \Phi \in L(\text{Hom}^N_{\omega_1}) \cap \text{PCR} \) 

Then use the tree production lemma.

The hypothesis of the TPL holds for \( \Phi \). We get \( A \in \text{Hom}^N_{\omega_1} \).

**Proof of Sub lemma**

Let \( \kappa \in \omega_1^N \) is a limit of Woodin in \( N \)

\( \nu \text{ is } \omega \text{ e.t. } \text{in } N \)

Then \( \text{let } \nu < \kappa \text{ be a Woodin.} \)
Find $\mathcal{G}_\omega \subseteq \mathcal{P}_\omega^{<\omega}$ that is generic over $\mathcal{N}$ s.t.
$G_i = \mathcal{G}_\omega \cap \mathcal{P}_\omega^{<\omega}$ is $\mathcal{N}$-generic for $\mathcal{P}_\omega^{<\omega_i}$. Let
$s_i = \cap i R_{\mathcal{N},\omega}^{<\omega_i}$. There are
$j_i : N \rightarrow \mathcal{M}_i$ be the generic embeddings.
Let $\mathcal{M}_*^i$ be the direct limit. $\mathcal{M}_i$ is embeddable
into $\mathcal{M}_\omega$, hence well-founded. We have
$j_i(\text{Hom}_{\mathcal{N}_\omega}^{<\omega_i}) = \text{Hom}_{\mathcal{N}_\omega}^{<\omega_i}.$

Let $j : N \rightarrow \mathcal{M}_*$ be the direct limit map.
We get $j^*(\text{Hom}_{\mathcal{N}_\omega}^{<\omega_i}) = \text{Hom}_{\mathcal{N}}^{\mathcal{N}(\omega)}$.

Let $N[G]^{(\omega)}$ be the symmetric extension of $N[G]^{(\omega)}$
for Col$(\omega_1 < \omega)$ s.t. $N(\omega) = N[G]^{(\omega)}$. We have
$j^* : L(\text{Hom}_{\mathcal{N}_\omega}^{<\omega_i}) \rightarrow L(\text{Hom}_{\mathcal{N}[G]^{(\omega)}}^{<\omega_i})$ and
$j^*(\text{Hom}_{\mathcal{N}_\omega}^{<\omega_i}) = \text{Hom}_{\mathcal{N}[G]^{(\omega)}}^{<\omega_i}$

Also: $j^* : L(\text{Hom}_{\mathcal{N}[G]^{(\omega)}}^{<\omega_i}) \rightarrow L(\text{Hom}_{\mathcal{N}(\omega)}^{<\omega_i})$
and
$j^*(\text{Hom}_{\mathcal{N}[G]^{(\omega)}}^{<\omega_i}) = \text{Hom}_{\mathcal{N}(\omega)}^{<\omega_i}$. (for some different $j^*$)

Now we fix points + use trees to show that the
two maps move sets of reals correctly. Then
this can be used to embed $L(\text{Hom}_{\mathcal{N}_\omega}^{<\omega_i}) \rightarrow L(\text{Hom}_{\mathcal{N}_\omega}^{<\omega_i})$.
We assumed $\nu^* \leq \sigma_1$.

$G$ generic over $L[x]$, for $\mathcal{C}_0$ ($w_1, c_1$), $\nu^* = \text{1st inacc of } L[x]$.

In $L[x, G]$ defined a DLS $T$:

Indizes: $(N_i, s)$ where $N \in M_\mu$-like, $\delta^N < w_1$, $s \in c_0$.

$N$ is strongly $s$-iterable:

Given a good stack $(T_0, \ldots, T_n)$ on $N$ (Each $T_i$ maximal on else has a last model without dropping on the main branch.) $T_{i+1}$ as the last model of $T_i$ or as $L(M(T_i))$ (if maximal).

Let $T_i$ be on $N_i$. We demand that there are $b_0 \ldots b_n$ s.t.

\[ \nu_e (\text{Type } (s^- \cup S_{i_k}^{T_{i_k}})_{i_k=1}^{N_{i_k}}(s)) = \nu_e (\text{Type } (s^- \cup S_{i_k}^{T_{i_k}})_{i_k=1}^{N_{i_k}}(s)) \]

We then define strongly $s$-iterable as before.

---

Need this notion in order to get absoluteness.

$(N_i, s)$ indexes $H^N \equiv \text{Hull } N_{\text{max}}(s)$.

$(N_i, s) \leq^* (P, t)$ iff there is a good stack on $N$ with last model $P$ and $s \leq t$.

\[ \pi (N_i, s)(P, t) = b_0 \ldots b_n \mid H^N \]

for any such good stack.

$M_\nu = \text{dir lim } M_{\nu^+}$

$M_{\nu^+} = \text{dir lim of all iterates of } M_\mu$ by its canonical strategy $\Sigma M_\mu \in HC$.

We have $\pi: M_\nu \rightarrow M_{\nu^+}$ and $\pi \upharpoonright (\Sigma M_{\nu^+}) = \Omega$.

For any $s \in M_\nu$ we let $s^k = \pi (N_i, s)$, then the map $s \rightarrow s^k$ is OI in $L[x, G]$.

Claim: $s_\omega = \nu^+(L[x, G])$ ($= L[w_2]$, $\in L[x, G]$).
Proof \( S_\omega \leq n^+ L[\kappa, G] \): Take \( \gamma < S_\omega \). Say

\( \pi_1(N, \kappa, \gamma) = \xi \) from \( \gamma < \kappa^+ \). The DLS of all \((N_\gamma, \kappa)\)

s.t. \((N, \kappa) \leq (N_\gamma, \kappa)\) gives us a map from \( H \in L[\kappa, G] \) onto

sup \( \pi_1(N, \kappa, \gamma) \) in \( L[\kappa, G] \).

To see \( n^+ L[\kappa, G] \leq S_\omega \). Pick \( \alpha < n^+ L[\kappa, G] = n^+ L[\kappa] \). Let

\( s = \langle \kappa^\beta \rangle \) and \( \alpha \) a term s.t. \( (\gamma, \alpha) \in \beta \) and \( \beta < \alpha \).

Let \( \eta < \alpha \). Have \( \eta = \min \{ \max(s), \kappa^\beta, \beta, \alpha \} \) s.t. \( \eta < \alpha \).

Let \( N \) be \( \xi_{\beta+1} \)-iterable s.t. \( \beta < \kappa^\eta \) measurable in \( \kappa_1 \),

and \( x \) being \( \beta_{\kappa^\eta} \)-generic \( \in N \) (Extender algebra)

\( \pi_1(N_\gamma, \kappa, \gamma) \leq \kappa \leq \kappa_1 \).

Let \( \pi_1(N_\gamma, \kappa, \gamma) \leq \eta < \alpha \).

Note \( \ot_p(\pi_1(N_\gamma, \kappa, \gamma)) \leq \delta^\gamma \) (\( \delta^\gamma \).

\( \pi_1(N_\gamma, \kappa, \gamma) \leq \eta < \alpha \).

\( \o_p(\pi_1(N_\gamma, \kappa, \gamma)) \leq \delta^\gamma \).

Let \( \gamma^\gamma = \pi_1(N_\gamma, \kappa, \gamma) \).

Show \( (\alpha) \gamma^\gamma \) does not depend on \((N_\gamma, \kappa)\).

(\( \beta \)) \( \gamma < \beta < \alpha \implies \gamma^\gamma = \gamma^\beta \).

Proof: Exercise.

\( \text{Claim: } L_{\kappa^\gamma} \in \mathcal{H} \).

Proof: Given \( \eta \) normal on \( M_\alpha \), every \( \Gamma \) short:

\( \text{let } \lambda_{\eta} = \sum_{\chi \in \lambda} \chi \text{ is free in } M_\alpha \mid M_\alpha \).

\( \text{Let } \lambda_{\eta} = \kappa^* \text{ the least inacc } > \delta_{\eta} \text{ of } M_\alpha \).

Claim: \( \lambda_{\eta} \in \mathcal{H} \).

Proof: Given \( \eta \) normal on \( M_\alpha \), every \( \Gamma \) short:
If $T$ short: $M \models (T)$ is the unique $b$ s.t. $Q(T) \subseteq M^b$.
If $T$ maximal: Note for $s \in On^{\omega}$

$M_b \models \forall \theta s^{**}$-iterable for good stacks in $M_b[\kappa_{b^*}]

Why: Pick $(N,s') \in F(s.t. x \in \mathbb{R}_{\omega}^n - \text{generic } N.)$ $s' = s$

Then $N[N_{s'} \models HC(L_{\omega_1^C})]$. So

$N[\text{max}(s')] \models I$ am $s$-iterable

$N[s'] \models \text{same}$, $(N_{s'}, \omega) : (N, \omega) \rightarrow M_{b^*}$. So
$M_{b^*} / s^{**} \models I$ am $s^*$-iterable.

More precisely:

$M_{b^*} \models I$ am $s^*$-iterable in $L[\gamma, H]$ where $(\gamma, H)$ is

$\text{Col}(w_1, \delta_0) \times \text{Col}(w_1, <\kappa_0)$ for $\ldots$.

For each $s^*$ pick a branch $(in V)$ $b_{s^*}$ which

witnesses $s^*$-iterability for $T$. (Is cofinal and

$\text{type } \text{max}(s^*) (s^* - \delta_0^*) = (\text{type } (s^* - \delta^C))$

Let $b = \text{max}(T)$. Then $L[M^b] = M_b$ and $s(T) = b_*(s^*)$

Then

$b = \lim_{s^*} b_{s^*}$

Because $\gamma_{s^*}$ are cofinal in $\delta_0^*$.

$b$ is independent of how $b_{s^*}$ were generically chosen in

$\text{Col}(w_1, \gamma_0)$. Hence $b \in \text{HOD}[L_{\omega_1^C}^\omega]$. So $\kappa_\alpha \in \text{HOD}[L_{\omega_1^C}^\omega]$

Claim: $\text{HOD}[L_{\omega_1^C}^\omega] \subseteq L[M_{\omega_1}, \delta_0]$. (Hence $\subseteq$)

Proof: We can find an $A \in \delta_0^* = \omega \times L_{\omega_1^C}^\omega$ s.t.

(1) $\text{HOD}[L_{\omega_1^C}^\omega] = L[A]$

(2) $A$ is definable without paramaters over $L_{\omega_1^C}^\omega$

(like Vopěnka.)
Claim \( M^*_\infty = \lim_{n \to \infty} F M_n \) is dense in \( \mathcal{L}[\mathcal{Y}, \mathcal{H}] \).

Proof:
Given \( (N_1) \in F[\mathcal{L}[\mathcal{Y}, \mathcal{H}]) \).

Let \( M_\infty = \lim_{n \to \infty} F M_n \).

For \( \psi \in M_\infty \) and \( \mathcal{L}[\mathcal{Y}, \mathcal{H}] \),

For \( q \leq p \) let \( \phi_q = (\pi \circ \text{dom}(q)) \circ q \).

Let \( \phi_q \).

Let \( q \in \mathcal{L}[\mathcal{Y}, \mathcal{H}] \).

\( (N_1, \xi) \in \mathcal{F}[\mathcal{L}[\mathcal{Y}, \mathcal{H}]) \).

There are only finitely many.

Now compare all \( N_q \) simultaneously and also with \( P \).

The calculation terminates at \( R \).

So \( \xi \mapsto R \).

\( (N_1, \xi) \in \mathcal{F}[\mathcal{L}[\mathcal{Y}, \mathcal{H}]) \) by symmetry.

Similarly, if \( M_\infty \)

\( M_\infty \rightarrow^L \mathcal{L}[\mathcal{M}_\infty, \mathcal{L}_\infty] \)

By \( \mathcal{N}_m \)

\( \ni \)

also by \( \mathcal{E}_{M_1} \).

Obtain by companion.

tree in \( \mathcal{L}[\mathcal{Y}, \mathcal{H}] \).

Then we can find \( \xi \) in \( \mathcal{L}[\mathcal{M}_\infty, \mathcal{L}_\infty] \).

Use a tree searching for \( \xi \).

\( j \in \text{dom}(\xi) \).

\( L \) is unique with making the diagram commutative.

Because it moves types of indiscernibles correctly.

(This needs some elaboration.)

So \( F M_\infty \) is dense in \( \mathcal{L}[\mathcal{Y}, \mathcal{H}] \).

\( \lim_{n \to \infty} F M_n = M^*_\infty \).

Let \( \iota : M_\infty \rightarrow M^*_\infty \) be the map given by \( M_\infty \).

(\( \xi < M_\infty \) need this since we only want countably many \( N_q \)')
Claim. For $f \leq S_\omega^1$, \[ \text{Col}(w, S_\omega^1) \times \text{Col}(w, < \kappa) \] \[ \exists \alpha \in \mathbf{A} \iff M_\alpha \models ( \text{Col}(\omega_1, S_\omega^1) \times \text{Col}(\omega, < \kappa) ) \]

\[ \Rightarrow M_\alpha \models ( \text{Col}(\omega_1, S_\omega^1) \times \text{Col}(\omega, < \kappa) ) \iff \varphi(\bar{\alpha}, \bar{\beta}) \]

Proof. Fix $\bar{z}$. Let $\bar{z} = \bar{u}$, $(\bar{v}, \bar{w})$ with $s \subseteq \omega^\omega$. Choose $N$ s.t.

\[ \bar{v} \subseteq N \nexists \bar{z} \in \mathbb{R}_N^\omega \text{ - generic over } N, \text{ hence } L[\bar{v}, \bar{w}] \text{ is a } \text{Col}(\omega_1, S_\omega^1) \times \text{Col}(\omega, < \kappa) \text{-generic finite extension of } N. \]

Then \[ \bar{v} \in A \iff L[\bar{v}, \bar{w}] = \varphi(\bar{z}) \]

\[ \Rightarrow N \equiv ( \text{Col}(\omega_1, S_\omega^1) \times \text{Col}(\omega, < \kappa) ) \]

Note that $\Pi : N \rightarrow M_\alpha$ is an elementary map via $\Sigma_1$. Then $\Pi \upharpoonright S_\omega^1 = \Pi_{(\omega_1)^{N \cap \omega_1}}$. So they agree on $\bar{z}$. Hence:

\[ \Rightarrow M_\alpha \models ( \text{Col}(\omega_1, S_\omega^1) \times \text{Col}(\omega, < \kappa) ) \]

\[ \Rightarrow M_\alpha \models ( \text{Col}(\omega_1, S_\omega^1) \times \text{Col}(\omega, < \kappa) ) \]

as $\bar{v}(\bar{z}) = \Pi_{(\omega_1)^{M_\alpha \cap \omega_1}}(\bar{z}) \Rightarrow \varphi(\bar{z})$.

Exercise: \[ \exists \gamma \text{ s.t. } \gamma < \delta_\omega \subseteq M_\alpha. \text{ Hence } \gamma \upharpoonright \text{Ord} \text{ is } M_\alpha \text{-closed.} \]

Thus, Woodin $(\text{PD})$. For a cone of $x$:

\[ L[\bar{v}, \bar{w}] \models \omega_2 \text{ is Woodin.} \]

Hence $L(M_\alpha, S_\omega^1) = \delta_\omega$ is Woodin.

Exercise: \[ \delta_\omega \subseteq j(M_\alpha, S_\omega^1) \text{ for } j \text{ generic over } M_\alpha. \]

\[ \exists \gamma \text{ s.t. } S_\omega^1 \subseteq j(M_\alpha, S_\omega^1) \text{ and } \gamma \text{ is Woodin.} \]

Hence $L(M_\alpha, S_\omega^1) = \delta_\omega$.
We can use this to show:

Let $\mathcal{L} = \Sigma_{M_1} \Gamma$ (trees in $M_1$) where $\nu$ is the first inaccessible $> \delta_{M_1}$ in $M_1$.

Then $L[M_1, M] \models L[M_1, M] = M_1 \upharpoonright \delta_{M_1}$ and $L[M_1, M] \models \delta_{M_1}$ is Woodin.

Sketch: Let $M_0$ be the adic limit of $\mathcal{F}^{M_1}$ where $\mathcal{F}^{M_1}$ is the DSL for $M_1$ up to $\nu$

$M_0^* = $ adic limit $\mathcal{F}^{M_0}$ of $\mathcal{F}^{M_0}$

$M_1 \xrightarrow{i} M_0 \xrightarrow{i(i)} M_0^*$

(Note: Adding $\delta_{M_0}$ to $M_0$ does not add bounded subsets of $\delta_{M_0}$)

$i(i)$ maps $L[M_0, M_0] \rightarrow L[M_0^*, M_0^*]

Use this to show:

$\text{Hull } L[M_0, M_0] \models \text{rng}(i(i)) = L[M_1, M]$

Point: Definitions are allowed to act on $\delta_{M_0}$

$i(i)$ preserves $\delta_{M_0}$-definitions.
Lemma let $B \subseteq \mathcal{L}$, $\rho < \lambda$ and $B = (B_\beta | \beta < \rho)$ be s.t.
$B_\beta \subseteq B$ for each $\beta$. Then $B \subseteq \mathcal{L}$

Lemma $\mathcal{L}$ is closed under ultrapowers

Lemma $\Sigma_1$ is closed under ultrapowers.

Lemma $\Sigma_1$ is a Suslin cardinal, $S(\Sigma_1) = \Sigma_2$ and scale$(\Sigma_2)$

Remark We can show that $\Delta_1$ (and $\Sigma_1$, $\Pi_1$) is closed under measure quantification by measures on $\lambda$.
Using this one can show that every $\Pi_1$ set admits a semi-scale with curves in $\Pi_1$.

Question Do we have $\text{scale}(\Sigma_0)$, $\text{scale}(\Pi_1)$?

Definition A tree on $w \times w$ is strongly homogeneous if there are measures $\mu_x$ on $T_x$ s.t.
- $\mu_x$ witnesses the homogeneity of $T_x$
- There are measures $\lambda_x$ on $A_x$ s.t. for all $x$ with $T_x$ well-founded, the ranking function $T_x \upharpoonright A_x$ has minimal values $[+\xi]$ where $T_x$ is the function on $T_x$ induced by $\xi$.

Fact If every $\kappa$-LS is strongly $\kappa$-LS then we can fill the gap with the where we have only semi-scale instead of a scale.
A non-selfdual, closed under quantifiers, \( \Gamma = S(n) \)
where \( n = o(A) \). Let \( A \in \mathcal{B}'(\Gamma) \) and let \( A = \mathcal{P}(T) \)
where \( T \in \text{on} \ \omega \times \nu \).

**Definition (Steel)**: \( \text{Env}(\Gamma) \) is the set of all
\( A \in \omega^\omega \) s.t. for some \( z_0 \in \omega^\omega \), for any countable set
of reals \( z \) containing \( z_0 \) we have \( A \in \text{EL}(T, z) \).

\( \text{Env}'(\Gamma) = \) the set of all \( A \in \omega^\omega \) s.t. for some
\( z_0 \in \omega^\omega \) : for any countable set of reals \( z \) containing \( z_0 \),
we have \( A \in \text{EL}(T, z) \) from finitely
many ordinals, \( T \) and \( z \).

**Remark**: We can consider the variations \( \text{Env}^\sim, \text{Env}'^\sim \) where
we consider "\( z \geq z_0 \)" instead of "\( z \) containing \( z_0 \)".
Clearly \( \text{Env} \leq \text{Env}^\sim, \text{Env}' \leq \text{Env}'^\sim \).

**Theorem**: For \( \Gamma \) as above : \( \lambda (\Gamma, \nu) = \text{Env}(\Gamma) = \text{Env}'(\Gamma) = \text{Env}^\sim(\Gamma) \).
Analyze $E_0^L(\omega_1)$ on the assumption $M^*_w$ exists. Let $E_0^L$ be the unique FS of $M^*_w$.

Actually, it is important to do it under the weaker $D^L(\omega_1)$. Let $M_\omega$ be the linear set $E_0^L$-iterates of $M_w$ via trees in $M_w \upharpoonright \delta^M_w$, so that there is no drop on the main branch.

Recall: $M_w = \text{Hull}^w(M_w)$ whenever $\Gamma$ is a proper class.

So $M_w$ is sound. This soundness can be used to show that the system of iterates is directed.

$L_\omega = E_0^L$ trees in $M_\omega$ based on $M_\omega \upharpoonright \delta^M_\omega$.

(We $E_\omega = \sup_{i\in\omega} E_i^{M_\omega}$.)

Then: $\text{HOD}^{L(\omega_1)} = L(M_\omega, L_\omega)$

Approximate via a DSL defined over $L(\omega_1)$.

**Def** $WG(M_w)$ as:

1. $\varphi_0, \varphi_i$ (only $\varphi_i$ on $M_{w-1}^{T_i}$ where $M_{w-1}^{T_i} = M$)
2. $b_0, b_i$ (if $\varphi_i$ normal)
3. win iff $\lim_i M_{w_i}$ exist and is w.f.

"It has a winning strategy in $WG(M_w)$ if $\delta^{\omega_1}_1 = \sum_i (\varphi)$

If it has a w.s. \(\Rightarrow\) it has a w.s. in $L(\omega_1)$.

**Fact** If $M, N$ are $\delta^\omega_1$-closable project to $w$ one sound and w-small then $M \models \text{N} \models \text{N}^*$. 


So the *Mouse-set-conjecture* holds in $L(\mathbb{R})$: in $L(\mathbb{R})$, TFAE for a countable transitive and $\kappa \leq \alpha$:

1. $b \in OD(\alpha \cup \{\lambda\})$
2. $b$ is $C_{\Sigma_3^1}(\alpha)$
3. $b$ is in some $\omega_1$-iterable mouse over $\alpha$
4. $b$ is in some $\omega$-small, $\omega_1$-iterable mouse over $\alpha$

Remark: every every $\omega_1$-iterable mouse is $(\omega_1+1)$-iterate.

The proof (1) $\Rightarrow$ (2) is just an abstract computation.

(3) $\Rightarrow$ (4): Define $b$ to be from its state constructed in any $M_\alpha$

(1) $\Rightarrow$ (4): This is the "correctness" of $M_\alpha$. Enough to show (ETS): $b \in M_\alpha(\alpha)$. But then $b \in M_\alpha(\alpha) \cup w_1^{M_\alpha}(\alpha)$ and this is iterable in $L(\mathbb{R})$: the iteration strategy: $\mathbb{R} \to T$ to the unique cut $b \in \{M_\alpha \cap W \in (\mathbb{R},\mathbb{R})\}$-iterable.

To be $M_\alpha(\alpha)$: iterate $M_\alpha(\alpha) \to \cdots M_i \to \mathbb{N}$ via $\Sigma_0^{M_\alpha(\alpha)}$ so that for some $G$ queue $\mathcal{G} = \lambda \cap \mathcal{G}$: $1^{\mathcal{G}}_\mathbb{R} \in \mathbb{R}$.

So $b \in OD(\alpha \cup \{\lambda\}) \cap L(\mathbb{R})$. So $b \in \mathcal{N}$, so $b \in M_\alpha(\alpha)$.

**Def.** A premouse $M$ is *full* off $(\mathbb{V}_\theta,\mathbb{V}_{\theta+})$ if

$b \in OD(\mathbb{V}_\theta,\mathbb{V}_{\theta+}) \Rightarrow b \in M$

So: $M_{\omega+1}$ is and its iterates are full.

**Def.** A premouse $M$ is le-suitable off there are $S_0 \cdots S_\ell$

Woodins s.t. $M = S_i$'s are the unique Woodins and

$\mathcal{O}_n = S_\ell^{+\ell+\ell+\ell+}$ and $\mathbf{E}$

and $M$ is full and $\omega$-small.

(To be safe, add the requirement: no $\mathcal{M}_\mathcal{Y}, \mathcal{Y} \in \mathcal{O} \mathcal{Y}$ has this property.)
We write \( k = k(M) \) \( (k \text{ as above}) \)

**Crucial Definition** Let \( ASR, M \) be a \( \text{pointwise} \) \( M \models \text{ZFC} - \text{Powerset} \) and \( M \models \delta^*_\omega \). Let \( \tau \) be a \( \text{Col}(\omega_1^M, \delta^*_\omega) \) term. Then \( \tau \) captures \( A \) over \( M \) iff for every \( q \), \( \text{Col}(\omega_1^M, \delta^*_\omega) \)-generic \( /M \)
\[ \tau^q = A \cap M[g] \]

**Example** Let \( ASR \) be \( \text{OD}^{L(\mathbb{R})} \), \( \delta = \delta^*_\omega \). Then there is \( \tau \in M_\omega \) s.t. \( \tau \) captures \( A \).

Exercise using genericity iterations.

For \( \tau \) a term, \( \delta \) as above let
\[ \tau^* = \{ (p, \sigma) \mid p \in \text{Col}(\omega_1^M, \delta) \text{ and } \sigma \in \text{Col}(\omega_1^M, \delta) \times \mathbb{R} \text{ and } \tau \models p, \sigma \} \]

Assuming \( B \models \sigma \in IR \); (we assume such terms always exist)

1. \( \tau = \tau^* \)
2. \( \tau^* = \tau^* \)

**Definition** \( \tau \) is invariant iff for all \( q, h \) generic for \( \text{Col}(\omega_1^M, \delta) \)
\[ M[g] = M[h] \Rightarrow \tau^q = \tau^h \] \( (M \text{-definable}) \)

For invariant \( \sigma, \tau \) TFAE:

1. \( \sigma^* = \tau^* \)
2. \( \sigma^q = \tau^q \) on all \( M[g] \), \( q \) \( \text{Col}(\omega_1^M, \delta) \)-generic \( /M \)
3. \( \neg \exists \text{some } \neg \)

**Proof**: Exercise
\( \tau^* = \text{the unique standard invariant term capturing } A \text{ over } M \text{ if exists} \)

We write: \( \tau^* = \bigotimes_{A, i, j} \tau^*_{ij} \)

\( \rho \in M^\omega(\delta^*) \)

\( \text{Denote extended } \text{let } M \text{ be } k\text{-suitable, } A \in OD^{L(\mathbb{R})} \)

then \( \tau^*_{ij} \text{ exists for all } k \leq n \).

\[ \mathbb{R^R} \]

**Definition** Let \( A \) be OD, \( A = \langle A_0, \ldots, A_n \rangle \). Let \( M \) be \( k \)-suitable then \( \Sigma \) is an \( A \)-iterative strategy for \( M \) iff \( \Sigma \) is a strategy in \( WG(M, \omega) \) for \( \Sigma \) s.t. of \( M \rightarrow N \) is an iterative map w.r.t \( \Sigma \) then

1. \( N \) is \( k \)-suitable
2. \( \pi(\tau^*_{ij}, \delta^*) = \tau^*_{ij} \) for all \( i, j \) \( j \neq k \) is enough.

\( \Sigma \) is \( A \)-iterative iff \( \pi \) has such a strategy.

**Lemma** if \( A \in (OD^{L(\mathbb{R})})^\omega \) then for any \( \Sigma_0 \)-iterate \( N \)

of \( M^\omega \) there is a \( \Sigma_0 \)-iterate \( P \) of \( N \) s.t. for all \( k \leq n \)

\( \Sigma_0 \) is an \( A \)-iterate \( P \) s.t. for \( \Pi^R \).

**Proof** (by picture) Assume \( N = N_0 \rightarrow N_1 \rightarrow N_2 \rightarrow \ldots \)

and \( \Sigma_i \) moves \( \tau^{N_i} \) incorrectly. Update each \( N_i \) to \( N^*_i \)

to make \( D(N_i, \delta^*) A_{i, j} \delta^* \).

\( \mathbb{R^R} \)

The map - composition: bottom row + right column is an iterative map, \( \exists N^*_w \) is w.f. in each \( N_0 \rightarrow N_1 \rightarrow N_2 \rightarrow \ldots N_n \) \( D(N_i, \delta^*) : A \in OD \) so as to

then for sufficiently large \( i \), \( \tau^*_w(i) \) is fixed, \( \text{our } N^*_w \)

would be ill-founded. From that point on, the terms are moved correctly.

[8]
Def. For $M, N$ $k$-suitable $\pi : M \rightarrow N$ is an $A$-iterable iff $\pi$ arises from a play according to an $A$-iterable strategy.

Def. $M$ is strongly $A$-iterable iff whenever $\pi : M \rightarrow N$, $\sigma : M \rightarrow N$ one $A$-iterable then $\pi \cap H^M_A = \sigma \cap H^M_A$. (Here $M$ is $k$-suitable, $A \in OD^{\omega_0}(\mathcal{P}(\beta))$)

Here: for $P$ $k$-suitable over $A \in OD^{\omega_0}(\mathcal{P}(\beta))$, $P = L(P \cup \text{crit}(P))$

$\delta(P, A) = \sup \{ \xi < \delta^P \uparrow \eta \mid \Delta_4 \text{ definable over } P \text{ from parameters} \}

\delta(P, A) < \delta^P

Similar: for $A = \{ A_0, \ldots, A_k \}$

$\delta(P, A) < \delta^P$

$H_{\pi, A} = \text{Hull}P(\pi, P \cup \{ \xi \in \varepsilon^P \uparrow \eta \})$

$H_{\pi, A} \cap \delta^P = \delta(P, A)$

Lemma. Let $N$ be a $\Sigma_0$-iterable of $M_0$ s.t. $\Sigma_0 \vdash \text{an } A\text{-IS for } P = N \uparrow \delta^P \uparrow \eta$. Then $P$ is strongly $A$-iterable.

Proof:

This is like the $M_1$ argument before - check this.

Also appeal to the $D-\psi$ property of $\Sigma_0$.

$\pi \uparrow H^P_A = \Sigma_0 \uparrow H^P_A$ using

$P \not\rightarrow \eta$ $D-\psi$ property of $\Sigma_0$. 
Let
\[ I^* = \xi(N, A) \quad \text{if} \quad N \text{ is } k \text{-suitable and } N \text{ is strongly } A \text{-iterable} \]
\[ (N, \bar{A}) \leq (P, \bar{B}) \quad \text{if} \quad \text{there is a } \bar{A} \supseteq \text{IM} \quad \text{and} \quad k(N) \leq k(P) \quad \text{and} \quad \bar{A} \text{ is an initial segment of } \bar{B}. \]

\[ P(N, A), (P, B) : H^N_A \rightarrow H^P(B, A) \]

be the common value of all \(k\)-iteration maps.

\( \mathcal{F} = \text{the corresponding DLS,} \)

\( \mathcal{F} \) is definable over \( L(R) \)

**Claim:** \( M_\omega = \text{chlin lim of } \mathcal{F} \)

**Def:** \( \mathfrak{T}_k = T^{L(R)}(d_0, \ldots, d_k) \) where \( d_i \) are \( R \)-indiscernibles
coded as set of reals.

**So:** \( \mathfrak{T}_k \) is \( \text{OD}(L(R)) \)

**Lemma:** Suppose \( B \subseteq R \) is \( \text{OD}(L(R)) \) and \( A \) is \( \text{OD}(L(R)) \) and \( A \leq_w B \). Then there are densely many \((N, \bar{C}) \in I \) s.t. \( A_1B \subseteq \bar{C} \) and \( \mathfrak{c}^N_{A_1} \subseteq H^N_B \) and hence \( \mathcal{F} : H^N_A \rightarrow H^P_B \) is \( \mathcal{F}(N, B), (P, B) \) then \( \mathcal{F}(\mathfrak{c}^N_{A_1}) = \mathfrak{c}^P_{A_1S_w} \).

**Proof:** Choose any \((N, \bar{C}) \) s.t. \( A_1B \subseteq \bar{C} \) and \( \Sigma_0 \) is a \( \bar{C} \)-iterable \( \Sigma \)-iteration of \( M_\omega \). Also make sure \( x \in B^N_{\Sigma_0} \text{-generic}/N \) where \( A \leq_w B \text{max} \).

For \( x \) a standard \( \Sigma \)-iteration of \( \text{Col}(\mathfrak{c}, k) \).

When \( k = k(N) \) pick an \( \tau_c \in B^N_{\Sigma_0} \) s.t.
Let \( k = q \ast h \) as \( \text{cel}(w, s_k) \) - generic by rearrangement of generics. Then
\[
\sigma \circ \sigma \Rightarrow \sigma \text{ is } \sigma
\]
Since there are \(< s_k^N \) such \( \sigma \)'s and \( \tau \)'s \( (\tau \text{ determines}) \)
So \(< s_k^N \) many. So all \( \tau \in H^N_B \). So \( \tau \in H^N_B \). \( \square \)

**Corollary**: \( \text{discrim } \overline{f} = \lim \text{ of all } H_{(N, \overline{F}_w)} s.t. \)
\[
N = P \ast \delta_{j}^N \text{ for } P \in \Sigma \text{ iterate of } M_w.
\]

To show this limit is \( M_w \! \lambda_w \):

**Lemma**: Let \( \overline{f} \) be a \( \Sigma_0 \) - iterate of \( M_w^N \) and
\[
N = N^N \ast \delta_{k}^N \text{ for } \delta_{k}^N.
\]
Let
\[
S_j = \text{Th}(M_w(N), \ldots, \delta_j) \cup N(1_{\overline{F}_w})
\]
very large cardinals.

Then
\[
(1) \ U_j \ast P_s_j \in H_N(\delta_{k}^N \cup S_j)
\]
\[
(2) \ U_j \ast P_s_j
\]

**Proof**: (2) is easy - use the theory \( \text{Th}(M_w(N), \ldots, \delta_j) \)

(1) Given \( j \) take \( \overline{f}_j = \overline{f}_{j+5} \). Idea: Induction force prevents

Idea: Once \( L(\overline{F}_w) \) Induction force a remove whose derived

(\( \overline{F}_w \) is \( L(\overline{F}_w) \).
For a countable transitive well-founded self well-ordered add a Turing degree above a

letting

\[ T = \text{tree of a scale on universal } \Sigma^2_1 \text{ set on } \mathcal{L}(\varphi) \]

in \( \mathcal{L}[T_{\varphi}] \) : take all \( \varphi \)-suitable \( \varphi \) s.t. \( \varphi \) is \( \varphi \)-suitable and \( \varphi \leq_T \varphi \). \( \varphi \) over \( \alpha \).

\( \varphi(\alpha) \) = result of comparing all of them and making all \( \varphi \leq_T \varphi \) generic \( \mathcal{L}[T_{\varphi}] \).

They can be compared via \( \mathcal{L}[T_{\varphi}] \).

Given \( d_0 < d_1 < \ldots < d_n \)

\[ \varphi_0 = \varphi_0 \]

\[ \varphi_{i+1} = \varphi_{d_{i+1}} \]

Let \( \langle d_i \mid i \leq n \rangle \) be Prikey. Can show for

\[ \varphi_{\omega} = \bigcup \varphi_i \]

\[ \mathcal{L}[\varphi_{\omega}] \vdash \varphi(y) \leq_T \varphi_i \text{ for any } y < \varphi(\varphi_i). \]

(all \( \varphi_i \) are OD-finite)

Moreover: There is an iterate of \( M_\omega \) via \( \Sigma_0 \) s.t.

it is of the form \( \varphi_\omega \) some Prikey-generic \( \varphi_\omega \).

Can then define \( S_j \) from \( \varphi_{j+5} \) using the Prikey forcing.

The rest is similar to the \( M_\varphi \)-argument.
Core model induction in $L(R)$

**Def.** Let $\kappa \geq \aleph_1$ be a cardinal and $A \in H_\kappa$. A model operator over $A$ on $H_\kappa$ is a partial function $F : H_\kappa \rightarrow \kappa$.

$$M = (\mathcal{M}, \epsilon, A, E, B, S) \rightarrow F(M) = \eta$$

where

$$M = (\mathcal{M}, \epsilon, A, \bar{E}, B, \bar{S})$$

such that

- $\mathcal{M}$ is an end-extension of $M^\#$, $M \in \{\mathcal{M}\}$
- $F(M) = \text{Hull}_{\mathcal{M}} \{\mathcal{M} \cup \{\mathcal{M}\}\}$
- $F(M)$: $\text{the least ordinal above } \text{cut}_{\mathcal{M}}$.
- $\mathcal{M}$ is a Woodin + Sharp.
- $F$ is feeding in info about $\Sigma_1$ on $\mathcal{M}$ for $\eta$ coded by $A$.

**Examples**
- $\eta = F$
- $F = M^\#$

**Theorem:** (Heath's shakeup + condensation relative to $F$).

Let $(\mathcal{M}, \bar{M})$ be given. Suppose $\pi : \bar{M} \rightarrow F(\mathcal{M})$ is either $\Sigma_0$ cofinal or $\Sigma_1$. Then $\bar{M} = F(\pi^{-1}(\mathcal{M}))$.

* $F$ condenses well
Exercise Assume \( F : H_\omega \to H_\omega \) is a model operator
which condenses well. Let \( \kappa > \nu \). Then there is at most one
extension \( \bar{F} \) of \( F \), \( \bar{F} : H_\kappa \to H_\nu \) s.t. \( \bar{F} \) also condenses well.

**Def** Let \( F : H_\nu \to H_\nu \) be an MO. A model
\( M = (M_1, V, A, F, B, S) \) is a potential premodel iff
there is \( \bar{M} = (M_1, \bar{V}, \bar{A}, \bar{F}, \bar{B}, \bar{S}) \) a sequence of models, write
\( M_0 = M_1 \) satisfying
\[
M_{n+1} = (F(M_n), \bar{M}_{n+1})
\]
and \( E \) is a coherent extension sequence.

\( \iff \) \( M \) is a premodel off all proper initial segments are sound.

**Def** \( K^C, F_\kappa(\nu) \) - construction. This is like an ordinary
\( K^C \) construction with the exception that the step
\( M_3 \mapsto \begin{cases} \bar{M} & \text{if } \bar{M} \in \text{M}_1 \text{ satisfying } M_3 \mapsto F(M_3). \end{cases} \)

**Example** Don't add any extenders, \( \nu = \omega \). Then
\( K^C, F_\omega(\omega) = L^F(\omega) \). Point: If \( F \) condenses well
then \( L^F(\omega) = GC \& \text{ et c.} \) (\( \omega \)-countable \( \bar{F} \) in \( L^F(\omega) \)).

As usual: Countable substructures of models \( N_1 \) from the
\( K^C, F_\kappa(\nu) \) construction are \( \omega_1 \)-iterable in this sense:
if \( Y \) is a countable tree on \( W \) with \( \bigcup_{\alpha < \epsilon} Y_\alpha \)
and \( \sigma : W \to N_\kappa \) then \( F \) has a last model embeddable
in \( \sigma \) some \( W_2 \) - \( \sigma \leq \sigma \) or else there is or else there
is a maximal branch \( b \) s.t. \( \bar{M}_b \) is embeddable into \( W_2 \).
Def. A premouse $M$ is $F$-small iff $M \Vdash \kappa$ is Woodin where $\kappa = \mathsf{cf}(E^M_{\alpha})$ some $\alpha$.

$M^F_2(\kappa) = \text{the least } (\kappa+1)\text{-iterable premouse}$

Def. Assume $\mathcal{Y}$ is an IT on an $F$-pm which does not have a definable Woodin card. We say that $\mathcal{Y}$ is guided by $L^F$ iff $\forall \lambda < \mathsf{lh}(\mathcal{Y}) : \mathcal{Y}(0, \lambda)^* = \mathsf{unfGd}$.

Def. Assume $\mathcal{Y}$ is an IT on an $F$-pm which does not have a definable Woodin card. We say that $\mathcal{Y}$ is guided by $L^F$ iff $\forall \lambda < \mathsf{lh}(\mathcal{Y}) : \mathcal{Y}(0, \lambda)^* = \mathsf{unfGd}$. Let $\mathcal{Y}$ be the unique cofinal branch $\mathcal{Y}$ of $\mathcal{G}$ for $\mathcal{G}$ s.t. for some $\langle \mathcal{G} \rangle \subseteq M^\mathcal{Y}$ s.t. $\mathcal{G}$ either projects below $S(\mathcal{Y})$ or else $S(\mathcal{Y})$ is not definably Woodin over $\mathcal{G}$. Briefly $\mathcal{G}$ kills Woodinness of $S(\mathcal{Y})$ and $\mathcal{G} \subseteq L^F(M(\mathcal{Y}))$.

Plan: $K^F_r(\mathcal{G})$ is fully iterable via the strategy of producing trees which are guided by $L^F$.

Theorem ($K^F_r$ existence dichotomy). For simplicity assume $\mathcal{G}$ is a measurable cardinal.

Let $\mathcal{F}$ be a model on $V_\mu = V_{\kappa2}$ s.t. $K^F_r(\mathcal{G})$ be the result of the $K^F_r(\mathcal{G})$-construction $K^F$ inside $V_\kappa2$.

Let $\mathcal{E}$ be the partial strategy of producing IT's which are guided by $L^F$. Then:

1. If $\mathcal{E}$ produces a model with a Woodin, i.e.,

   there is a tree $T$ of limit length on $K^F_r(\mathcal{G})$ guided by $L^F$ s.t. $L^F(M(T)) \in \mathcal{S}(\mathcal{G})$ is Woodin, then $K^F_r(\mathcal{G})$ reaches $M^F_r(\mathcal{G}) = M^F(\mathcal{G})$ is iterable.

2. If one of the hypotheses is false, then $K^F_r(\mathcal{G})$ is $\kappa+1$ iterable.
If (3) applies, isolate \( \mathcal{K}^F(P) \) and use it to get a contradiction from the favorite background hypothesis.

**Proof**

This is like the proof in the classical case when \( F = \text{nd}^+ \) and uses that \( F \) condenses well.

Remember one more "local" version of the \( \mathcal{K}^F \)-existence axiom (for instance:)

**Applications**  Show PD from various hypotheses

**Theorem**  \( T \Omega \Rightarrow \mathcal{V} \) is closed under \( M^\#_{\text{suitable}} \)

**Theorem**  There are \( \omega \) pairs of successor cardinals with the tree property with sup \( \delta \cdot \omega \cdot 2^\delta < \delta \).

Then \( H_\delta \) is closed under \( M^\#_{\text{suitable}} \)

**Theorem**  Let \( \kappa \) be singular, \( \text{cf}(\kappa) > \omega \). Suppose \( \exists \kappa \in \kappa \setminus \kappa^+ \delta \) is stationary. Then \( H_\kappa \) is closed under \( M^\#_{\text{suitable}} \).

**Theorem**  Suppose \( CH \) and there is a precipitated ideal on \( \omega_1 \).

Then PD holds. (i.e., \( H_\omega \) is closed under all \( M^\#_m \))

**Theorem** (Woodin) There is \( \omega_1 \)-dense ideal on \( \omega_1 \). Then PD.
Theorem 9.40

Suppose \( G \subseteq \mathcal{P}(\omega_1) \) is a pointclass, \( V = L(\omega_1, \mathcal{R}) \), and \( \omega_1 \) is \( \Theta \)-regular.

Let \( G_0 \subseteq P_{\omega_1} \) be \( L(\omega_1, \mathcal{R}) \) generic and let \( H_0 \subseteq \text{Col}(\omega_3, \langle G_0 \cup \mathcal{R} \rangle) \) (here \( \mathcal{R} \) is essentially \( \text{Hod} \)).

Let \( \mathcal{L}(\omega_1, \mathcal{R}) \langle G_0 \cup \mathcal{R} \rangle \) be \( \mathcal{L}(\omega_1, \mathcal{R}) \)-generic. Then

\[ L(\omega_1, \mathcal{R}) \langle G_0 \cup \mathcal{R} \rangle \models \exists X \mathcal{L}(\omega_1, \mathcal{R}) \langle G_0 \cup \mathcal{R} \rangle \]

\( MM^{++}(\mathcal{R}) \) is:

- \( MM \) for pales of size \( 2^{\aleph_\omega} \) plus
- For any collection \( \langle \tau_x, \omega_1 < \tau_x \rangle \) of \( \mathcal{R} \)-names for stationary subsets of \( \omega_1 \), each \( \tau_x \) is stratified.

Define \( P_{\omega_1} \) is the set of \( \langle (M, I), a \rangle s.t. \):

- \( M \) is a countable transitive model of \( \text{ZFC} + \text{MA}_{\omega_1} \)
- \( I \) is a precipitous ideal on \( \omega_1 \) in \( M \)
- \( (M, I) \) is iterable by repeated application of generic ultrapowers by \( I \).
- \( a \in \mathcal{P}(\omega_1)^M \) and \( \forall x \in \mathcal{P}(\omega_1)^M s.t. \omega_1^M = \omega_1^M \).

Ordering:

\[ \langle (M, I), a \rangle \leq \langle (N, J), b \rangle \]

iff

\[ \langle (N, J), b \rangle \in H(\omega_1)^M \]

\[ \forall j : (N, J) \rightarrow (\omega_1^*, j^*) \text{ in } M \text{ s.t. } j(\bar{b}) = a \text{ and } \]

(so \( j \) is an iteration map of length \( \omega_1 \))

Note: \( j \) is uniquely determined by \( j(\bar{b}) \).

Facts

(1) If \( G \subseteq P_{\omega_1} \) is a filter but

\[ A_G = \bigcup \{ a : \langle (M, I), a \rangle \in G \} \]

For all \( p \in G \), \( p = \langle (M, I), a \rangle \) there is unique

\[ j_p : (M, I) \rightarrow (M^*, I^*) \text{ s.t. } j_p(a) = A_G. \]
Let
\[ \mathcal{B}_k = \bigcup \{ j_p (\mathcal{P}(\alpha, \mu)) : P = (\mu, i, \alpha) \in G \} \]
and
\[ \mathcal{R}_{\text{max}} \in L(KR) \).

**Theorem 9.33/35** Suppose that \( \Gamma \subseteq \mathcal{P}(\aleph_1) \) is a pointclass and \( L(\Gamma, \mathcal{R}) \models \Delta^+ \). Let \( G \in \mathcal{R}_{\text{max}} \) be \( \mathcal{R}(\Gamma, \mathcal{R}) \)-measurable. Then in \( L(\Gamma, \mathcal{R})[G] \):

1. \( \mathcal{P}(\{y\}) = \mathcal{P}(\omega_1) \subseteq L(\mathcal{R})[G] \)
2. \( L(\mathcal{R})[G] \models c = \omega_2 \)
3. \( \forall A \in \mathcal{P}(\mathcal{R}) \cap L(\Gamma, \mathcal{R}) : L(A, \mathcal{R}) \not\models [G] \models \neg \text{FC} \)
4. \( \forall A \in \mathcal{P}(\omega) - L(\mathcal{R}) : G \in L(\mathcal{R})[A] \).

**Proof of Theorem 9.40** \( L(\Gamma, \mathcal{R})[G_0] \models \omega_2 - \text{DC} \) so \( \text{ETS} \)

\[ L(\Gamma, \mathcal{R})[G_0] \models \text{MM}^+ (c) \]

Let \( \pi_1, \pi_2, \pi_3 \) be \( \mathcal{R}_{\text{max}} \)-names for:
- \( \pi_1 \) a point on \( \omega_2 \) preserving stationary subsets of \( \omega_1 \)
- \( \pi_2 \) an \( \omega_1 \)-sequence of dense subsets of \( \pi_1 \)
- \( \pi_3 \) an \( \omega_1 \)-sequence of \( \pi_1 \)-names for stationary subsets of \( \omega_2 \).

Fix a coding of elements of \( H(\omega_2) \) by reals
- first code elements of \( H(\omega_2) \) by subsets of \( \omega_1 \)
- then, since each subset of \( \omega_1 \) is on \( L[K] \) for some \( x \in \omega \) code this by \( x^\# \) and the relevant \( \tau \).

Letting \( B_{\pi_1}, B_{\pi_2}, B_{\pi_3} \) be the set of codes for elements of \( \pi_1, \pi_2, \pi_3 \) we have that for any \( \text{TCM} \) of \( \text{ETC} \) and closed under \( \text{E} \) daggers for reals: of \( \omega_2^M \).

\( B_{\pi_1} \) \( \text{M} \) decodes as a \( \mathcal{R}_{\text{max}} \)-name for a \( p.o. \) on a subset of \( \omega_1^M \).

\( B_{\pi_2} \) \( \text{M} \) decodes as an ...
Let $T_0$ be a tree on $\omega^3 \times \omega^3$ set
\[ p(T_0) = B_\omega \times B_\omega \times B_\omega \quad \text{and} \quad p(T_1^*) = \text{its complement}. \]
This is possible due to $\text{AD}^+$; it implies reflection to $\text{Thm}^+$ below.

If $j: M \rightarrow M^*$ when $M \models \text{ZFC transitive and } T_0, T_1 \in M$ the
\[ p(T_i^*) \subseteq p[j(T_i^*]) \quad i = 0, 1. \]

**Theorem 9.38**: Assume $\mathcal{P} \in \mathcal{P}(\mathcal{R})$ is a pointclass and
$L(\mathcal{P}, \mathcal{R}) \models \text{AD}^+$. Then $\forall X \in \mathcal{P} \cap \text{On} \cap L(\mathcal{P}, \mathcal{R})$ satisfies
\[ 0 \in L[\mathcal{Y}] \]
\[ L[\mathcal{Z}] \subseteq \mathcal{Y} \quad \text{and} \quad \forall \mathcal{Z} \in \mathcal{Y} \subseteq \mathcal{R} \text{ s.t. } L[\mathcal{Z}] \subseteq \mathcal{R} \]

- $L[\mathcal{Y}] \subseteq \mathcal{N}$ and
- $L[\mathcal{Y}] \cap \mathcal{N} = \mathcal{N} \cap \mathcal{V}_\kappa$ for the least strongly inaccessible
- $\exists \mathcal{D} \leq \omega_1^\mathcal{N}$ s.t. $\mathcal{N}$ is Woodin in $\mathcal{N}$.

**Proof sketch**
- Let $S$ be this set $\mathcal{S}$ for $T_0, T_1$
- Let $\mu$ be the club measure on $\mathcal{P}(\mathcal{R})$

\[ \text{normality: if } f: \mathcal{P}(\mathcal{R}) \rightarrow \mathcal{P}(\mathcal{N}) \text{ is such that } \]
\[ f(0) \leq \sigma \quad \text{for } \sigma \neq \delta \text{ then } \exists \mathcal{X} \in \mathcal{R} \text{ s.t. } \]
\[ \exists \sigma, 1 \in \mathcal{X} \Leftrightarrow f(\sigma) \downarrow \mu. \]

Take $\prod_{\sigma \in \mathcal{P}(\mathcal{R})} L(\mathcal{S}, \sigma)(\mu = L(\mathcal{S}, \mathcal{R}))$

Let $T_0, T_1^*$ be the images of $T_0, T_1$ under the up map.
Then $p(T_0^*) = p(T_0)$ and $p(T_1^*) = p(T_1)$. So
$L(\mathcal{S}, \mathcal{R}) = p(T_1^*)$ decides as... .

So: $\forall \sigma \in \mathcal{P}(\mathcal{R}) \ L(\mathcal{S}, \sigma)$ also thinks this.

Force over $L(\mathcal{S}, \sigma)$ with $\prod_{\text{max} \sigma} L(\mathcal{S}, \sigma)$

denote the forcing generic $g$. Then $L(\mathcal{S}, \sigma) \models 2^{\mathcal{R}} = \mathcal{R}$, etc let $t$ be an enumeration
of $\sigma$ in $L(\mathcal{S}, \sigma)[g]$. Then $L(\mathcal{S}, \sigma)(g) = L(\mathcal{S}, t)$. 
let \( N \) be as for \( L(S, \mathcal{M}) \) (T.9.36). Let \( P \) be the realization of \( \mathcal{M} \) by \( g \). \( N \vDash P \) preserves state subset \( \mathcal{M} \).

Let \( h \subseteq P \) be \( N \) - generic.

Let \( d \) be winning on \( N \). Let \( K \) be \( N \) - generic for \( \text{Coll}(\omega_1, \delta) \). Force over \( N \vDash K \) with \( \mathcal{C} \) forcing to get \( M \vDash \mathcal{M} \). Call this extension \( N^+ \). Let \( g \) be the least strongly inaccessible of \( N^+ \). Then \( \langle (N^+, N^+) \rangle \vDash \mathcal{M} \) and is above all \( \langle (M, L) \rangle \) for all \( \langle (M, L) \rangle \) in \( g \).

Let \( p_0 \in G \subseteq \mathcal{P}_{\text{max}} \), \( L(P_{\text{max}}) \) - generic. Then \( j_{p_0} : (N^+, N^+) \rightarrow (N^+, N^+) \).

\[ j_{p_0} : (N^+, N^+) \rightarrow (N^+, N^+) \]

\[ j^* = N^+, N^+ \]

Assume \( \Gamma \) is a pointclass, \( L(P_{\text{max}}) \vDash \mathcal{M} \). \( G \subseteq \mathcal{P}_{\text{max}} \) is \( L(P_{\text{max}}) \) - generic. Then \( L(P_{\text{max}}) \vDash \omega_2 - \text{DC} \).

Proof: It suffices to prove \( \omega_2 - \text{DC}_P \). Suppose \( R \subseteq \Gamma \times \Gamma \). Work in \( L(P_{\text{max}}) \). Find \( n < \Theta \) s.t. all \( w \)-sequences from \( \text{tr}(w, n) \) have extensions in \( v\). For all \( n < \Theta \), \( w(n) \) is \( \leq \Theta \). Why:

- \( c = \mathcal{P}(w_1) = \mathcal{P}(w_2) \)
- \( \exists B \subseteq R \) coding \( R \setminus w(c) \), \( FA \subseteq R \) coding \( R_{\text{max}} \) name for \( B \).

So in \( L(C_{\text{max}}) \) can find \( \omega_2 \) - sequence through \( R \).