Working in $N^*_x[I]$ denote the picture to make $R^*$ generic.

So far we have: For every $B \in B$ there is a bad pair $(P,E)$ s.t. $E$ has BC and $F$ is FPR and strongly respects $B$.

Note: given $<B,E>$ we can get a bad pair $(P,E)$ s.t. $E$ is FPR, has BC and strongly respects $\bigoplus B_i$ for all $i$.

Do this by comparing $(P_i,E_i)$'s.

Backtrack. Take $\tau = -1$, i.e. $\text{HOD} \cup \emptyset$.

Let $<A;i;E>$ be a semiscale on $\mathbb{N}^2$. By the above, get $(P,E)$ s.t. $P$ is a bad pair, $\chi(0) = 0$, $E$ is FPR and has BC and strongly respects $\bigoplus A_i$ for all $i$.

Then $<A;i;E>$ guides $E$.

Proof. Given $T$ on $P$. We have: if $E(\tau) = 0$ then

$(\tau^\sigma_a) \mid A_i, \quad \tau^\sigma_a(a_i) = \tau^\sigma_a$ but if $E(\tau) = 1$ then $\tau^\sigma_a = 1$. 
Why: If not then letting $S = \text{Hull}_{\mathcal{A}_1} (\varepsilon_1, \ldots, \varepsilon_\alpha, \pi)$ after collapsing with $\iota: S \to M^*_b$, then $\iota_*(\pi) = \pi$ and $S \models \text{Woodin}$. Also $S$ is full because $\langle A_i | i \in \omega \rangle$ is a semiscale on $\Pi^2_1$. So $\iota_*(S_1) = \pi$ is Woodin, hence $M^*_b \models \pi$ is Woodin. \qed

We can now show $\Sigma^0_n (\pi, \Sigma) = \Theta_{\omega_1}$.

Proof: \begin{itemize}
  \item comes from the covering system $\mathcal{J}$.
  \item because otherwise $\pi$ would be $\mathcal{J}$-ladder-
  \item so $w(\Sigma) < \Theta_{\omega_1}$. But we know $w(\Sigma) \geq w(A_i)\omega_1$.
\end{itemize}

Why is $\Sigma < \Theta_{\omega_1}$ Suslin:

\begin{align*}
\pi &
\xrightarrow{\Theta_{\omega_1}} M_{\omega_1} \leftarrow \Theta_{\omega_1} \Theta_{\omega_1} \downarrow \Theta_{\omega_1}\uparrow \Theta_{\omega_1}
\end{align*}

John says: Enough to look at the tree of all suitable attempts and is significantly simpler (since we are merely looking for a Suslin representation, and not for a scale).

Back to $\Theta_\alpha$ and $\Theta_{\alpha+1}$ for $\alpha \geq 0$.

We have: $\langle A_i, \Sigma \rangle$ with $\Sigma \in \mathcal{P}_2 + BC$

$M_{\omega_1} (\pi^{-1}, \Sigma^{-1}) \models \Theta = \epsilon_{\omega_1}^{\omega_1}$. Also

$w(\Sigma^{-1}) = \Theta_{\omega_1}$. For all $B \in \mathcal{B}$: some subset of $\langle A_i, \Sigma \rangle$

strongly respects $B$.

Consider $N (\pi, \Sigma) = \text{dir lim} \alpha$ of all $\Sigma$-iterates of $\pi$ above $\alpha$.

This is like $\Theta = \Theta_{\omega_1}$ case. Show by the same argument that $N_{\omega_1} (\pi, \Sigma) = \epsilon_{\omega_1}^{\omega_1} \Sigma^{-1}$ and get $\langle A_i, \Sigma^{-1} \rangle$ semiscale on $\Pi^2_1 (\Sigma^{-1}) \in \text{OD}_{\omega_1} \Sigma_{\omega_1}^{-1}$. \qed

\[ \text{sup} \sup_{\varepsilon \in \omega} \delta_{A_i, \varepsilon} = \Theta_{\omega_1} \]
For $\gamma < \Theta_{\Psi_0}$ let

$$B_\gamma^* = \{(\alpha, \lambda), x, (y, z)\} \text{ s.t.}$$

1. $(\alpha, \lambda)$ is a had pair with FPR $+ BC$ and $M_\alpha(\alpha, \lambda) = \theta_\alpha$
2. $x$ codes $\alpha$
3. $y$ codes $R_\gamma$ s.t. $(R_\gamma, \lambda)$ is a suitable pair
4. $z$ is in the least OD $\Theta_{\Psi_0}$-set $C$ s.t. $(R_\gamma, \lambda)$ is $C$-iterable in the sense of getting $\Theta_{\Psi_0}$ and

$$\pi_{R_\gamma, \lambda}(y^{R_\gamma}) \geq \gamma$$

Let $(\gamma_i : i \in \omega)$ be a sequence s.t.

$$\sup_{i \in \omega} \gamma_i = \Theta_{\Psi_0}$$

Let $\beta_i = B_{\gamma_i}$

Let $(P, E)$ be a had pair with FPR $+ BC$

$$M_\alpha(P, E_{\beta_i}) = V_{\Theta_{\Psi_0}} \upharpoonright \beta_i$$

Then $\sup_{i \in \omega} \gamma_i, E_{\beta_i} = \Theta_{\Psi_0}$

Let $h_{\Theta_{\Psi_0}}$ be the branch, $\Pi = H_{\Theta_{\Psi_0}}$

In $\Theta_{\Psi_0}$-limit, the $\beta_i$'s are cofinal in $\Theta_{\Psi_0}$ and

$$M_\alpha(P, E_{\beta_i})$$

is a suitable pair which is $\oplus$-iterable by

$$\pi_{\beta_i}((y^{\beta_i}) \geq \gamma$$

Claim $E$ is graded by $(B_i : i \in \omega)$

Proof: ETS: If $T$ is on $P$ and $b \in b$ is the branch, $\Pi = H_{\Theta_{\Psi_0}}$

Then $\sup_{i \in \omega} \gamma_i, E_{\beta_i} = \Theta_{\Psi_0}$

Then $\sup_{i \in \omega} (\beta_{\Theta_{\Psi_0}}) = \Theta_{\Psi_0}$

Let us backup.

Suppose $\gamma = \sup_{i \in \omega} (\beta_{\Theta_{\Psi_0}})$. Let $H = \text{Hull}_{\Theta_{\Psi_0}}(\prod_{i \in \omega} E_{\beta_i})$.

If there is $\Pi$ s.t. then it is done. Because $H$ is full

$\pi \upharpoonright H$ by our requirement on $P$ and $H \upharpoonright \gamma$ results in

So $\Phi \upharpoonright \gamma$ coding $b$. 

$\pi \upharpoonright H$
It exists if \( P = \mathbb{R} \). To get such a \( P \) just consider a countable version of \( \text{HOD} \): look for \( N_{\omega}^{L(\mathbb{R})} \) where \( T \) is the tree for \( \mathbb{E} \) on a core of \( d \). Let \( N_{\omega+1}^{\mathbb{E}} \).

Let \( T \) be a good pointed, \( w(T) > \Theta_{\omega+1} \). Work in some \( N_x \) that can see everything we want. Then let \( \Phi \) be the \( N_x^\omega \) sets of Wadge rank \( < \Theta_{\omega+1} \).

\[ N = \mathcal{W}_{\omega} (\mathbb{E}, \mathbb{P}) \]

14:00

\[ \text{Remark} \]

Step 1: \( \mathcal{V}_{\text{HOD}}^{\Theta_{\omega}} \)

Step 2: \( \mathcal{V}_{\text{HOD}}^{\Theta_{\omega}} \)

Step \( w \): \( \mathcal{V}_{\text{HOD}}^{\Theta_{\omega}} \)

If \( \Theta_{\omega+1} \) exists then we can get \( (\mathbb{P}, \mathbb{E}) \) s.t. \( \mathcal{X}^{\mathbb{P}} = \omega \), \( \mathbb{E} \) is FPR + BC

\[ \forall n \mathcal{M}_n (\mathbb{P}(n), \mathbb{E}(n)) \mid \Theta_{\omega+1}^{\mathbb{E}} = \mathcal{V}_{\Theta_{\omega+1}}^{\text{HOD}} \]

Compare all of them.

If \( \Theta_{\omega+1} \) does not exist take the union of \( \mathcal{V}_{\Theta_{\omega+1}}^{\text{HOD}} \)

Thus generalizes to arbitrary cf \( w \) step.

Now assume \( \Theta_{\omega+1} \) exists.

Step \( w_1 \): Let \( T > w(\Theta_{\omega+1}) \). Let a STIR with \( w(A) = \omega_1 \).

\[ B \] is the set of all \( \sigma \) s.t. \( \sigma^{-1}[CA] = \langle \mathbb{P}, \mathbb{E} \rangle \), \( \mathbb{E} \) is FPR + BC

\[ \mathcal{M}_\omega (\mathbb{P}, \mathbb{E}) = \mathcal{V}_{\Theta_{\omega+1}}^{\text{HOD}} \] all \( d < \omega_1 \).

Get some \( N_x^\omega \) capturing \( A, B \), then do had fun construction on side \( N_x^\omega \). We will get \( (\mathbb{P}, \mathbb{E}) \) s.t.
$\Sigma \leftrightarrow \text{FPR} + \text{BC}$

$x^\varphi = \text{the least measurable in } P$

$(\forall \varphi < x^\varphi) \quad M_\varphi(P,\Sigma,\pi_P)|\Theta_\varphi = V_{\Theta_\varphi}^{\text{HOD}}$

What about $M_\varphi(P,\Sigma)$?

$B(P,\Sigma) = \{\Theta | \Theta \text{ is a } \varphi \text{-iterate of } P\}$

$\leq^* \text{ in } B(P,\Sigma):$

$s <^* s_R \iff (\exists s < x^R) (s_R \in I(s,\Sigma_s))$ and

$(s,\Sigma) \rightarrow R(s)$ the corresponding embedding

$\text{let } M_\varphi^- (P,\Sigma) = \lim_{\varphi \rightarrow \varphi_R} (B(P,\Sigma),\leq^*) \text{ inside } \text{HOD}.$

Exercise $M_\varphi^- (P,\Sigma) = M_\varphi (P,\Sigma)|\theta^{M_\varphi (P,\Sigma)}$

It follows: $M_\varphi^- (P,\Sigma)|\theta_{x^\varphi} = V_{\Theta_{x^\varphi}}^{\text{HOD}}$

Claim $M_\varphi^- (P,\Sigma) = V_{\Theta_{x^\varphi}}^{\text{HOD}}.$

Proof (Sketch) Let $\alpha < \omega_1$. By induction:

$M_\varphi^- (P,\Sigma)|\theta_{\alpha} = V_{\Theta_{\alpha}}^{\text{HOD}}$ for $\alpha < \omega_1$.

Want this for $\alpha + 1$. Let $R \in B(P,\Sigma)$ be s.t.

$M_\varphi (R,\Sigma_R) = M_\varphi^- (P,\Sigma)(\alpha + 1)$

WTS: $M_\varphi (R,\Sigma_R)|\theta_{\alpha + 1} = V_{\Theta_{\alpha + 1}}^{\text{HOD}}$

For this: Need that the HOD holds for $(R,\Sigma_R)$.

So, let $R^* \in I(R,\Sigma)$ be s.t. $R^* \leq_{\text{med}} R^*.$

Then use the following fact about derived models:

Notation: $D^*(P,\Sigma) = D(\text{Ult}(P,\Sigma)(x^\varphi), \Sigma)$ where

$\mu$ is the order 0 measure on cf$(x^\varphi)$. 
One can show:

\[ D^*(P, E) = \{ A \mid w(A) < \Theta_{XP} \} \]

Using this,
\[ P = \{ m \in \text{my desired model} \mid \forall d < X^P (\text{Pla}_d) \} \]

is "good" (i.e., good for computing WOD)
\[ R^* = \{-1\} \]

Then
\[ D^*(R^* \subseteq P) = \{ A \mid w(A) < \Theta_{XP} \} \rightarrow (R, E) \text{ is good} \]

An idea of the proof of (1): Introduce a trim of

strategies to generic extensions.

Given a bad pair \((P, E)\) and \(d < X^P\) we can find

\[ T_s \in P \text{ s.t. } (T_s, s) \text{ in } \text{generic extension, i.e., } p \left[ T_s \subseteq P \right] = 1 \text{ and } \forall s \subseteq P \left[ T_s \subseteq P \right] \}

So for all \(d < X^P: E \subseteq P(d) \subseteq D^*(P, E) \Rightarrow \) for any

\[ A \in D^*(P, E) \exists d < X^P \text{ s.t. } (T_s, s) \text{ in PCg}\] that gives \(A: \text{ when } d < X^P \text{ be}

A \subseteq \text{Code}(P(d)) \text{ the dimension of } T_s\), then show that \(E\)

Also: \(\forall d < X^P w(\text{Code}(P(d))) < \Theta_{XP} \) so

\[ D^*(P, E) = \{ A \mid w(A) < \Theta_{XP} \} \]

Exercise: Show

\[ D^*(P, E) \not\subseteq \text{ Pla}_d \] is not OD from \(\text{ Pla}_d \).
An idea toward (2). Unable to record.

**THE PROOF OF MSC**

**WTS:** $x \in E_y \Rightarrow \exists y$-naive $M$ s.t. $x \in E M$.

**Assumptions:** $T (A_{T_0} + \Theta \text{ regular})$

**Assume not:***

Suppose: $\Pi$ is the largest s.t.

$$L_\mu (\Pi_1 \Theta) = MC \land L_{\mu + 1} (\Pi_1 \Theta) = TMC$$

Then there is $(\Pi_1 \Theta)$ s.t.

Let $\Pi$ be the largest initial of $\nu$ where $MC$ holds.

**Proof:**

Let $\Gamma = \Pi (\Theta)$ be the largest of $\Pi (\Theta)$ and $L_{\mu + 1} (\Pi_1 \Theta) = TMC$.

Fix $x \in E_0 \cap \Pi (\Theta)$, $x$ is not in any $y$-naive.

Then there is $(\Pi_1 \Theta)$ s.t.

Either $x^\rho$ is limit and

1. $\forall \xi \in B_1 (\Pi_1 \Theta): \xi \in \Pi$
2. $\Pi$ is $\Gamma$-FPR and has BC
3. $L (\Pi_1 \Theta) = x \cap \delta D$
4. $\Pi$ has good properties one needs to compute $\Pi (A)$

or else $x^\rho$ is a successor ordinal and

1. $\xi^\rho \in \Pi$
2. $\Xi$ is $\Gamma$-FPR and has BC
3. $L (\Xi_1 \Theta) = x \cap \delta D$
4. As above

By assumption: $\Pi \cap (\Xi_1 \Theta) = L (\Xi_1 \Theta)$. Some have
a good $\Gamma^* \in \Gamma$ s.t. $(P, \Sigma) \in \Delta$.

We have: For any $\Sigma$-inequality $Q$ of $P$ s.t. $Q \in \Sigma$-meet $M$,

$q$: $L(\Sigma, R) = L(\Sigma, R)$ by $MC$ in $L(\Sigma)$.  

We can assume:

$A = \{(Q, M_Q) \mid Q \in E(P, \Sigma) \} \in \mathcal{P}^*$.

Let $N^*$ be for $\Gamma^*$ s.t. $(N^*_x, \Sigma_x, \Sigma_x)$ satisfies $Q$ in $(P, \Sigma)$, $A$. Let $N = L(E)N^*1_{\Sigma_x}$. ETS: $x \in N$.

For the above, it is enough to prove:

**Lemma**: There is $Q \in E(P, \Sigma) \cap N$ s.t. $\Sigma Q \in L[N]$.

$q$: let $N^* = (L(E, \Sigma Q)1_{\Sigma Q})^N$. By universality $M_Q \subseteq N^*$. But $x \in M_Q$ so $x \in N^* \subseteq N$.

**Proof of the Lemma**

By induction on $B(P, \Sigma) \cap N^*1_{\Sigma_x}$.

**Step 1** $N$ captures a tail of $(P(0), E_P(0))$.

Proof: By what we did before: the least strong cardinal of $N$ is a limit of Woodin cardinals.

Let $y$ be one of them s.t. $y$ is a successor Woodin and a cutpoint.

Let $y$ be a Woodin in $N$ that is not a limit of points $z$ with $L^P(N^{13}) \models E$ is Woodin.
Then $L[E \uparrow \uparrow]$ is an iterate of $P(\omega)$ above the largest $\xi$ as above. We want to construct a direct limit of all iterates of $P(\omega)$ on $\mathbb{N} \setminus \kappa$ (where $\kappa$ is the least strong). This is like the construction of $M_{\kappa+2}$ in $L[\kappa]$. Working on $N$ define the following system.

$I^* = \{ \xi \in \text{SEN}(\mathbb{N}) \mid S \in L[E] \text{ is a suitable name}
\text{ (S has} \ 1\)-weak and \(w\)- and \(v\)-c)
\text{ (S is full (i.e. as certified by \(L[E]\)-construction of N))}
\text{ (for some cutpoint} \ z : S \in H + w + 1 \ \text{and}
H_z \text{ is generic for the extended algebra of S)}

Let
$I = \{ (S, A) \mid S \in I^*, A \in D(N, \kappa) \text{ and } D(N, \kappa) \to S \text{ is A-iterable}
\text{ (new derived model} \}

\text{Let } \mathcal{F} = \{ H_A^S | (S, A) \in I \}$

$R^* = \text{dirlim} (\mathcal{F}, \leq^*) \text{ under the } A\text{-iteration maps}
(\leq^* \text{ as before})$

We showed $R^* = (V_{\mathcal{H}0})^{D(N, \kappa)}$.

\text{Change the definition of } I \text{ !}
\[ I = \text{the set of all } (s, \tau) \text{ s.t.} \]
\[ \tau \text{ is a term for a set of reals in } D(N, \langle \cdot \rangle) \]
\[ s.t. \| s \| \tau \text{ is } \tau \text{-iterable} \]

\[ c \in N \]
\[ \mathcal{T} = \{ h \in c \mid (s, \tau) \in \mathcal{T} \} \]

\[ R^* = \text{dir lim of } \mathcal{T} \text{ under the partial iteration map} \]

Let \( R = Lp_\omega(R^*) \). \( R \in N \) as \( N \) is hull.

Claim: \( R \) is an iterate of \( P(0) \)

Proof: Let \( g \in \text{Col}(\omega, \langle \cdot \rangle) \). In \( N^* \langle g \rangle \) construct

\[ \langle \mathcal{T}_i \rangle_{i \in \omega} \text{ s.t.} \]
\[ P(0) \xrightarrow{\mathcal{T}_0} \mathcal{T}_1 \xrightarrow{\mathcal{T}_1} \ldots \text{ R = dir lim along } \langle \mathcal{T}_i \rangle \text{.} \]

\( \mathcal{T}_i \in (\mathcal{T}_i \text{ on } N^* \langle g \rangle \text{ s.t. } \mathcal{T}_i \text{ is a cutpoint of } N \text{, } Lp(N, \mathcal{T}_i) \text{ is Woodin and } \mathcal{T}_i \to \tau \).

\( \mathcal{T}_i \) is not a limit of \( Lp\)-Woodins. Let

\[ i^* = \sup \{ \beta \mid \beta < \mathcal{T}_i \text{ and } \beta \text{ is } Lp\text{-Woodin} \} \]

\( P_i = L \cap \omega_{i^*+1} \). We have \( P_i \) iterates to \( P_{i+1} \)

(by universality). Let \( \mathcal{T}_i \) be the corresponding tree,

and \( \mathcal{T}_i \) the tree from \( P(0) \) to \( P_0 \).

Let \( \mathcal{T}_i : \mathcal{T}_i \to P_i \) and \( P_\omega = \text{dir lim } P_i \)

under \( \mathcal{T}_i \). Then \( P_\omega = R \).

- \( \mathcal{T}_i \) is guided by some \( \mathcal{T} = \langle A_i, \mathcal{T}_i \rangle \)
- \( A_i \in D(N, \langle \cdot \rangle) \)
- \( \mathcal{T}_i = \bigcup_{j < \omega} (P_i \oplus A_j) \cup (P_{i+1} \oplus A_j) \text{ for } j < n \)
Claim \( \exists R \mid N \in \mathcal{E}[W] \)

Proof: Given \( \Gamma \) on \( N \). If \( \Gamma \) is short then \( N \in \Gamma \) is short (by universality) as \( N \) finds the branch as the unique branch with the \( \mathcal{Q} \)-structure.

Now what if \( \Gamma \) is maximal? Let \( E \) be an extender with \( \alpha(E) = \kappa \). Strength of \( E \implies \forall \mathcal{E}, \mathcal{V}, N \geq \alpha(E) \implies \text{rank}(\mathcal{E}) \geq \text{rank}(\mathcal{V}) \). So \( \mathcal{E} \in \mathcal{A}(N) \). Let \( \mathcal{Q} = L_{\Theta_0}(M(\mathcal{G})) \).

Claim \( E(E) = b \implies \exists \mathcal{Q}: \mathcal{Q} \rightarrow \mathcal{E}(R) \) s.t. \( \mathcal{E} \models \Sigma_0 \).

Proof: Note: \( \mathcal{E}(R) \) is an infinite of \( \mathcal{Q} \).

We have \( \mathcal{Q} \) is \( \Theta \) iterable from the point of view of \( \mathcal{A}(N) \) and \( \mathcal{E}(R) \).

Let \( E^* \) be the background certificate of \( E \)

\[
\begin{align*}
E^* & \rightarrow \mathcal{A}(N_{\mathcal{E}[E]^*}) \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{Q} \rightarrow \mathcal{E}(R) \quad \text{Let} \\
\mathcal{Q} \rightarrow \mathcal{E}(R) \quad \text{Let} \\
\end{align*}
\]

Now \( \mathcal{Q}^* = \mathcal{Q} \mathcal{E}(R) \) is the iteration embedding.

So now \( \mathcal{Q}^* = \tau^* \mathcal{Q} \) (pushed by \( \sigma \)) so \( \mathcal{E} = \tau^* \mathcal{Q} \).

**CASE 1**
We prove \( \text{AD}^{\text{1}(\text{IC})} \Rightarrow \text{AD}^{\text{1}(\text{IC})} \). Let \( A \in \text{1}(\text{IC}) \). Let \( \text{AD}^{\text{1}(\text{IC})} \Rightarrow \text{AD}^{\text{1}(\text{IC})} \), in fact a consequence:

1. \( \text{Pmax axiom (1)}, \) in fact a consequence:
   - \( \text{I}_{\text{NS}} \) (i.e., NS ideal on \( \omega_1 \)) is \( \omega_2 \)-saturated
     and quasi-homogeneous.
   - Quasi-homogeneity: if \( X \in \mathcal{P}(\omega_1) \) and \( X \in \text{OD}_{\text{NS}} \),
     then if there is a stationary-costationary set in \( X \),
     then every stationary-costationary \( A \in \omega_1 \) is \( \equiv_{\text{NS}} \) to some \( A \in X \).

2. Stationaryly many amenable closed hulls \( X \times H_{\omega_3} \):
   - Let \( H \subseteq X \), \( H \text{ transitive} \).
   - If \( A \in \omega_4 \) is amenable to \( H \),
     then \( A \in H \).

3. Every \( f : \omega_2 \rightarrow \omega_2 \) is bounded by a canonical function
   on a stationary set. (This is implied by \( \text{I}_{\text{NS}} \) weakly presaturated.
   \( \text{I}_{\text{NS}} = \text{I}_{\omega_2} \cap \text{cof}(\omega) \)).

First: \( (\text{1}) + (\text{2}) \Rightarrow \text{AD}^{\text{1}(\text{IC})} \)

\[ (\text{1}) \Rightarrow \quad j : V \rightarrow \mathcal{M}[\mathcal{G}] \text{ with } j(\omega_1) = \omega_2 \text{ and } \mathcal{V}[\mathcal{G}] \text{ is a transitive model}\]
\[ \forall x \in R \quad \text{HOD}_x \text{ is amenable to } V. \text{ What}\]
\[ j \text{ says about OD}_{\mathcal{R}} \text{-sets is independent of generics}\]

\[ (\text{2}) \Rightarrow \text{mouse (strategy reflection on } \omega_2 \text{ (external from } \text{OD}_{\mathcal{R}}\text{))}
   \text{ (Reference: Steel : PFA } \Rightarrow \text{AD}^{\text{L}([\mathbb{R}])}\text{.)}\]

\[ (\text{3}) \Rightarrow \quad H \ni X \times Y \text{ (large) is } \text{AC (amenably closed)}
   \text{ and } X \times \omega_2 \in \omega_2 \text{, } 1 \times 1 = \omega_1 \text{ then } A \in (H_{\omega_3})^+ \]
\[ \text{then } L^+(A) \in H.\]

To get \( \text{AD}^{\text{1}(\text{IC})} \). Using \( j \) extend mouse reflection
at \( \omega_1 \). Important point: \( j(K) \in V \) for local \( K = K^{\text{HOD}}(\mathcal{R}) \).
Now to get past $\kappa(\aleph)$:

- **Required for $\theta > \theta_0 = \Theta H(\aleph)$:**
  - $\text{cof}(\theta_0) = \omega$
  - Strategy with condensation at $\theta_0$

Let $H = HOD_{\kappa(\aleph)}(\theta_0)$, $\overline{H}_{\kappa(\aleph)}$ collapses in AC hull $\mathcal{H}^{\mathcal{H}}$

**Claim:** $\delta^+ = L_{\kappa(\aleph)}(\overline{H}_{\kappa(\aleph)})$ is suitable in $j(\kappa(\aleph))$.

**Proof:** Let $T$ be a tree for $(\mathbb{Z}, \leq)$ in $(\kappa(\aleph), j(\kappa(\aleph)))$.

$L_\kappa(\kappa(\aleph)) = L_{\kappa(\aleph)}(\overline{H}_{\kappa(\aleph)}) = L_{\kappa(\aleph)}(L_\kappa(\overline{H}_{\kappa(\aleph)})) = L_\kappa(\kappa(\aleph))$

But $L_{\kappa(\aleph)}(\overline{H}_{\kappa(\aleph)}) = \theta_0$ is the only worldly $HOD$

**Claim:** $\text{cf}(\theta_0) = \omega$ (Under our hypothesis $2^{\theta_0} = \aleph_2$).

**Proof:** $\text{cof}(\theta_0) = \omega_1, \omega_2$. Otherwise $\text{cof}(\theta_0) = \omega_1$ by amenable closure. Let $\gamma = \sup j(\theta_0) < j(\theta_0)$.

Let $E_{\theta_0} = \{ \gamma \} J_{\theta_0}$. $\mathcal{L}[\gamma, \theta_0] \xrightarrow{\gamma} \mathcal{L}[i(\gamma), i(\theta_0)] \xrightarrow{\gamma} \mathcal{L}[j(\gamma), j(\theta_0)]$

$\emptyset = \mathcal{Q}(i(\theta_1)) \in \mathcal{U}_{\theta_1} \emptyset$.

To see $\theta_0 < \theta_3$:

Let $h: \mathbb{w}_2 \rightarrow \mathbb{w}_1$.

Define $f: \mathbb{w}_2 \rightarrow \mathbb{w}_2$

$f(h) = \mathbb{w}_2 \wedge L_\kappa(\mathbb{w}_2 \wedge h[\{\gamma\}])$

Take $g: \mathbb{w}_2 \wedge \beta \rightarrow \mathbb{w}_3$ s.t.

$f(g) \leq \text{otp}(q[\{\gamma\}])$ for some $\beta$.

Take $Y \subset \mathbb{w}_3$. Let $\pi: X \hookrightarrow Y$ with $X$ transitive.

Here $\mathcal{L}[\mathbb{w}_1, \theta_1]$

$d = \mathbb{w}_1 \wedge \mathbb{w}_2$ s.t. $f(d) \leq \text{otp}(q[\{\gamma\}])$

$\pi^{-1}(\theta_0) \leq \mathbb{w}_1 \wedge L_\kappa(\mathbb{w}_1 \wedge h[\{\gamma\}]) \leq \text{otp}(q[\{\gamma\}]) = \pi^{-1}(\theta)$

Take $A \leq \text{OD} \cap P(\mathbb{w}_1)$ countable, $\omega$-closed, assume $A \in X$ (Big AC hulls).
In $K(R)$ we have $F$, $I$ giving $M_\alpha | \sigma_\alpha = HOD_{K(R)} | \sigma_0$.

Collapse $\overline{F}, \overline{I}, \overline{M}_\alpha$ ($HCM$).

In $j(K(R))$ let $F_{\mathcal{A}} ((N_i, A_i) \cup \emptyset)$ be increasing definable in $\overline{I}$. Let $M^*_\omega \triangleleft \text{quasi-limit } N_i$ wrt all $\mathcal{M}(K(R))$.

Quasi-limit: maps are only defined on reasonable hulls

$$\overline{M}_\alpha \equiv H_{\mathcal{A}}$$

$\overline{M}_\alpha$ is suitable and $\mathcal{A}$-iterate strongly $\mathcal{A}$-iterable

for $\overline{M} \in \text{Card}(K(R)) \forall \mathcal{A}$

$$\overline{M}_\alpha = M^*_\omega = H^+ \quad (H^+ = L_{\mathcal{A}} (HOD_{K(R)} | \sigma_0))$$

One can check: For every quasi-iterate $W$ of $\overline{M}_\alpha$:

$$W = H^+_{\mathcal{A}}$$

($\mathcal{A}$-c.c. for extending algebra)

So $\mathcal{A}$ guides a strategy $\Xi$ for $\overline{M}_\alpha$.

**Lemma** $\Xi$ has weak condensation, that is: if

$e : \overline{M}_\alpha \to S$ factors into a $\Xi$-iteration map there $\overline{M}_\alpha \to W$ then $S$ is suitable.

(We assume $S$ is countable in $\overline{M}$.)

Show: $\pi = j (\overline{M}_\alpha)$ (Exercise)

(This is because the elements of the direct limit system are countable, so they do not move.) So $e$ factors in $j$:

$$e^* : L[e(c), S]$$

$$L[e(c), j(M_\alpha)]$$

$$j : L[e(c), j(M_\alpha)]$$
By elementarity of \( j: \mathbb{N}^* \rightarrow K(\mathbb{R}) \) s.t. \( A \) guilts a strategy \( \Sigma \) for \( \mathbb{N}^* \) with weak condensation + Dodd-Jensen property.

**Claim** A tail of \( \Sigma \) has branch condensation.

**Proof** Otherwise denote \( N_0 \xrightarrow{\Sigma} N_{n+1} \xrightarrow{\Sigma} N_0 \). This is an instance of a "bad" sequence and can be obtained in \( K(\mathbb{R}) \) for all \( d < \omega_1 \):

\[
S: N_0 \xrightarrow{\Sigma} N_{n+1} \xrightarrow{\Sigma} N_{n+2} \quad N_d \text{ is suitable.}
\]

Assume: \( S \) is in a \( p_{\max} \) extension of \( K(\mathbb{R}) \).

\((\xi) \Rightarrow \forall X \leq \omega_1, X \in L(\mathbb{R}) \text{ or } X \text{ is in a } p_{\max} \text{ extension of } K(\mathbb{R}) \)

Take \( p \in K(\mathbb{R}) \) least s.t. \( \frac{1}{p} \) "bad sequence of length \( \omega_1 \)."

Take \( U \) a universal \( (\Sigma^2)^P \) set. Let \( B \in P \) be a \( \xi \)'s, \( U \in B \), \( T \) a tree for \( \oplus B \) given by \( B \).

In \( p_{\max} \) let \( P = X \times H(\omega_2), X \text{ is } U \)-iterable, phase \( P \) has a bad sequence of length \( \omega_1 \). Let \( x \) code \( P \). In \( L[T, x] \) iterate \( P \). Get a bad sequence \( (N_d, b_{0,1} \leq b_{1,2} \leq \cdots, 2 < \beta < \omega_1) \).

Let \( p = \omega_1 [\beta, x] \) and \( N_p = \lim \lim_{\alpha \in \beta} N_{\alpha} \) (suitable.)
$L[T \times]$ knows sequence of $T^N_{\mathcal{B}_i}$'s. \( \exists \delta_0 < 2 \) s.t.
\( \exists n \in \omega \land V \subseteq \text{rng}(j_{\delta_0}) \) By term condensation.
\( \forall x > \delta_0 \delta_{x+1} \) maps terms correctly (since it comes from a collapse) \( \delta_0 \).

Show: \( AD^L(\mathbb{R}) \Rightarrow (L^2(\mathbb{R}) \Rightarrow L(\Sigma, \mathbb{R})) \)

By quasi-homogeneity get \( j(T') \in V \), \( T' \) a tree for \( \mathcal{B}_0 \).
Based at \( \delta_0 \) guiding \( \Sigma \). In \( V_{\text{Coll}(v,w)} \) use \( T' \) to identify good branches (since \( j(T') \in V \)). Then continue as for \( AD^L(\mathbb{R}) \) in Steel-Zobele: Determinacy from strong reflection.
RECALL \((P,E)\) captured by \(N^*\) where \(N = L[E]^\cup \Delta^x\).
\[\exists \mathcal{Q} \in \mathcal{I}(P,E) \cap N \text{ s.t. } \mathcal{Q} \uparrow N \in L[N].\]

By induction on \(I(P,E) \cup B(P,E)\) we showed capturing for \((P_0, \mathcal{Q}, \mathcal{R}_0)\), \(\exists \mathcal{Q} \in N \cap \mathcal{I}(P_0, \mathcal{R}_0) \text{ s.t. } \mathcal{Q} \uparrow N \in L[N].\)

This is the general case

Suppose \(Q \in \mathcal{I}(P,E) \cup B(P,E)\) \& \(x^*\) is a successor \& \(\exists \mathcal{Q} \in \mathcal{I}(Q^*, E_{\mathcal{Q}^*}) \cap N \text{ s.t. } \mathcal{Q} \uparrow N \in L[N].\)

Want: Find \(R \in \mathcal{I}(Q^*, N^*) \cap N \text{ s.t. } \mathcal{Q} \uparrow N \in L[N].\)

Let \(N^* = (\mathcal{Q}, E_{\mathcal{Q}}, N)\).

As before, the least strong of \(N^*\) is a limit of Woodins.

Let \(x\) be this strong. Repeat the construction from above.

Let \(R^* = \bigcup \mathcal{Q} \uparrow E_{\mathcal{Q}}\) \& \(R = L[x]_{\mathcal{Q} \uparrow E_{\mathcal{Q}}}(R^*).\)

\(R \uparrow N^*\) because of unirelatedness.

Then use extenders with cr.pt. \(x\) to get \(\mathcal{Q} \uparrow N^* \in L[N].\)

So: Given \(T\) on \(R\) let \(E\) be on \(N^*\) an extender with \(cr(E) = x\) \& \(E \uparrow x^* > \text{rank}(T).\)

Let \(b \uparrow \mathcal{T}\) be the branch of \(T \in \mathcal{T} \cap N \text{ s.t. } j(b) \uparrow T = \sigma \uparrow i_b.\)

Need \(\mathcal{Q} \in N \in L[N]\)

Let \(T\) be a tree in \(N\). Let \(x = \text{rank}(T).\)

Let \(N^{**} = (\mathcal{Q}, E_{\mathcal{Q}}, N)\). Use extenders with cr.pt. \(x\)

Then let \(\lambda^x\) be the least strong of \(N^{**}\). We have \(R^{**}\), the version of \(R\) in \(N^{**}\). Also, \(T\)

is generic over \(N^{**}\) at the least Woodin.

Let \(U\) be the situation (map from \(R\) to \(R^{**}\), \(U \in N^*\).

Then, by the same proof as before,
This finishes the general successor case.

**Limit Case (General)** Suppose \( \mathcal{Q} \in (\mathcal{T}(\mathcal{P}, \mathcal{E}), \mathcal{B}(\mathcal{P}, \mathcal{E})) \) and \( \mathcal{R} \models \mathcal{Q} \) and \( \mathcal{Q} \in \mathcal{D}(\mathcal{R}_1, \mathcal{E}_1) \) if \( \mathcal{N} \) s.t. \( \mathcal{E}_1 \models \mathcal{N} \) [\( \mathcal{N} \)]

**NTS** \( \mathcal{R} \in (\mathcal{Q}, \mathcal{E}_1) \) and \( \mathcal{N} \) s.t. \( \mathcal{E}_1 \models \mathcal{N} \) [\( \mathcal{N} \)]

Assume \( \mathcal{Q} \) is countable (otherwise we have to do the same we did in general case above.)

Let \( \kappa \) be the least strong of \( \mathcal{N} \) that reflects the set of strong cardinals.

**Lemma:** If \( \mathcal{S} \in \mathcal{B}(\mathcal{Q}_1, \mathcal{E}_1) \cap \mathcal{N} \) and \( \kappa \) then \( \mathcal{S} \) is "captured" below \( \kappa \). This means:

\( \mathcal{E}_1 \models \mathcal{R} \in (\mathcal{S}, \mathcal{E}_1) \) \( \cap \mathcal{N} \) s.t. \( \mathcal{E}_1 \models \mathcal{N} \) [\( \mathcal{N} \)]

**Proof:** Suppose mot. Let \( \mathcal{R} \in (\mathcal{S}, \mathcal{E}_1) \) \( \cap \mathcal{N} \).

**NTS:** \( \mathcal{R} \) \( \cap \mathcal{N} \) [\( \mathcal{N} \)]

Let \( \kappa_1 > \text{rank}(\mathcal{R}) \) a strong cardinal of \( \mathcal{N} \). Let \( \mathcal{E} \) an extender on \( \mathcal{N} \) s.t. \( \mathcal{E}(\mathcal{E}) = \kappa_1 \),\n
\( \mathcal{E}(\mathcal{E}) > \kappa_1 \), \( \mathcal{E} \) reflects the set of strongs.

Then \( \kappa_1 \) is strong in \( \mathcal{L} \mathcal{U}(\mathcal{N}, \mathcal{E}) \). Moreover:

because \( \mathcal{V}_{\kappa_1}^\mathcal{N} \subseteq \mathcal{L} \mathcal{U}(\mathcal{N}, \mathcal{E}) \) \( \cap \mathcal{E} \) \( \cap \mathcal{L} \mathcal{U}(\mathcal{N}, \mathcal{E}) \) (by capturing). Now let \( \lambda \) be the strategy

of \( \mathcal{R} \in \mathcal{L} \mathcal{U}(\mathcal{N}, \mathcal{E}) \) we get by stretching \( \mathcal{E} \) \( \cap \mathcal{V}_{\kappa_1}^\mathcal{N} \)

using the strategies with \( \mathcal{R} \) s.t. \( \in \mathcal{L} \mathcal{U}(\mathcal{N}, \mathcal{E}) \).

**NTS:** If \( \mathcal{E}_1, \mathcal{E}_2 \in \mathcal{L} \mathcal{U}(\mathcal{N}, \mathcal{E}) \) and \( \mathcal{E}(\mathcal{E}_1) = \kappa_1 = \mathcal{E}(\mathcal{E}_2) \)

then \( \mathcal{J}_{\mathcal{E}_1} (\mathcal{E} \cap \mathcal{V}_{\kappa_1}^\mathcal{N}) \cap \lambda_1 = \mathcal{J}_{\mathcal{E}_2} (\mathcal{E} \cap \mathcal{V}_{\kappa_1}^\mathcal{N}) \cap \lambda_1 \)

where \( \lambda_1 = \mathcal{L}(\mathcal{E}_1) \leq \mathcal{L}(\mathcal{E}_2) \). To see hull condensation (Possibly branch condensation ².)
Claim: \( \Lambda = \sum \Pi \Gamma \text{wt}(N, E) \)

Proof: Let \( E^* \) be the background certificate of \( E \). Let \( N \rightarrow j_\pi^*(N) \)

\[ \sigma(N) = \text{the strategy on } j_\pi^*(N) \text{ of } \sigma(R) = R \text{ given by stretching } \sum \pi_0 \Gamma \text{wt}(N) \text{ using extended with } \sigma = R. \]

Note: \( V_{\pi_0}^N = V_{\pi_0}^{j_\pi^*(N)} \) \( \sigma \Gamma \) \( V_{\pi_0}^N = \Gamma \)

ETS: \( \sigma(N) = \sum \Pi \Gamma j_\pi^*(N) \) and then use hull and extend.

However, the equality is clear since the extenders \( j_\pi^*(N) \) do move \( E \) to itself as they are actual extenders. \( \square \)

Now working in \( N \) form the following limit. Let \( \mathcal{F} \) be the set of all \( (\xi, \Lambda) \) s.t.

- \( (\xi, \Lambda) \) is a local pair
- \( \Lambda \) is FPR + BC
- \( \xi \in V_{\pi_0}^N \)

Here "FPR" means that it is certified by background construction. Let \( \preceq \) the natural relation \( M_\infty = \text{dlc}(\mathcal{F}, \xi \leq \xi) \) under the iteration maps. \( M_\infty \in H_{\text{ht}} \).

Let \( \Lambda^* = \bigoplus_{\Delta \in \Lambda_\infty} \Lambda_{M_\infty(\Delta)} \) where \( \Lambda_{M_\infty(\Delta)} \) is the common IS coming from some \( (\xi, \Lambda) \) s.t. \( \xi \) iterates to \( M_\infty(\xi) \) via \( \Lambda \).
Claim. For some $x \in X^I$, $R^*(x)$ is a $\Sigma_\varphi^1$-iterate of $\varphi$.

Proof. $M = \operatorname{dir lim} \text{ of all } \Sigma_\varphi^1 \text{-iterates of } \varphi$

that are in $U_{N^*}$.

Then $M \subseteq R^*$. Why? Let $F^* = \{ F \subseteq \mathbb{B}(\alpha, \varepsilon_0) \cap U_{N^*} \}$

Let $M^* = \operatorname{dir lim} \text{ of } F^*$, then $M^* \cap M = M^*$.

But $M^* \subseteq R^*$ because everything in $\mathbb{B}(\alpha, \varepsilon_0) \cap U_{N^*}$

is captured below $\varepsilon_0 \in \mathbb{N}$.

Let $x \in R^*$ be s.t. $R^*(x)$ an iterate of $\varphi$.

Let $R = R^*(x)$. WTS: $\exists \mathbb{N} \varepsilon \exists \mathbb{N} N \varepsilon \mathbb{L}[\mathbb{N}]$

Case 1. $(x^R)$ is not measurable.

Pt Easy: $\exists \mathbb{R} = \bigoplus_{\alpha < \mathbb{R}} \mathbb{R}_{\mathbb{R}(\alpha)}$ and $\exists \mathbb{R} \varepsilon \mathbb{L}[\mathbb{N}]$ \forall \alpha \in \mathbb{R}

Case 2. $(x^R)$ is measurable.

Notation. Given $S \subseteq \mathbb{B}(\alpha, \varepsilon_0) \cap N^* \mathcal{L}_x$ we let

$(R^S, \xi^S) \in \mathbb{L}[\mathbb{N}]$ be the captured tail of $S$.

(\text{i.e. } R^S \in \bigoplus (S, \xi_S), S^S = \xi^S \cap \mathbb{N})$

The challenge is to guess

```
\varepsilon \rightarrow x_{\varepsilon}(R) \quad \text{The actual embedding}
```

\[ M^* \rightarrow j_{\varepsilon}(R) \]

Let $E = \text{the set of all extenders on } R$

with cut pt $\varepsilon$ that reflect the set of all strays.

+ Working in $\mathbb{W}$ define $A$ a strategy for $R$

as follows, given $F$ on $R$ with essential components
\[ \tilde{\tau} = \langle M_\alpha, M_\beta^w \rangle \to \nu \beta \ (\nu \beta < \gamma) \] is via \( \lambda \) via

\[ \exists (\tilde{R}_\alpha, \tilde{M}_\alpha) \text{ s.t.} \]

4. \((\tilde{R}_\alpha, \tilde{M}_\alpha)\) is a had pair

2. If \( \gamma \in \mathbb{E} \) s.t. there is a strong cardinal between \( \text{rank}(\tilde{R}_\alpha), \text{lh}(E) \)

\[ \lambda_\alpha \upharpoonright \mathcal{U}(N, E) \in \mathcal{U}(N, E) \]

3. If \( \lambda \) is a strong cardinal s.t. \( \lambda > \text{rank}(\tilde{\tau}), \text{rank}(\tilde{R}_\alpha) \)

\[ \forall \gamma \in \mathbb{E} \text{ with } \text{lh}(E) > \lambda \exists \tilde{\tau}^{E \upharpoonright \lambda \leq \gamma} \text{ s.t.} \]

- \[ \tilde{\tau}^{E \upharpoonright \lambda} = j_E \upharpoonright R = M_\delta \]
- \[ \tilde{\tau}^{E \upharpoonright \lambda} : M_\delta \to j_E(R) \]
- \[ \tilde{\tau}^{E \upharpoonright \lambda} = \tilde{\tau}^{E \upharpoonright \lambda} \circ \tilde{\tau}^{E \upharpoonright \lambda} \]
- \( R_\alpha \) iterates via \( \lambda_\alpha \) to \( \tilde{\tau}^{E \upharpoonright \lambda} (M_\delta^E) \)
- \( \tilde{\tau}^{E \upharpoonright \lambda} \) is according to \( \lambda_\alpha \upharpoonright \tilde{\tau}^{E \upharpoonright \lambda} (M_\delta^E) \) (\( \tilde{\tau}^{E \upharpoonright \lambda} \) is tame on \( M_\delta^E \))

5. Moreover, if \( \tilde{\tau}^{E \upharpoonright \lambda} \) is a stack on \( M_\delta^E \), \( \lambda \leq \gamma \), \( \mathbb{E} \in \mathbb{E} \)

as in 3 and \( \langle \tilde{\tau}^{E \upharpoonright \lambda}, \lambda \leq \gamma \rangle \) are as in 3 also

\( \lambda \) a strong cardinal as in 2 then

\( b = \lambda^{+E \upharpoonright \lambda} (\tilde{\tau}^{E \upharpoonright \lambda}) \leftrightarrow b \) is the unique branch

s.t. \( \exists \tilde{\tau}^{E \upharpoonright \lambda} = j_E(R) \) s.t. \( \tilde{\tau}^{E \upharpoonright \lambda} = \tilde{\tau}^{E \upharpoonright \lambda} \).

Claim: \( \tilde{\tau}^{E \upharpoonright \lambda} \) is also \( \tilde{\tau} \) up to the last branch \( \tilde{\tau} \) according to \( \tilde{\tau} \).

Proof: Suppose \( \tilde{\tau}^{E \upharpoonright \lambda} \) according to \( \tilde{\tau} \), \( \tilde{\tau} = \langle h_\alpha, M_\alpha^w \rangle \to \nu \beta \) \( \mu \beta \)

\[ \text{WTS: } \tilde{\tau} \text{ is according to } \tilde{\tau} \].

Fix \( \langle \tilde{R}_\alpha, \tilde{M}_\alpha \rangle \ (\lambda \leq \gamma) \) for \( \tilde{\tau} \). [let \( \lambda \) be a strong cardinal > rank \( \{ h_\alpha^E \upharpoonright \lambda \leq \gamma \} \), \( \mathbb{E} \in \mathbb{E} \) with \( \text{lh}(E) > \lambda \). We have: \( \tilde{\tau}^E \upharpoonright \lambda = N_\lambda \), \( \tilde{\tau}^E \upharpoonright \lambda \), \( \mathcal{U}(N, E) \)]

DELETE
Let \( \langle \pi_x \mid x \leq \gamma \rangle = \langle \pi^E_x \mid x \leq \gamma \rangle \) and let \( E^* \) be the background certificate of \( E \). Let \( \tau : \text{ult}(N,E) \rightarrow \text{ult}(N,E^*) \).

Want to define a strategy for \( M^*_x \) on \( N \). Let
\( \lambda > \text{ rank} \{ R^M_x \mid x < \gamma \} \), a strong. Given \( \bar{u} \) on \( M^*_x \) let \( \tau \) be a strategy for \( M^*_x \) be defined as:
\[
\tau_x (\bar{u}) = a = \forall \xi \in E \text{ with } \text{ul}(E) > \lambda
\]
\[
\pi^E_x (M^*_x (\bar{u}^\lambda)) = a \quad \text{for } x \leq \gamma
\]

Claim: \( \tau_x = \Sigma_{M^*_x} \cap N \)

Proof: Induction on \( x \).

\( x = 0 \): Given \( \bar{u} \) on \( M^*_0 \) we need to \( \text{ult}(N, \Sigma_{M^*_0} \cap N) \)
\( \tau_0 (\bar{u}) = \Sigma_{M^*_0} (\bar{u}) \quad \text{(but } \Sigma_{M^*_0} \cap N \in L[N]) \)
Let \( E \in E : \Sigma_{M^*_0} \cap \text{ult}(N,E) \in \text{ult}(N,E) \)

So by the above picture for \( x = 0 \):
\[
(R_0, R_0) \cap \text{ult}(M^*_0, \Sigma_{M^*_0}) \quad \text{compare } \lambda_0 \text{ and } \lambda^* \text{ common place before going to } \tau_0 (M^*_0).
\]
So \( \Sigma_{\tau_0} = \Sigma_{\tau_0} \cap N \) by hull condensation.

\[
(\Delta_{\tau_0})_{\tau_0 (M^*_0)} = \Sigma_{\tau_0 (M^*_0)}
\]
Case 2+1 Suppose \( \mathcal{Y}_\alpha = \mathcal{E}_{M^*_\alpha} \).

WTS: \( \mathcal{Y}_{\alpha+1} = \mathcal{E}_{M^*_{\alpha+1}} \).

The problem if \( \pi^E: M^*_\alpha \rightarrow j^E(R) \) is an iteration embedding via \( \mathcal{Y}_\alpha \) then the proof for \( \alpha = 0 \) works here.

Things got messed up now. Go back to 1 and start again.

Define a strategy \( \mathcal{A} \) for \( \mathcal{R} \) in \( N \). Given

\[ \mathcal{T} = (\mathcal{T}_2, \mathcal{M}_2, \mathcal{M}^*_2, \mathcal{Y}_2) \]

a tree according to \( \mathcal{A} \) iff \( \exists (\langle R, \lambda_\gamma \rangle, 1 \leq \gamma \leq \eta), \langle \mathcal{Y}_\beta, 1 \leq \gamma \rangle \in \mathcal{T} \)

1. \( \langle R, \lambda_\gamma \rangle \) are hod pairs
2. \( (\mathcal{M}^*_2, \mathcal{Y}_2) \) is a hod pair s.t. \( \mathcal{Y}_2 \) is according to \( \mathcal{Y}_2 \)
3. \( \mathcal{Y}_\beta \) extends \( \mathcal{T}_2 \) for \( \beta > \alpha \)
4. For every \( E \in \mathcal{E} \) s.t. \( E \) is strong cardinal

\[ \lambda \in \text{rank}(R, 1 \leq \gamma), \lambda \in \mathcal{E} \] there is a sequence

\[ < \pi^E_\alpha, 1 \leq \gamma > \]

5. \( \pi^E_\alpha: M^*_\alpha \rightarrow j^E(R) \)
6. \( \pi^E_\alpha = \pi^E_\beta \circ \mathcal{Y}_\beta \)
7. \( \pi^E_\alpha \) is the iteration map by \( \mathcal{Y}_\alpha \)
8. \( \mathcal{R}_\alpha \) iterates to \( \pi^E_\alpha(M^*_\alpha) \) in \( \mathcal{M}_2 \)
9. \( (\mathcal{Y}_\alpha)^{\pi^E_\alpha}_{\mathcal{R}_\alpha(M^*_\alpha)} = \mathcal{Y}_\alpha \)
10. \( \mathcal{Y}_\alpha \) is \( \mathcal{R}_\alpha \)

5 For every \( \alpha \leq \eta \) and \( E \in \mathcal{E} \) s.t. \( \lambda \in \mathcal{E} \) with \( \lambda > \text{cf}(\lambda_\gamma) \)

\[ \mathcal{Y}_\alpha(\lambda) = 0 \rightarrow 1 \text{ is unique s.t. } \exists \pi^E_\alpha(M^*_\alpha) = \mathcal{Y}_\alpha \]
Claim \( \lambda = \Sigma \)

Proof Let \( \delta \) be according to \( \lambda \) and \( \gamma \) without its last branch be according to \( \Sigma \)

Let \( \langle \alpha_\xi, \lambda_\xi \rangle \mid \alpha \in \gamma \rangle \), \( \langle \tau_\xi, \delta \leq \gamma \rangle \) be as in the definition

Claim \( \forall \xi, \exists H_\xi \succ N \)

Proof By induction on \( \alpha \).

\( \alpha = 0 \) Trivial

\( \alpha = n + 1 \) We have \( \Sigma_{\delta} \succ N = \Sigma_{\tau} \). Let \( \bar{t} \)

be a tree of limit length according to both \( \tau_{n+1} \) and \( \Sigma_{\delta_{n+1}} \), WTS \n
\( \tau_{n+1}(\bar{t}) = \Sigma_{\delta_{n+1}}(\bar{t}) \). Let \( E \in \mathcal{E} \n
be s.t. there is a strong cardinal in the interval \( (\text{rank}(R_{\delta_{n+1}}(x)), \text{lh}(E)) \).

Define \( \tau : \mathcal{H}_{n+1} \rightarrow \mathcal{H}_{n} \) is the embedding

Claim \( \forall \xi, \exists H_{\delta+1} : H_{\delta+1} \rightarrow H_{\delta} \succ (\Sigma_{\delta}) \times \Sigma_{\delta_{n+1}} \)

is the iteration embedding according to \( \xi \).

Proof Let \( x \in H_{\delta+1} \). Then

\( x = \iota^{-1}_{\delta, \delta+1}(e)(a) \) where \( a \) is a generator of \( \delta \).

So \( \tau_{\delta+1}(x) = \tau_{\delta+1}(\iota_{\delta, \delta+1}(e)) = \tau_{\delta}(e) \tau_{\delta+1}(a) \)

But \( \tau_{\delta+1}(a) \) comes from the iteration embedding according to \( \xi \), \( \tau_{\delta}(e) \times \tau_{\delta+1}(a) \).

By the IH : \( \tau_{\delta}(e) \times \tau_{\delta+1}(a) = \tau_{\delta_{n+1}}(e) \times \tau_{\delta_{n+1}}(a) \)

Now we have \( \forall \xi, \exists H_{\delta+1} \in \text{Ult}(N, E), \n
\tau_{\delta+1} \succ \text{Ult}(N, E) \in \text{Ult}(N, E) \n
= \Sigma_{\delta+1} \succ \text{Ult}(N, E) \)
Since $\rho_{d+1}^* M_{d+1}^*$ is the iteration embedding according to $\Sigma_{M_{d+1}^*}$, we get $\forall_{d+1} = (\Sigma_{\rho_{d+1}(M_{d+1}^*))}^\rho_{d+1} = \Sigma_{M_{d+1}^*}$

Thus shows that $\mathcal{G}$ is $\forall_\alpha\Sigma_\alpha$ because $\mathcal{G}_{\forall_\alpha}$ is according to $\forall_\alpha = \Sigma_{M_{\forall_\alpha}}$.

To finish, need to show that given $\mathcal{G}$ according to $\mathcal{L}$ s.t. $\mathcal{G} = \langle M_\alpha, M^\forall_\alpha, \vartheta_\alpha, \vartheta_\beta \mid \alpha, \beta \leq \gamma \rangle$ with $M^\forall_\gamma$ undefined, then we can find $M_{\forall_\gamma}$ continue $\mathcal{G}$. Let $M^\forall_\gamma \leq \mathcal{G} M^\forall_\gamma$ be the least limit segment for which we haven't defined $\mathcal{L}$. Let $\langle R_\alpha, \lambda_\alpha \rangle \mid \alpha < \gamma$, $\langle \lambda_\alpha, \lambda_\beta \rangle$ witness that $\mathcal{G}$ is according to $\mathcal{L}$. Define a strategy for $M^\forall_\gamma$ as follows. Given $\mathcal{G}$ on $M^\forall_\gamma$ let $E \in E$ be an extender with $lh(E) > \text{rank}(\mathcal{L})$ and $E$ strong cardinal $\lambda \leq lh(E)$, $\lambda > \text{rank}(R_{\lambda} M^\forall_\gamma)$. Let $b$ be the branch of $\mathcal{L}$ if $\mathcal{G}$ is iff $\mathcal{G}^*: M^\forall_\gamma \rightarrow \rho_{\alpha} (M^\forall_\gamma)$ s.t. $\rho_{\alpha}^* M^\forall_\gamma = \mathcal{L} E b$. There is always such a $b$ and is according to $\Sigma_{M^\forall_\gamma}$. Why?
Let $\pi^*_y M^*_b$ be the image of $\pi^*_y M^*_b$.

By null condensation,

$b = \pi^*_y (\tilde{u}^*)$ and $\pi^*_y (M^*_b)$ is an iteration of $M^*_b$.

So let $\sigma: M^*_b \to \pi^*_y (M^*_b)$ be the iteration map.

Then $\pi^*_y M^*_b = \sigma \circ \tilde{u}^*$ (by commutativity). \qed
Derived models of H"{o}e

Direction 1

% on a mouse with \( \kappa \) a limit of Woodin. Study \( D(\kappa, \lambda) \)

Direction 2

Given a model \( V \) of \( \text{AD}^+ \) find (Pinky force)

a presiover \( M \) s.t. \( V = D(M, \lambda) \).

Connect the theory of \( M \) with that of \( D(\kappa, \lambda) \).

Mouse operators

Examples

(1) \( \kappa \) is countable transitive, most often self-well-ordered

\[
M^{\text{ad}}(\kappa) = \text{the minimal active mouse } M \text{ over } \kappa
\]

s.t. \( M \models \text{AD}_{\text{Ir}} \text{-hyp} \)

\[
\text{AD}_{\text{Ir}} \text{-hyp} = \exists X \times \text{limit of Woodins and } < X \text{ shows}
\]

Remark \( M \models \text{AD}_{\text{Ir}} \text{-hyp} \Rightarrow D(M, \lambda) \models \text{AD}_{\text{Ir}} \)

We already proved this without requiring \( \kappa \text{-mouse} \)

We will show: If \( \text{AD}_{\text{Ir}} \) holds then there is a "Pinky

general" \( G \) s.t. in \( V[G] \) there is a presiover \( M \models \text{AD}_{\text{Ir}} \text{-hyp} \)

\( \kappa \mapsto M^{\text{ad}}(\kappa) \) is an operator of interest.

\( 1 \) Means \( a \) has a well-ordering that is not dense.
(2) \( \mathcal{M}_c^* (\alpha) = \text{minimal mouse satisfying} \)
\[ D(\mathcal{M}_c^* (\alpha) \models AD_{\aleph_2} + \text{DC} \Rightarrow cf(\theta) > \omega) \]

(3) \( \mathcal{M}_c^* = \exists ! \lambda \text{ inaccessible limit of Woodins} \)
\[ \text{and } \lambda \leq \theta \text{-strong} \]
\[ D(\mathcal{M}_c^* \models \theta = \theta_{\omega_1}, \text{ still}) \]

(4) \( \mathcal{M}^* \models \exists ! \lambda \text{ minimal mouse with a Woodin limit of Woodins} \)
\[ \text{Will show: } D(\mathcal{M}^* \models \lambda \models AD_{\aleph_2} + \theta = \theta) \]
\[ \text{So } \theta_{\omega_1} < \theta. \]

Open question: Does \( D(\mathcal{M}^* \models \lambda \models AD_{\aleph_2} + \theta = \theta) \) regular?
Known some \( L(\Gamma_{1R}) \) satisfies \( AD_{\aleph_2} + \theta \text{ regular} \)
where \( L(\Gamma_{1R}) \models D(\mathcal{M}, \lambda) \text{, (Sargsyan.)} \)

Reference: Steel: Derived models associated to mice.

Operators have the form \( \alpha \mapsto (M(\alpha), \lambda(\alpha)) \)
The important property: are "tractable".
If \( M = M(\alpha) \) and \( \alpha \in \text{col}(\omega, \alpha) \) generically \( (\nu < \lambda) \)
and be \( \text{HC}^{R \theta} \), then \( \text{M} \gamma \theta \text{ can rebuild } M(\beta) \text{ using its sequence of certificates}. \)
So "\( M(\alpha) \text{ reconstructs itself below } \lambda \)".
Today: Start with one of these operators \( a \rightarrow (M(a), \lambda(a)) \) 
Investigate \( \mathcal{D}(M(a), \lambda(a)) \). We are assuming that each 
\( M(a) \) has a \( \text{s-Horn} \) iteration strategy. Since the \( M(a) \) 
projects to \( a \), such a strategy is unique. Assume 
there are arbitrary large Woodin cardinals in \( V \).

For such \( M = M(a) \) and \( I \) is \( \mathcal{R} \)-genericity situation 
of \( M : M = M_0 \xrightarrow{\tau_0} M_1 \xrightarrow{\tau_1} \cdots \xrightarrow{\tau_m} M_{\xi} \) 
where 
- \( \tau_0 \upharpoonright \xi \upharpoonright \cdots \upharpoonright \eta \upharpoonright \cdots \) is named by \( \mathcal{E} = \mathcal{E}_M \) (unique) 
- \( \mathcal{R}^G = \mathcal{R}^V \) for some \( G \) on \( \text{cof}(\omega, < \xi^V) \) 
- \( I \) itself is \( \mathcal{R} \)-generic over the natural part of 
\( \langle \tau_0, \cdots, \tau_m \rangle \)’s.

Then we have, letting \( \text{Hom}_I^* = \text{Hom}_G^* \) for any and all 
\( G \) s.t. \( \mathcal{R}^G = \mathcal{R}^V \).

(a) \( \text{Hom}_I^* \leq V \) (as \( \mathcal{E}_M \in V \))
(b) \( \text{Hom}_I^* = \text{Hom}_G^* \) even if \( I \) is for some \( M(b) \), \( b \neq a \).
See the paper.

**Lemma** For any of our \( a \rightarrow M(a) : \delta(M(a), \lambda(a)) \rightarrow M_C \) 
\( \text{v.e. } x \in \text{OD}(\delta) \) iff \( x \in M_{\lambda(a)}^V \) -\( \mathcal{E} \)-name.

\( M(a) \) satisfies: \( M(a) \upharpoonright \omega_1^M(a) \) is \( \omega_1 \)-iterable.

\( M(a) \vdash \mathcal{U} \) for the trees based on its extenders 
sequence; this uses the \( \mathcal{P} \)-iterability of \( M(a) \) in \( V \).

\( M(a) \) re-builds \( M(a) \upharpoonright \gamma \) (\( \gamma < \omega_1^{M(a)} \)) with background 
extenders arbitrarily high (this generalizes 
for \( M(a) \upharpoonright \gamma \) for small \( \gamma \).

(1) Denote it by \( \mathcal{E}_M(a) \)
So $M(a)$ can iterate using UBH and CBH for trees of size $< \kappa$.

Gives $\Sigma$ for $M(a)$ $\uparrow$ which is hom $\kappa$.

**Corollary** if $a \mapsto (M(a), \lambda(a))$ is a tractable operator s.t. $M(a) \models \lambda(a)$ is a limit of cutpoints then $D(M(a), \lambda(a)) \models \Theta_0 = \Theta$.

**Proof.** If not: have $f \in D(M_1^{\updownarrow}, \lambda_1) = L(Hom_1^x, R_1^x)$ s.t. $f: \mathbb{N} \rightarrow \mathbb{N}$, $f(x) \notin OD(x)^D(M_1^{\updownarrow})$ all $x$ and $f \notin Hom^x$.

Then have: a cutpoint of $M_1, (T_1, \nu) \in Hom_1^1$ with $p \subseteq \{ (x, \omega, \mu) \mid f(x) \uparrow \}$, let $z \in \mathbb{N}$ be a code of $<M_1, \Theta_1>$. Then $f(z) \in M(\nu)$:

$f(z) \in M(\nu)$ because of $T$.

$M(\nu) \uparrow \uparrow w_1^{M(\nu)} = M(\nu) \uparrow \uparrow M(\nu)$ so $f(z) \notin OD$ in $D(M_1^{\updownarrow}, \lambda_1)$.

$M(a) \uparrow \uparrow \lambda(a)$ can be computed in $D$ for any tractable countable transitive $a$.  

**Next goal.** Identify the sets with Wadge rank $\Theta$ for $\Theta < \Theta$ in $D(M_1^{\updownarrow}, \lambda_1)$.

For one of our $M$-operators:

Set $\Theta_0 :$ its operator $a \mapsto M^x(a)$ in $D$, otherwise $M^x(a) \notin M(a)$. Why it is in $D^2$? Use a $\Theta_0 + 4k$ that it be strong up to $\lambda$.

$M^x(\lambda)$ make element of $M_1^{\uparrow}$ $\uparrow$ an end of $<\omega, \chi>$

$\eta$ meet Wadin $> \kappa$. Now shift $M^x(M^x)$. (Do rebuilding of the sheath modeluple $M$.) Thus gives earlier strategy below the bottom Wadin.
Given $\Sigma$ an IS for $P$ and a countable self-well-ordered $\sigma \in P$, $\Sigma$ satisfies hull condensation and condenses well = support closed supertrees of $T$ by $\Sigma$ are also by $\Sigma$.

$F^{\Sigma}(M)$ is defined for $M$ a model over $\sigma$, and it is itself a model over $\sigma$ $(M, e, B, E, S) = \tilde{M}$ where $S^m_3$ is the $3$-rd level of $M$.

$F^{\Sigma}$ is a model operator.

Given a model $M$ over $\sigma$, we get $F^{\Sigma}(M)$ as follows:

Let $T$ be the least tree on $P$ in the canonical well-order of $M$ ($\sigma$ is self-well-ordered) such that:

1. For $\lambda$ limit $< \text{lh}(T)$, $\exists \beta, \gamma \in \text{lh}(T) \; \exists \beta < \gamma < \lambda, \beta < \gamma$ and no $B^{S^m_3}$ is itself a cofinal branch of $T$.
2. Let $\lambda = \text{lh}(T)$

Case 1: Let $y$ be least s.t. $P^I(T)(M) \neq P^I(M)$ or $F^{\Sigma}(M)$

$$F^{\Sigma}(M) = (\exists y \in \mathcal{P} \forall z < y \exists z \in \mathcal{P} \exists (T) \forall \varphi_1 \in \mathcal{P} \forall \varphi_2 \in \mathcal{P} S^m_3 < N)$$

Case 2: $y = \lambda$

$$F^{\Sigma}(M) = (\exists y \in \mathcal{P} \forall z < y \exists z \in \mathcal{P} \exists (T) \forall \varphi_1 \in \mathcal{P} S^m_3 < N)$$

This structure is amenable. Check conditions $F^{\Sigma}$

Here we assume $M = C/B(\lambda)$ not measurable.

With this, one can do $F^{\Sigma}$-constructions and $F^{\Sigma}$-dictatorship.

1. Things that take more: generic interpretability: Take an $F^{\Sigma}$-measure $M$. Does $M$ have a term for $Z_0 M$?
(2) Another related topic: translations of $E$-nice $M$ over some $q$ to $E$-nice over $M [< q, q']$ where $q$ is call $(w, q')$-generic.

(3) a most self-well-orderd, e.g. $E$-nice over $\mathbb{R}$.
One approach: "tell the mice it is $E$-mouse of $\text{HOD}$.
On it $(p, E)$ a had mouse.
Sample problem: let $(p, E)$ be a had pair. Show $\forall \theta \in M$ with $M \vDash (p, E) \in \text{HOD}(\theta)$. Show $T^2(\mathbb{R}) = V(\theta^+)$.
Special case we know this: assuming $\text{MBC}$:

\[ T^2(\mathbb{R}) = V(\theta^+) \]

Get fine structure of mice this way.

(\text{D36 FROM D36})

Remark: To see that $(p, \mathbb{R}, \delta)$ codes at most one real.

Suppose $(\mathbb{N}^3, \delta), (\mathbb{N}^3, \delta)$ witness code $r, \delta$.

If $\mathbb{N}^3$ is a limit pt of the club on which these sequences agree:

$\mathbb{N}^3 \rightarrow \mathbb{N}^3$ as $\gamma \rightarrow \xi$. This is the key.

\[ (\text{WITNESS}) \]

Corollary If $M$ is an inner model, both $M$, $V$ models of $\text{BPFA}$, $w^M = w^V$ then $P_\omega(w^M) \subseteq M$.

\[ \text{Definition (Moore). Mapping Reflection Principle, MRP}.
\]

Let $\theta$ be a regular cardinal, $X$ an uncountable set, $M < H_\theta$ countable and $[X]^\omega \subseteq M$. A set $\varepsilon(M) \subseteq \theta$ is $\text{M}$-stationary if

$\forall \xi \in [X]^\omega$ club of $\varepsilon(M)$ then $\varepsilon(M) \cap \varepsilon(M) = \emptyset$.

( $\varepsilon(M)$ is not required to be in $M$.)
SUMMARY

1. We defined MOD pairs.
2. We proved comparison (2 arguments: 0 and AD$^+$ using $W^*$).
3. We computed MOD under AD$^+$+SIC up to $\Theta_{+1}$ under the assumption that $\Theta_{+2}$ exists. This under the minimality assumption (AD$^+_{\omega_1}$ is regular).

4. We proved: AD$^+ + \forall \alpha (\text{AD}_{\omega_1} + \& \Theta_{\text{regular}}) \Rightarrow \text{SIC}.$
5. We proved: “Weak limit of Woodin” $\Rightarrow$ Con(AD$^+_{\omega_1}$+$\Theta_{\text{regular}}$).

Actually from divergent models.

Exercise (Not easy) The proof of (4) can be used to get (5) more directly.

DIALL NOT DO

1. Branch condensation in limit cases (Getting a pair $(P, E)$ of $\tau^\omega$ limit, $E$ with BC).
2. The internal theory of MOD more - how to interpret strategies.
3. Given $\Gamma$ c.b. $L(P, \alpha) \vDash \text{AD}^+ + \text{SIC}$ and more
   Stationary cardinals $> \Gamma$ then there is a pair $(P, E)$ s.t. $P(P, E) = \Gamma$. Here we want: $\Gamma = \delta(12) L(P, \Omega(12)).$

Theorem: $\text{BPFA} \Rightarrow \exists \Delta_1^1(A) \text{ w.o. of } IR \text{ for some } A \subseteq \omega_1.$

Conjecture 1: (Vaillant?) $\text{MM} \Rightarrow$ there is a definable w.o. of IR

Conjecture 2: Assume $\text{MM}.$ Let $M$ be an inner model with the same cardinals as then $\omega_1 M \leq M.$

Proof of Thm.: First, a coding of reals: let $\mathcal{C}$ be a C-sequence:

$\mathcal{C} = \langle C_\alpha \mid \alpha < \omega_1 \text{ lim} \rangle.$ $\omega_3 \leq \beta,$ $\check{\alpha} = \omega_1$ if $\check{\alpha}$ is a limit. $\text{sup} C_\alpha = \beta.$

Def. (Oscillation map) (Todorcevic) $\text{osc} : (\aleph_\omega)^3 \rightarrow 2^{\omega_1}.$

Let $x, y, z \leq \omega.$ Let $\equiv$ on $\omega - \omega$ be the equivalence relation given by $\equiv_{\omega_1 \omega}$ iff $(\min (m,n), \max (m,n)) = \check{\omega}.$

Let $(I_k)_{k < \omega}$ be the enumerating enumeration of the corresponding intervals. Equivalence classes that meet both $y, z.$ Define $\text{osc} \equiv (x, y, z) : \omega \rightarrow 2^{\omega_1}$ by $\text{osc} (x, y, z)(k) = 0 \Leftrightarrow \min ((I_k, y) \leq \min ((I_k, z)).$

Def: Let $\alpha < \beta < \gamma < \delta$ be limit ordinals. Suppose

(1) $N \subseteq M \subseteq S$ one countable set of ordinals

(2) $\text{sup} (\beta \cap N) < \text{sup} (\beta \cap M)$ for $\beta = \omega_1, \beta, \gamma, \delta$

(3) $\text{sup} (\alpha \cap M)$ is a limit ordinal, $\beta$ as in (2)

Under these conditions the pair $(N, M)$ codes $\equiv$ on a finite string (of ordinals) as follows.

Let $\pi : M \rightarrow S$ be the transitive collapse

$\delta_M = \pi (\omega_0), \beta_M = \pi (\beta), \gamma_M = \pi (\gamma), \delta_N = \text{sup} \pi [\omega_1 \cap N]$
The height of $\omega_n \text{ in } m \text{ vs } M = \omega(n, M) = \omega_\omega \cap C_{\omega_n} M$

Set $x = \{ \exists \eta \in \beta M \mid \exists \zeta \in \beta \cap \eta N \}$

$\eta = (\zeta, M) \omega M - \omega M$

$\zeta = (\zeta, M) \omega M - \omega M$

Note: $x_1, x_2$ are finite.

Define $S_{\beta_0 \omega_1}^\omega (N, M) = \text{osc} \left( \frac{\omega}{x_n} \right) \in \Gamma_n$

$= \text{osc} \left( \frac{x_n - y_n - z_n}{x_n} \right) \in \Gamma_n$

If $\text{dom} \left( \text{osc} \left( \frac{x_n - y_n - z_n}{x_n} \right) \right) \geq n$

otherwise set $S_{\beta_0 \omega_1}^\omega (N, M) = *$

Also let $S_{\beta_0 \omega_1}^\omega (\emptyset, N, M) = * \text{ if } \emptyset > \omega_1$.

The triple $(\beta_0 \omega_1, \delta)$ codes a real $r \in 2^{\omega_1}$ iff $\exists (N_3, 3 < \omega_1)$

continuous sequence of countable sets with union $S$ s.t.

$\forall \emptyset \text{ limit } S \in 2^{\omega_1}$ s.t. $r = \bigcup_{\emptyset \in S} S_{\beta_0 \omega_1}^\omega (N_3, N_3)$

Note $(\beta_0 \omega_1, \delta)$ codes at most one real.

Theorem (C-V) (BPFA)

(1) If $\omega_1 < \beta < \omega_1 < \delta < \omega_1$ are of cofinality $\omega_1$

then there is an increasing sequence $(N_3, \omega_1 < \xi)$

consisting of countable sets with $\bigcup N_3 = \delta$ s.t.

$\forall \emptyset \text{ limit } \exists \zeta < 3 \exists S_3 \in \text{iso}(\omega_1, \omega_1) \delta \text{ s.t.}$

$S_{\beta_0 \omega_1}^\omega (N_3, N_3) \cap m = S_3 \text{ for all } n \in (\omega_1)$

(2) For each real $r$ there are $\omega_1 < \beta < \gamma < \delta < \omega_1$ of

cofinality $\omega_1$ s.t. $(\beta_0 \omega_1, \delta)$ codes $r.$
Corollary (BPEA) There is a $\Delta_1$ w.o. of $\mathcal{P}(w_1)$ with a parameter a $\mathcal{C}$-sequence. The length of w.o. is $w_2$.

Proof: Let $\mathcal{C}$ be a $\mathcal{C}$-sequence. If $N$ is a finite model of enough $\text{ZFC}$ and $\mathcal{C} \in N$ then $w_1 = w_1$ and there is a $w_1$-sequence of a.d. reals $\mathcal{C}$ such that $N$ correctly identifies $\mathcal{C}$.

Let $\mathcal{C}$ be the theory $\text{ZFC} + \forall x (\exists y \exists z (\exists ! x \mathcal{C} \in N \wedge \mathcal{C} \cap x \neq \emptyset) \wedge (1 \times 1 \subseteq \mathcal{C})$.

Key: If $\mathcal{C} \in M_1, M_2 \models \mathcal{C}$ and $\text{On} \cap M_1 = \text{On} \cap M_2$ then $M_1 = M_2$.

Suppose $M \models \mathcal{C}$. Let $\beta_3, \delta \in \text{On}$ and suppose $(\beta_3, \delta)$ code $\mathcal{C}$ in $V$. Then $\beta_3 \in M$.

Let $(\beta_3, \delta)$ witness $(\beta_3, \delta)$ code $\mathcal{C}$.

Since $M \models (1)\beta \exists \delta \in M$ a sequence $\langle \beta_3, \delta \rangle$ witnessing $\mathcal{C}$ for $(\beta_3, \delta)$. Then $\beta_3 = \beta_3$ on a club of $\beta$.

Now by (2) $\mathcal{C} \models (2)$ follows:

$x M = \{ \alpha \mid \exists (\beta_3, \delta) \in M (\beta_3, \delta) \text{ codes } x \}$

Using a.d. coding: $\mathcal{P}(w_1)^M$ is characterized this way.

Similarly: if $M_1 \models \mathcal{C}$ and $M_2 \models \mathcal{C}$ then $M_1 = M_2$. We can define a w.o. of $1\mathcal{R}$ by $\alpha \mathcal{S} \iff$ letting $\mathcal{C}_2$ be the least s.t. $\exists M \models \mathcal{C}_2,$ $o(M) = \mathcal{C}_2 \in M = M_\alpha$ we have $\mathcal{C}_2 < \mathcal{C}_3$ or $\mathcal{C}_2 = \mathcal{C}_3$ and $M_\alpha$ is the least anti-lex.

This is $\Delta_1(\mathcal{C})$: $\alpha \mathcal{S}$ iff some/all model of $\mathcal{C}$...
The Ellentuck topology on $[X]^w$: A $\subseteq [X]^w$ is open if $A$ is a union of sets of the form $[x, B] = \{ Y | x \in Y \subseteq B \mid Y \text{ countable} \}$ for $x \in B$ finite, $B \in [X]^w$.

$E$ is an open stationary mapping $\forall \xi \in \omega_1 \times \omega_1$ s.t. $[X]^\omega \subseteq H_\xi$, $\text{dom}(E)$ is a club in $[H_\xi]^w$ consisting of countable elementary substructures and $\forall M \in \text{dom}(E): E(\mu, \xi) \subseteq [X]^w$ is $M$-stationary and open in the Ellentuck topology.

Example: Let $M < H_\xi$ be countable, $d = \omega_1 \cap M$. Let $d = \kappa_0 \cup \kappa_1$ be a partition. Then (for $X = \omega_1$) either $[\omega_1]^\omega - \kappa_0$ or $[\omega_1]^\omega - \kappa_1$ is $M$-stationary. Because $\omega_1 - \omega \subseteq [\omega_1]^\omega$ is club. Exercise.

Now note that if $s \in \text{proj}_6 (N, M) = 2^N$ then there is $\tau \in N$ finite s.t. $\forall Y \in [\tau, N] \ s \in \text{proj}_5 (Y, M) = s$.

Definition (MRP): If $E$ is an open stationary map then there is a reflecting sequence for $E$, i.e. a sequence $(N_\xi \mid \xi < \omega_1)$ continuous increasing, $N_\xi = \omega$ and each $N_\xi \in \text{dom}(E)$ s.t.

$\forall \xi \text{ limit } \exists \eta < \xi \text{ s.t. } N_\eta N_\xi \subseteq E(N_\xi)$ all $\eta \in (\omega, \xi)$

Theorem (Moore): PFA $\rightarrow$ MRP

Proof idea: Let $P_\xi^\omega = \text{ the set of } p : d + 1 \to \text{dom } E$ s.t.
$\omega_1$, $\kappa$ continuous, $\varepsilon$-increasing

$$\forall \beta \leq \gamma \lim \exists \gamma < \beta \forall \delta \in \varepsilon(\beta) \delta(\beta) \wedge \delta \in \Sigma(\varepsilon(\beta))$$

The part $P_{\omega_1}$ is proper. This gives: if $G$ is sufficiently generic then $UG$ has domain $\omega_1$.

Proof of the main theorem. In fact: MRP $\Rightarrow$ (1), (2).

So these are proper forcings giving us these statements and they are $\Sigma_1(\varepsilon)$, so BPFA suffices.

Proof of (1). Recall: $\omega_1 < \beta < \gamma < \delta < \omega_2$, $\beta_1, \beta_2 < \delta$.

WTF $(N_3, 13, \omega_1)$ countable $N_3$, increasing, continuous,

$$U N_3 = 5 \text{ s.t. } \forall \delta \in \varepsilon(\beta_1) \wedge \delta \in \varepsilon(\beta_2) \wedge \delta(\beta_1) \cap \delta(\beta_2) \neq \emptyset.$$

Given $(\beta, \gamma, \delta)$: $h \times (N_3, 13, \omega_1)$ incomplete, $N_3$ cba, $\lambda_1 = \omega_1$,

$$U N_3 = 5 \text{ s.t. } \sup(N_3 \cap \rho) \text{ is limit for } \rho \in \omega_1 \setminus \beta \wedge \delta(\beta) \wedge \delta(\beta_1) \wedge \delta(\beta_2) \text{ (Easy)}$$

Also want the steps are increasing.

Fix $m$: For $n$ a limit let

$$k^m_n = \{ \gamma < \delta \in \beta \subseteq (N_3, N_4) \cap \delta = \varepsilon \} \text{ for } \delta \in 2^m \wedge \varepsilon \in \gamma.$$  

Set $\Sigma(m) = \{ N \subseteq [5]^\omega \mid s_{\beta \delta} (N, m, \delta) = s_m \}$

where $s_m \in [5]$. For $d = M \omega_1$ with $k^m$ is $m$-standard.

MRP gives a sequence for $\Sigma$: $(N_3, 13, \omega_1)$

let $C \subseteq \omega_1$ be a club subset on which $N_3, N_4$ coincide.

Then $(N_3, 13, C)$ works.

Proof of (2). Let $\kappa \in \mathcal{H}_0$ regular large, $\kappa > \theta_0$ countable, $\nu \in \mathcal{H}_0$.

$$\Sigma^\kappa(\nu) = \{ N \subseteq [\kappa]^{\omega_4} \mid s_{\omega_2 \omega_3 \omega_4} (N, \kappa) \subseteq \nu \}$$
If $\Sigma^1$ is open stationary and $\langle N_3 \downarrow \beta < w_1 \rangle$ is sufficiently generic for $P_{\omega_1}$ then letting $(M_i \downarrow \beta < w_1)$ be the sequence of transitive collapses this gives $(\beta, \gamma, \delta)$ coding $\gamma$ as witnessed by the $M_i$. So $BFA$ gives this.

To see that $\Sigma^0(\Delta)$ is stationary use the game:

$$
\begin{array}{c|c|c|c}
I & \beta_0 & \gamma_0 & \delta_0 \\
--- & --- & --- & --- \\
I & \omega_1, \gamma_0 & \lambda, \delta_0 & \mu, \gamma_0 \\
\end{array}
$$

Let $F : I_{\omega_1} \times \Delta \to \Delta$, $F \in M_1$, $\Delta \subseteq w_1$

$\square F : \beta_i, \gamma_i, \delta_i$ are increasing below $w_2, w_3, w_4$

$\beta_i \leq \gamma_i \leq \delta_i < w_2 \quad \gamma_i < \beta_{i+1}$

$\gamma_i < \lambda_i \leq \gamma_i < w_3 \quad \lambda_i < \delta_{i+1}$

$\delta_i \leq \mu_i \leq \nu_i < w_4 \quad \nu_i < \delta_{i+1}$

$X = \bigcap_{\omega \in \omega_1} \{ \bigwedge_{\omega \in \omega_1} (\nu_i, \delta_{i+1}) \}$

$X$ wins off $X \cap \omega_1 = \emptyset$

$X \cap [\beta_i, \beta_{i+1}] \in [\beta_i, \beta_i]$.

The game is open for $I$.

**Lemma.** For club many $\alpha$ $I \downarrow \alpha$ has a winning strategy in $F_\alpha$.

**Standing the lemma.** Given $M$ we want $\Sigma^0(\Delta)$ be $M$-stationary. For this let $F \in M_1$; use the Club of

the lemma be in $\Gamma$. Let $\sigma : \Delta \to \text{winning strategy} \cdot$

Let $(w_0, \xi_0) = \sigma(\phi)$, $\xi_0 = \sup \xi_\delta \xi_\delta \in M_{\omega_2}$

Let $k = h_0(\beta_0) = h_{\xi_0}(\beta_0) \in M_{\omega_2}$

I was unable to record the game
We showed \( D(M^I_{\omega_1}, \bar{\omega}^I_{\omega_1}) \) via \( \Sigma^I_{\omega_1} \) for our model operators \( * \) independent of \( I, \alpha \).

**Question:** How general is this? Is this true for any sound mouse projecting to \( \omega \) with \( \omega^L \times \omega \) limit of Woodin? Also: \( \Sigma^I_{\omega} \& \Pi^I_{\omega} \). When \( \Sigma^I_{\omega} \) the Wedge least \( \& \Pi^I_{\omega} \)?

**Showed:** \( D(M\chi) \neq \text{MSC} \) - in fact \( \forall \alpha \exists \eta \in \text{any } \alpha, \delta \in \eta \)

(a ctd bs transition)

(1) \( b \in \varphi(D(M^I_{\omega_1}, X^I_{\omega_1})) \)

(2) \( b \in M(\alpha) \)

(3) \( b \in \text{an } \omega \text{-stable mouse over } \alpha \).

(2) \( \rightarrow \) (3) last time (3) \( \rightarrow \) (2) trivial, (1) \( \rightarrow \) (2) by

symmetry of the collapse.

**Defining:** \( M^* \) - one of our operators with hypotheses \( * \).

\( M^*(\alpha) = \text{Sharp of the minimal } M^\alpha \text{-closed model of } \varphi \)

\( M^{\alpha+1}(\alpha) = \text{Sharp of the minimal } M^\alpha \text{-closed model of } \varphi \)

\( M^\alpha(\alpha) = \bigcap M^\beta(\alpha) \text{ def limit, } \delta \in \alpha \). (Projects to \( \alpha \)?)

\( \beta < \delta \) here \( \alpha < \omega_1 \)

**Then:**

(1) For \( M = M^* \text{ ad } A \text{ new } \)

\( a \rightarrow M^*(a) \in D(M^I_{\omega_1}, X^I_{\omega_1}) \)

(a ctd transitive)

Moreover: \( F^\alpha \) has Wedge rank approximately \( \Theta^\alpha \) in \( D(M^I_{\omega_1}, X^I_{\omega_1}) \)
(2) For \( M \) one of the remaining three: For \( \alpha < \omega_1 \):

\[
F^\alpha : \alpha \rightarrow D_{M^\alpha} \setminus A \subseteq \omega^\alpha \text{ for } A \subseteq \omega^\alpha.
\]

(\( \alpha \in D(M^\alpha, X^\alpha) \)) of Wadge rank approximately \( \Theta \).

(a) In cases 2, 3 \( M^\omega \setminus M^\omega_{\text{lim}} \) these are \( \leq_w \) coheren.

(\( \text{So } D(M^\alpha, X^\alpha) \) is \( \Theta = \Theta_{\omega_1} \) in such cases)

(b) Let \( M^\omega_{\text{lim}}(\alpha) = M^\omega(\alpha) \) \( \omega_{\omega_1} \text{ is 1st admissible in a ordinal} \)

Let \( F : \alpha \rightarrow M^\alpha(\alpha) \) then for \( M = M^\omega_{\text{lim}} \): \( F \in D \)

and \( F \upharpoonright \omega \approx \Theta \omega_1 \). In particular: \( \Theta \omega_1 < \Theta \)

For instance: \( M = \text{one of the 4 operators: Consider } \alpha \rightarrow \Pi^0 \).

(1) Do the \( \Pi^0 \)-genericity iteration \( I \) s.t.

\( \eta_0 = \text{the least } \zeta \text{- strong critical pt. } > \eta_0. \)

\( \delta_1 = \text{the least Woodin } > \eta_0. \)

Let \( g \) be generic over the iterate \( N \) of \( M_{\text{iter}}. M \in \text{EN}_g \)

In \( N_{\text{iter}} \) compute \( M^\omega \)-operator \( \forall \langle \gamma, N_{\text{iter}} \rangle \)

(\( \alpha \in V_{\gamma +} \), \( \nu < \gamma \). Take \( p \in L(E) \) \( \text{where } E \text{ on the } \eta - \text{sequence with } E(\nu) = \nu, E(\epsilon) > \eta \).

\( \mathcal{P}_{\lambda, \theta} = \mathcal{P}_{\lambda, \theta} \).

So \( a \in PC_{g, \lambda} \). So use \( PC_{g, \lambda} \)-extenders

to rebuild \( M(\alpha) \in V_{\nu_{\gamma+}, g, \lambda} \).

Then: Exercise: \( N_{\text{iter}} \) \( B \) has \( T_1 \) that are \( \zeta - a.c. \)

s.t. in \( M^\zeta \), \( p \in \mathcal{E}(\gamma) \) is the operator \( \alpha \rightarrow M^\alpha(\alpha) \)

(by condensation.)

Note: \( \alpha \rightarrow M^\alpha(\alpha) \) cannot be \( OD(\alpha) \) in \( D \) as otherwise \( \mathcal{M}^\alpha(\alpha) \in M^\alpha(\alpha) \).
On the other hand: If

\[ T^m(x) = \text{the theory of first n indiscernibles of} \]

Then for any real \( x \): \( T^m(x) \equiv \bigoplus_{i=1}^n T^m(x) \). Each \( T^m(x) \in T^m(x) \)

Use this to see that \( T^m \) is OD(\( \mathbb{R} \)).

Why is \( a \rightarrow T^m(a) \) at \( \Theta_1 \) in \( \Delta(\mathbb{N}_A \times \mathbb{N}_A) \)?

Exercise 1: \( f \) is \( M \)-generic / coln(\( w, n \)) then \( M[G] \) does

not have a.c. \( (\text{TI}) \) with \( p(\text{I}) = M \)-operator.

M \( \models \text{GI} \). Notice: \( M[G] \equiv M^d(\text{W}1(\text{I} \times \mathbb{N}), g) \)

M \( \models \text{GI} \). Notice: \( M[G] \equiv M^d(\text{W}1(\text{I} \times \mathbb{N}), g) \)

\[ \delta_0 \rightarrow \delta_0 \]

Hint from R to L: From \( \sim \text{real} \) \( \mathbb{R} \)

\[ \mathbb{R} \]

2 we get \( g \) and \( M \text{I}1(\delta_0) \), ETS:

\[ M \text{I}1(\delta_0) \text{ can reconstruct extenders with crit pt. } 1_0 \]

These can be identified as follows:

reconstruct how subsets of \( n_0 \) are moved. For this use \( M \)

which is \( \omega \)-sound, so \( \tau \)-sound and the canonical

function \( f: \tau \rightarrow \delta_0 \) \( (\text{G}(\tau)) = \text{G}(\delta_0) \).

\( M^\delta(\tau) \) knows how

\( f \) should be shifted for it to become a \( \tau \)-operator,

as it knows \( a \rightarrow \mathbb{N}(a) \). Use this to repeat our previous

argument but with \( M^\delta(\tau) \) replacing \( M^\delta(\tau) \).

Now start with \( V = \mathbb{A}^\tau + \Theta_0 \subset \text{G} \). We get \( M \text{ s.t.} \)

\( M \models \text{X is \( \mathbb{A}^\tau \)-hyp.} \) (We use MSC here.)

First: Pick a forcing horn preface \( P \) with Woodin

cardinals \( \delta_0, \delta_1, \ldots \) such that \( (\text{L}^{\text{\text{W}}}(P\text{I}(\delta_0), \text{E}_0)) \) is a

bad pair. This can be done, as we assume \( \Theta_0 \subset \text{G} \).

\( \text{Think of } \text{E}_0 \text{ as } \text{a } M^\delta \text{-operator.} \) \( (W(\text{E}_0) = \Theta_0) \)

Let \( \delta_1 \) be the least cardinal in \( P \) strong to \( \delta_0 \). Build

\( \text{Wt: Build } \text{L}(E) \text{ on } P \text{ in which } \delta_1 \text{ is strong up to } \lambda = \text{sup} \delta_0 \).
Change inotation: Write $N^+$ for $P$.

Let $N = L[\mathbb{E}^\mathbb{N}]_{\mathfrak{N}^+\lambda}$ built over $\mathcal{N}^{\mathfrak{N}^+\lambda}$, Extend $N$ to a $L[\mathbb{E}^\mathbb{N}]$ model of height = ord $\mathfrak{s}$ s.t.

$N^+ \models \lambda$ is a limit of Woodin $+ \kappa_1 < \lambda$-strong

We will show: at sufficiently good $\eta < \mathfrak{z}$ there is a ''translation process'' where-by for any $\eta < \mathfrak{z}_1$ s.t. $\eta < \mathfrak{z}$ and $N \models "\eta \text{ is Woodin}"$, so $N^\mathfrak{z}_1\eta \equiv \mathfrak{n}$-generic over

$N^\mathfrak{z}_1\eta$ for $\eta$-generator extender algebra.

$(N^\mathfrak{z}_1\eta)(N^\mathfrak{z}_1\eta) \equiv \mathfrak{z}_{\mathfrak{z}_1\eta} \text{ - mouse over } N^\mathfrak{z}_1\eta$. for all $\mathfrak{z}_1 < \mathfrak{z}$

"ahm.

(proved by induction on $\mathfrak{z}$)

This goes in both directions. above done by $T_\mathfrak{z}$.

The translation is tight to the fixed $\eta$, so $T_{\mathfrak{z}_1}$ (has a definition from $\eta$). Now apply the definition of $T_{\mathfrak{z}_1}$ to levels of $N^\mathfrak{z}_1\eta$ past $\mathfrak{z}_1$, and $\exists \mathfrak{z}_1 < \mathfrak{z}$ with $T_{\mathfrak{z}_1} \mathfrak{z}_1 = \text{the result of replacing } 2 \mathfrak{z}_1$ is $\lambda$-strong $\eta$ by $\mathfrak{z}_1$ in the definition.

Then $s(\mathfrak{z}_1) = \eta$ is the resulting model by a reflecting argument.

The $\tau$-transform: Given $M$ $\omega$-sound, $\mathcal{P}_\mathfrak{z}^\mathfrak{z} = \mathcal{P}$, $(\mathfrak{z}_1, \tau)$-itable

(typically $M = M(\tau)$ for one of our operators.) Let $T$ be a tree on $M$ according to the strategy with the least model $N$.

Let $S$ be a cardinal in $N$ not measurable in $N$ . Let

$$(\mathfrak{z}_1, \tau) = \langle (\mathfrak{z}_1, \tau) \mid \mathfrak{z}_1 = \text{c} \in \mathcal{E} \rangle \text{ some } \mathcal{E} \text{ on } N$$

sequence with $\text{lh}(\mathcal{E}) \geq \mathfrak{z}(\mathfrak{z}_1)$

$$(\mathfrak{z}_1, \tau) = \langle (\mathfrak{z}_1, \tau) \mid \mathfrak{z}_1 = \text{c} \in \mathcal{E} \rangle \text{ some } \mathcal{E} \text{ on } N$$

Then one define for all $\mathfrak{z}_1 \leq \mathfrak{z}_1$:

$$(N^\mathfrak{z}_1\eta)(\mathfrak{z}_1, \tau) \equiv \text{ a mouse over } (N^\mathfrak{z}_1\eta, \mathfrak{z}_1)$$
s.t. $\phi(x)$ is "fine structurally equivalent" to $(\phi(x^{\dagger}))^\dagger$ modulo $<\neg\phi, \exists y>$. Given $\phi$ s.t. $\neg\phi \leq \phi$ and can be reached by extenders overlapping $\Delta$ from $\Delta$, define $\phi(\neg\phi^{\dagger})^\dagger$:

(a) if $\phi(\neg\phi^{\dagger}) = \omega \pm \omega \quad \phi(\neg\phi^{\dagger}) = \text{ and } \phi(\neg\phi^{\dagger})$,
(b) if $\neg\phi$ limit then $\phi(\neg\phi^{\dagger}) = \bigcup_{\phi(x^{\dagger})}$
(c) if $\phi$ is active with least extender $E$, say $\phi = \{E\}$,

\begin{itemize}
  \item (i) if $\phi(x) > \Delta$ then $\phi(\neg\phi^{\dagger}) = (R_{\neg\phi^{\dagger}})^{\dagger}$
  \item (ii) if $\phi(x) < \Delta$ then $\phi = R_{\neg\phi^{\dagger}}$
\end{itemize}

Details: Thesis of Eric Closen. Shows that the fine structure correspondence.

5.8.2010 9:30 JOHN STEEL

One application: For $M$ any of the first three operators:

(1) $D(M_1, \lambda) \models \theta = \theta_m$

(2) (3) $D \models \theta = \theta_m$.

Proof sketch: If $A \in D(M_1, \lambda)$, $A \in \mathbb{R}$.

$M_0 \rightarrow \cdots \rightarrow M_n \rightarrow \cdots$ generically obtained

$\uparrow \quad (M_n, \omega, \Sigma_{M_n})$ Suslin captures $A$.

$\uparrow (\omega, \Sigma_{\omega})$ Generators of $\Sigma_{\omega}, \ldots, \Sigma_{M_n}$ are below $\Delta$.

$\uparrow \quad (\omega, \Sigma_{\omega})$ (i.e., the relevant $\Sigma_{\omega}, \ldots, \Sigma_{M_n}$ have been made generic earlie, so it is in $M_n^{\neg\phi^{\dagger}}$), let $y$ be the next Woodin $> \Delta$.

We can read off $\Sigma_{M_n} \setminus \text{trees in the window } (\Delta, \lambda)$

using the $*$-transform. non-dropping
Find $\mathcal{Q}(\mu)$. (= the $\mathcal{Q}$-structure)

The common part model.

$L_{\mathcal{N}} \to S$

Coeval $(S, \mathcal{Q})^+, \mathcal{Q}(\mu)$. This is a $\mathcal{Q}$-structure

in the other hierarchy; and is $\preceq M^\beta(M(\mu))$ for $\beta \leq \delta$

where $\delta$ = the number of cardinals strong past $\gamma$ in $M_\gamma$.

Gives: $A$ is projective in the $\mathcal{M}^\delta$ operator for this $\delta$.

$\mathcal{C}(\kappa)$

2nd application For $M = M^{	ext{Ord}^\kappa}$: $D(M(\lambda)) \subseteq \Theta^\kappa < \Theta$.

For this want to show: $\kappa \to M^\kappa(\omega) \subseteq$ on $D(M^\kappa(X))$

some $A$. Let $M \models \mathcal{A}$ - reflecting on $\lambda \equiv \text{top} $ Woodin

for $A = Th_M(1, \lambda)$

$\lambda = \text{least Woodin} > \kappa$. Choose $\kappa$ s.t. $M \models \mathcal{C}(q)$.

Claim For any cardinal $\gamma$ of $\mathcal{N}(q)$

and any $\alpha \leq \omega_1$

$M^\alpha(N\mathcal{C}(q)) \in \mathcal{N}(q)$

Given this: can then use the proof to see $M^\alpha \models \mathcal{N}(q)$; then get $M^\alpha \models D$ using condensation.

Proof of Claim Show by induction on $\alpha < \lambda$. For any such $\alpha$:

$M^\alpha(N\mathcal{C}(q)) \in \mathcal{N}(q)$, in fact:

$M^\alpha(N\mathcal{C}(q)) \subseteq Q[\mathcal{G}]^{<\gamma}$ for some proper initial segment $Q$ of $\text{Ult}(N, F)$ where $F_\alpha = \text{1st} \text{ extender overlapping}$

$\gamma$ from $N$ with $\alpha(E) > \kappa$ or $D$. Of DNE.

Exercise 2 limit: Show the induction step at $\alpha$.

(Stacking the things)

Now do the successor step $\alpha \to \alpha + 2$. 
Let $A = \text{Th}^{\mathcal{P}_2}(k(\lambda))$. Let $E$ be on $\mathcal{P}_2$ sequence witnessing $\kappa$ is $A$-reflecting past $\eta^+$. Let $\mathcal{Q} = \mathcal{P}_2 \upharpoonright \text{lh}(E)$. Note $\mathcal{Q}[\mathcal{Q}]^\kappa_{\eta^+} = \text{Ult}(M_1, E)[\mathcal{Q}]^\kappa_{\eta^+}$. Now $\text{Ult}(M, E)$ reaches "Woodin limit of Woodin" hypothesis. So it is enough to see the transfer $\mathcal{Q} = \text{RHS}$ is $M^\kappa$-closed. ETS:

$\text{Ult}(N, E)[\mathcal{Q}]^\kappa_{\eta^+}$ is $M^\kappa$-closed. But $N[\mathcal{Q}]^\kappa_{\eta^+}$ is so closed and we have enough reflection via $E$:

$P = \mathcal{P}_2 \upharpoonright \eta^+ \\
\mathcal{Q}[\mathcal{Q}]^\kappa_{\eta^+} = \mathcal{P}_2 \upharpoonright \eta^+ \\
P[\mathcal{Q}]^\kappa_{\eta^+} = \forall \beta \leq \alpha. \forall \gamma \in \mathcal{Q}, k(\gamma) < \gamma < k(\lambda), \gamma$ successor card

with $P \leq \mathcal{P}_1[\mathcal{Q}]^\kappa_{\eta^+}$ has a proper $\kappa$-s satisfying "I am $E$-closed $M^{\mathcal{P}_1[\mathcal{Q}]^\kappa_{\eta^+}}$". Call this sentence $\varphi(k(\gamma), M, P, \mathcal{P}_1[\mathcal{Q}], k(\lambda), \alpha)$. Let $p$ be a condition in $\text{Col}(\omega, k(\gamma))$  s. t.

$P \Vdash_{\mathcal{P}_2} \varphi(k(\gamma), \kappa, \eta^+, k(\lambda), \alpha) \\
\Rightarrow \mathcal{P}_2 \upharpoonright \eta^+ = M$

$P \Vdash_{\mathcal{P}_2} \varphi(k(\gamma), \kappa, \eta^+, \mathcal{P}_2, \mathcal{P}_1[\mathcal{Q}], k(\lambda), \alpha)$  \\
\therefore \mathcal{P}_2 \upharpoonright \eta^+ = M$

$P \Vdash_{\mathcal{P}_2} \varphi(k(\gamma), \kappa, \eta^+, \mathcal{P}_2, \mathcal{P}_1[\mathcal{Q}], k(\lambda), \alpha) \\
\Rightarrow \mathcal{P}_2 \upharpoonright \eta^+ = M$

Here we can replace $\text{Ult}(P, E)$ by $\text{Ult}(M, E)$, since we have $M \rightarrow P$. Also $\sigma : \text{Ult}(M, E) \rightarrow \text{Ult}(P, E)$ with $\text{lh}(\sigma) \geq \eta$. Also note $i^M_{\mathcal{P}_2}(2) \geq \alpha$.

Rem. We actually needed less about $M$.

Let $\lambda$ be a limit of Woodins and $\kappa < \lambda$.

Say $\kappa$ is strong $\text{cf}(\aleph_1)$ and $\kappa > \lambda$ (for $\lambda(\lambda)$.

(1) With the top extender $E$
The proof above can be refined to show: if $M \prec V$ is an iterable mouse projecting to $\omega_1$ sound (i.e. $M$ is a sharp mouse) with

$$ \text{then } D(M, \kappa) = \Theta_{\omega_1} < \Theta. $$

For the minimal such $M$, we get

$$ D(M, \kappa) = \Theta_{\omega_1 + 1} = \Theta. $$

**Open** Do we need a mouse here? Could we do get the result for $V$ instead?

**Probably:** $ \text{Con} (2^{\mathfrak{c}} = \aleph_1 ) \Rightarrow \text{Con} (AD^+ + \Theta_{\omega_1} = \Theta) $

**Probably:** We conjecture

**The Reverse Direction**

**Conjecture/ Theorem:** Assume $AD^+ + \text{SMC} + \Theta_{\omega_1} = \Theta$ for some $\omega_1 \leq \omega_1$.

Then in some $V[G]$ there is a premouse $M$ with $M \models \kappa$ is Woodin such that $V = L[A_M, \mathbb{R}^*_M]$ for some $H$ on $Col(\omega, \kappa)$ generic over our $M$.

**Proof:** in the case $\Theta = \Theta$: We pick only sparse mice. Let $\mathcal{A}$ be a countable sequence $\langle \mathcal{A}_n : n \in \mathbb{N} \rangle$ and $\mathcal{K}$ be a countable and $\mathbb{R}$-closed by a real recursive in $x$. Let
$F_a^x$ = the set of all $P_2$ s.t. $2 \leq_T x$ and $P_2$ is O-suitable (i.e. Woodin + $\exists^2$ - full) premouse over $\Delta_0$ and $P_2$ is short tree iterable

If $T$ is a tree for universal $\exists^2$ set, we can simultaneously compare all $P_2$ in $\mathcal{F}_a^x$ with the $L[T, x]$. At the same time, making all reals $y \leq_T x$ generic for the extenders algebra at our common Woodin. $L[T, x]$ can find correct branches for short trees $U$. It has all mice over $M(U)$ projecting to $D(U)$ and their IS restricted to itself. We get a O-suitable premouse $Q^a_\epsilon$ with $\delta Q^a_\epsilon = \omega_1 [T, x]$. (This is like the $L[U]$ argument given earlier.) Also, $Q^a_\epsilon = Q^a_d$ for the $d = [x]_T$. $Q^a_\epsilon$ = the stack of all iterable mice over $M(U)$ projecting to $D(U)$; $U$ any of those $\lambda n = \omega_1 [T, x]$ trees.

For a.e. $T$-degreces $d_0 < d_1 < \ldots < d_n$ we have

$Q^0 = Q^{d_0}$

$F(d^0) = \text{this stack}$

$Q^0 = Q^{d_0}$

Define a measure $\nu_\epsilon$ and $\chi_\epsilon$:

$A \in \nu_\epsilon$ iff for a.e. $d_0$ for a.e. $d_1$...

$F(d^0) \in A$

$\chi_\epsilon$ = the measure with a stem

For any $a$: $x \in \chi_\epsilon$ iff for a.e. $d$: $Q^d \in X$ such that:

$\forall (Q_0, \ldots, Q_n) \in T : \{Q_{n+1} \leq Q_0, \ldots, Q_n\} \in T \forall y \in \omega^\omega$

Measure one tree is a tree s.t.:

$\forall (Q_0, \ldots, Q_n) \in T : \{Q_0, Q_1 \leq Q_0, \ldots, Q_n\} \in T \forall y \in \omega^\omega$
Let $T < S$ off $T$ is a subtree of $S$

Let $P_0 = this$ forcing on $L(U, IR)$ where $U$

is the universal $E_1$ set

$P = this$ forcing in $V$

Easy: Every dense set in $P_0$ is predense in $P$.

Let $G$ be $P$-generic in $V$ (hence $G$-${\mathcal c}_2$-generic in $L(U, IR)$).

Let $<\mathcal{Q}_i : i \in w>$ = union of stems on $T \in G$. Show:

$L(U, IR^V)$ is a derived model of $L[\mathcal{Q}_0]$. Let

$x = sup$ of the Woodins of $\mathcal{Q}_i$'s = $0(\mathcal{Q}_0)$

Note: As before, no bounded subsets are added to $L[\mathcal{Q}_0]$

i.e. $G(\mathcal{Q}_0)$ in $L[\mathcal{Q}_0] \subseteq \mathcal{Q}_0$

(1) There is $h \in \text{Col}(u, \langle x \rangle)$ s.t. $IR^h = IR^x$

Remark: $\exists x \exists \beta \exists y boys < x$ and a cone of $x$.

This is done by a reflection argument to $E_2$.

If not, $V \not\models \psi$ some $\psi$ saying "no". Get

$L(\mathcal{P}(1)) \not\models \exists \eta = \Theta_0 \exists \exists \theta + \eta$

with $L(\mathcal{P}(1)) \subseteq \Delta^2_1$; $\mathcal{M}_{\Delta^2_1} \models \exists \eta$.
Using $N^*$ capturing good, for our purpose we can take inductive-like point class $\Gamma$ where scale $(\mathbb{F}^*)$ and $L_\alpha(P, \mathbb{R}) \subseteq \mathbb{A}$ we can construct by full backgrounds $L \subseteq \mathbb{V}^{\mathbb{R}^x}$, and show it must reach a 0-suitable short tree iterable $\mathcal{Q}_\alpha$

These "short" means "short in the sense of $L_\alpha(P, \mathbb{R})$". Note: there are club many $P$-values here. (Exercise?) Then $L_\alpha(P, \mathbb{R}) \models \mathcal{Q}_\alpha$ is iterable + 0-suitable.

\[ \mathbb{R}^* = \mathbb{R}^\mathbb{U} = \mathbb{R}^y \downarrow \text{generic} \] for p.o. whose conditions are $h_n \downarrow \text{cof}(\mathbb{U}, \beta_n)$

Claim: $\mathcal{N} \upharpoonright \mathbb{R}^* = (\Delta^2_\lambda)^\mathbb{U}$

PT Easy: \[ \Delta^2 \] Easy: Every $\mathcal{N} \upharpoonright \mathbb{R}^*$ is Suslin co-Suslin in the derived model. Or we get a $\beta$ for $\mathbb{V}$ by forcing.

2. For any $A \subseteq (\Delta^2_\lambda)^\mathbb{U}$ let $\mathcal{Q}_A$ be scales for $A$, $T_A$ which are $(\Delta^2_\lambda)^\mathbb{U}_A(x)$. 

Because let $x \in \mathcal{Q}_A[\mathcal{N}]$ at each $\delta_j, j > i$ we have capturing times for $\mathcal{Q}_A[\mathcal{N}]$ in $\mathcal{Q}_A[\mathcal{N}]$, by $\mathbb{E}^2_\lambda$-fullness. These give a.c. trees $(T_{i,0}) \in \mathcal{L}(\mathcal{Q}_A[\mathcal{N}])$ projecting to $A$, $T_A$ on $\mathbb{R}^\mathbb{U}$. So we are done in $\mathcal{L}(\mathbb{U}, \mathbb{V})$. 