Chapter 12
Analytic determinacy

Let X be a non–empty set, and let $A \subset {}^{\omega}X$. We associate to A a game, called $G(A)$, which we define as follows. In a run of this game two players, I and II, who alternate playing elements x_0, x_1, x_2 of X as follows.

I	x_0		x_2		$\dots$	
II		x_1		x_3		$\dots$

After ω moves they produced an element $x = (x_0, n_x, x_2, \dots)$ of ${}^{\omega}X$. We say that I *wins* this run of $G(A)$ iff $x \in A$, otherwise *II wins*. A *strategy for I* is a function

$$\sigma \colon \bigcup_{n<\omega} {}^{2n}X \to X,$$

and a *strategy for II* is a function

$$\tau \colon \bigcup_{n<\omega} {}^{2n+1}X \to X.$$

If σ is a strategy for I and $z \in {}^{\omega}X$, then we let $\sigma * z$ be the unique $x \in {}^{\omega}X$ with

$$x(2n) = \sigma(x(0), x(1), x(2), \dots x(2n-1)) \text{ and}$$

$$x(2n+1) = z(n)$$

for $n < \omega$, i.e., x is the element of ${}^{\omega}X$ produced by a run of $G(A)$ in which I follows σ and II plays z. We say that σ is a *winning strategy for I in* $G(A)$ iff

$$\{\sigma * z \colon z \in {}^{\omega}X\} \subset A.$$

Symmetrically, if τ is a strategy for II and $z \in {}^{\omega}\omega$, then we let $z * \tau$ be the unique $x \in {}^{\omega}X$ with

$$x(2n+1) = \sigma(x(0), x(1), x(2), \dots x(2n)) \text{ and}$$

$$x(2n) = z(n)$$

for $n < \omega$, i.e., x is the element of ${}^{\omega}X$ produced by a run of $G(A)$ in which *II* follows τ and *I* plays z. We say that τ is a *winning strategy for II in* $G(A)$ iff

$$\{z * \tau : x \in {}^{\omega}X\} \subset {}^{\omega}\omega \setminus A.$$

Of course, at most one of the two players can have a winning strategy.

Definition 12.1 *Let X be a non–empty set, and let $A \subset {}^{\omega}X$. We say that $G(A)$ is* determined *iff player I or player II has a winning strategy in $G(A)$. In this case, we also call A itself determined.*

We shall mostly be interested in the case $X = \omega$ in which A is a set of reals. It can be shown in ZFC that if $A \subset {}^{\omega}\omega$ is Borel, then A is determined, cf. [1, Chap. 20, pp. 137–148]. We here aim to show that every analytic set is determined, cf. Thorem 12.7. It turns out that this cannot be done in ZFC, though, cf. Corollary 12.13. We shall prove later (cf. Theorem 13.23) that in fact every projective set of reals is determined.

Recall that a set $A \subset {}^{\omega}\omega$ is coanalytic iff there is some map $s \mapsto <_s$, where $s \in {}^{<\omega}\omega$, such that for all $s, t \in {}^{<\omega}\omega$ with $s \subset t$, $<_t$ is an order on $lh(t)$ which extends $<_s$, and for all $x \in {}^{\omega}\omega$,

$$x \in A \iff <_x = \bigcup_{s \subset x} <_s \text{ is a wellorder.}$$

(Cf. Lemma 7.5 and Problem 7.1.) Obviously, such a map $s \mapsto <_s$ can be coded by a real x. Our first goal is to show that if $x^{\#}$ exists then A is determined. In order to achieve this we need to consider simpler "auxiliary" games, albeit in a space which is slightly more complicated than ${}^{\omega}\omega$.

We may construe ${}^{\omega}X$ as a topological space as follows. For $s \in {}^{<\omega}X$, set $U_s = \{x \in {}^{\omega}X : s \subset x\}$. The sets U_s are declared to be the basic open sets, so that a set $A \subset {}^{\omega}X$ is called open iff there is some $Y \subset {}^{<\omega}X$ with $A = \bigcup_{s \in Y} U_s$.

Theorem 12.2 (Gale, Steward). *Let $A \subset {}^{\omega}X$ be open. Then A is determined.*

Proof. Let us suppose *I* not to have a winning strategy in $G(A)$. We aim to produce a winning strategy for *II* in $G(A)$. Let us say that *I* has a winning strategy in $G_s(A)$, where $s \in {}^{<\omega}X$ has even length, iff *I* has a winning strategy in $G(\{x \in {}^{\omega}X : s^\frown x \in A\})$. By our hypothesis, *I* doesn't have a winning strategy in $G_\emptyset(A)$.

Claim 12.3 *Let $s \in {}^{<\omega}X$ have even length, and suppose I not to have a winning strategy in $G_s(A)$. Then for all $y \in X$ there is some $z \in X$ such that I doesn't have a winning strategy in $G_{s^\frown y^\frown z}(A)$.*

Proof of Claim 12.3. Otherwise there is some $y \in X$ such that for all $z \in X$, *I* has a winning strategy in $G_{s^\frown y^\frown z}(A)$. But then *I* has a winning strategy in $G_s(A)$: he first plays such a y, and subsequently after *II* played z, follows his strategy in $G_{s^\frown y^\frown z}(A)$.

Let us now define a strategy τ for II in $G(A)$ as follows. Let $s=(x_0,\dots,x_n)$ be a position in $G(A)$ where it's II's turn to play, i.e., n is odd. Then let $\tau(s)$ be some $z\in X$ such that I doesn't have a winning strategy in $G_{s^\frown z}(A)$, if some such z exists; otherwise we let $\tau(s)$ be an arbitrary element of X.

Claim 12.4 *τ is a winning strategy for II in $G(A)$.*

Proof of Claim 12.4.

$$\begin{array}{c|cccccc} I & x_0 & & x_2 & & \dots & \\ \hline II & & x_1 & & x_3 & & \dots \end{array}$$

be a play of $G(A)$ which is according to τ. Claim 12.3 can easily be used to show inductively that for each even $n<\omega$, I does not have a winning strategy in $G_{(x_0,\dots,x_{n-1})}(A)$.

Now suppose that II looses, i.e., $x=(x_0,x_1,x_2,\dots)\in A$. Because A is open, there is some basic open set U_s such that $x\in U_s\subset A$. We may assume $lh(s)$ to be even. Then I has a trivial winning strategy in $G_s(A)$: he may play as he pleases, as every $s^\frown x', x'\in {}^\omega X$, will be in A. But this is a contradiction! □

Lemma 12.5 *Let $A\subset {}^\omega X$. Suppose that for every $y\in X, \{x\in {}^\omega X : y^\frown x\in A\}$ is determined. Then ${}^\omega X\setminus A$ is determined.*

Proof. Let us first suppose that there is a $y\in X$ such that II has a winning strategy τ^* in $G(\{x\in {}^\omega X : y^\frown x\in A\})$. We claim that in this case I has a winning strategy σ in $G({}^\omega X\setminus A)$. We let $\sigma(\emptyset)=y$, and we let $\sigma(y^\frown s)=\tau^*(s)$, where $lh(s)$ is odd. It is easy to see that if $x\in {}^\omega X$ is produced by a run which is according to σ, then $x\in {}^\omega X\setminus A$.

Now let us suppose that for all $y\in X$, I has a winning strategy σ_y^* in $G(\{x\in {}^\omega X : y^\frown x\in A\})$. We claim that in this case II has a winning strategy τ in $G({}^\omega X\setminus A)$. We let $\tau(y^\frown s)=\sigma_y^*(s)$, where $lh(s)$ is even. It is easy to see that if $x\in {}^\omega X$ is produced by a run which is according to τ, then $x\in A$. □

Corollary 12.6 *Let $A\subset {}^\omega X$ be closed. Then A is determined.*

Theorem 12.7 (D. Martin) *Suppose that $x^\#$ exists for every $x\in {}^\omega\omega$. Then every analytic set $B\subset {}^\omega\omega$ is determined.*

Proof. Let us fix an analytic set B, set $A={}^\omega\omega\backslash B$, and let $s\mapsto <_s$ be as above. We have to consider the game $G(B)$,

$$\begin{array}{c|cccccc} \mathrm{I} & n_0 & & n_2 & & \dots & \\ \hline \mathrm{II} & & n_1 & & n_3 & & \dots \end{array}$$

in which I and II alternate playing integers $n_0,n_1,\dots$ and I wins iff $x=(n_0,n_1,\dots)\in B$. We have to prove that $G(B)$ is determined.

The key idea is to first consider the following auxiliary game, $G^*(A)$.

$$\begin{array}{c|cccccccc} \mathrm{I} & n_0 & & n_2 & & n_4 & & \dots & \\ \hline \mathrm{II} & & n_1, \alpha_0 & & n_3, \alpha_1 & & n_s, \alpha_2 & & \dots \end{array}$$

In this game, I and II also alternate playing integers $n_0, n_1, \dots$. In addition, II has to play countable ordinals $\alpha_0, \alpha_1, \dots$ such that for all $k < \omega$,

$$(k+1, <_{(n_0,\dots,n_k)}) \cong (\{\alpha_0, \dots, \alpha_k\}, <).$$

The first player to disobey one of the rules looses. If the play is infinite then II wins.

Notice that what II has to do is playing a witness to the fact that $<_x$ is a well–order, where $x = (n_0, n_1, \dots)$.

Notice also that $G^*(A)$ is an open game in the space ${}^{\omega}\omega \times {}^{\omega}\omega_1$ (which we identify with ${}^{\omega}(\omega \times \omega_1)$) and hence by Theorem 12.2 $G^*(A)$ is determined in every inner model which contains $s \mapsto <_s$.

Fix a real x such that the map $s \mapsto <_s$ is in $L[x]$. Let us first assume that II has a winning strategy for $G^*(A)$ in $L[x]$, call it τ. Obviously, $\tau \in L[x]$ is then also a winning strategy for $G^*(A)$ for all plays in V (not only the ones in $L[x]$). But then II will win $G(B)$ in V by just following τ and hiding her "side moves" $\alpha_0, \alpha_1, \dots$. If $x = (n_0, n_1, \dots)$ is the real produced by the end of a play then $<_x$ must be a well–order, and thus $x \in A$, i.e. $x \notin B$.

Let us now suppose that I has a winning strategy for $G^*(A)$ in $L[x]$, call it σ. Whenever $\alpha_0, \dots, \alpha_k$ and $\alpha'_0, \dots, \alpha'_k$ are countable x–indiscernibles with

$$(\{\alpha_0, \dots, \alpha_k\}, <) \cong (\{\alpha'_0, \dots, \alpha'_k\}, <),$$

then

$$L[x] \models \varphi(\sigma, \alpha_0, \dots, \alpha_k) \Longleftrightarrow L[x] \models \varphi(\sigma, \alpha'_0, \dots, \alpha'_k)$$

for every $\mathscr{L}_{\in}$–formula φ (cf. Problem 11.3). In particular, then,

$$\sigma(n_0, n_1, \alpha_0, \dots, n_{2k}, n_{2k+1}, \alpha_k) = \sigma(n_0, n_1, \alpha'_0, \dots, n_{2k}, n_{2k+1}, \alpha'_k)$$

for all integers $n_0, n_1, \dots, n_{2k+1}$. We may therefore define a strategy $\overline{\sigma}$ for I in $G(B)$ as follows. Let

$$\overline{\sigma}(n_0, n_1, \dots, n_{2k}, n_{2k+1}) = \sigma(n_0, n_1, \alpha_0, \dots, n_{2k}, n_{2k+1}, \alpha_k)$$

where $\alpha_0, \dots, \alpha_k$ are countable x–indiscernibles with

$$(k+1, <_{(n_0,\dots,n_k)}) \cong (\{\alpha_0, \dots, \alpha_k\}, <).$$

We claim that $\overline{\sigma}$ is a winning strategy for I in $G(B)$.

Let us assume that this is not the case, so that there is a play of $G(B)$ in which I follows $\overline{\sigma}$ and which produces $x = (n_0, n_1, \dots) \in A$. Then $<_x$ is a well–order and there is a set $\{\alpha_0, \alpha_1, \dots\}$ of countable x–indiscernibles such that

$$(\omega, <_x) \cong (\{\alpha_0, \alpha_1, \dots\}, <),$$

i.e.

$$(k+1, <_{(n_0,\dots,n_k)}) \cong (\{\alpha_0,\dots,\alpha_k\},<)$$

for all $k < \omega$. This means that for every $k < \omega$,

$$n_{2k} = \sigma(n_0, n_1, \alpha_0, \dots, n_{2k-2}, n_{2k-1}, \alpha_{k-1}),$$

that is, $n_0, n_1, \alpha_0, n_2, n_3, \alpha_1, \dots$ is a play of $G^*(A)$ in which I follows σ.

Let us now define the tree T of attempts to find an infinite play of $G^*(A)$ in which I follows σ as follows. We set $s \in T$ iff $s = (n_0, n_1, \alpha_0, \dots, n_{2k-2}, n_{2k-1}, \alpha_{k-1}, n_{2k})$ for some n_0, n_1, ..., $n_{2k} \in \omega$ and α_0, ..., $\alpha_{k-1} \in \omega_1$ such that for all $l \leq k$,

$$n_{2l} = \sigma(n_0, n_1, \alpha_0, \dots, n_{2l-2}, n_{2l-1}, \alpha_{l-1}).$$

If s, t ∈T, then we let $s \leq t$ iff $s \supset t$. Notice that $(T; \leq) \in L[x]$.

Now $(T; \leq)$ is thus ill–founded in V by what was shown above. Hence $(T; \leq)$ is ill–founded in $L[x]$ as well. But there cannot be such a play in $L[x]$, as σ is a winning strategy for I in $G^*(A)$. Contradiction! □