Woodin's axiom $(*)$, bounded forcing axioms, and precipitous ideals on ω_1

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Abstract

If the Bounded Proper Forcing Axiom BPFA holds, then Mouse Reflection holds at $\aleph_2$ with respect to all mouse operators up to the level of Woodin cardinals in the next ZFC–model. This yields that if Woodin's $\mathbb{P}_{\max}$ axiom $(*)$ holds, then BPFA implies that V is closed under the "Woodin-in-the-next-ZFC–model" operator. We also discuss stronger Mouse Reflection principles which we show to follow from strengthenings of BPFA, and we discuss the theory BPFA plus "NS_{ω_1} is precipitous" and strengthenings thereof. Along the way, we answer a question of Baumgartner and Taylor, [2, Question 6.11].

0 Introduction.

Let Γ be a class of forcings, e.g. the class of all c.c.c., proper, semi–proper, or stationary set preserving forcings. The *bounded forcing axiom for* Γ says that

$$(1) \qquad (H_{\omega_2}; \in) \prec_{\Sigma_1} ((H_{\omega_2})^{V^{\mathbb{P}}}; \in)$$

whenever $\mathbb{P} \in \Gamma$. The bounded forcing axiom for c.c.c., proper, semi–proper, and stationary set preserving forcings is called MA_{ω_1} ("Martin's axiom"), BPFA (the "Bounded Proper Forcing Axiom"), BSPFA (the "Bounded Semi–Proper Forcing Axiom"), and BMM ("Bounded Martin's Maximum"), respectively. (Cf. [8] and [1].) This paper will be concerned with variants of BPFA.

The formulation (1) is not how the bounded forcing axioms were presented in the first place (cf. [8] and [1]). In section 1, we shall study variants of forcing axioms which are located between bounded and unbounded forcing axioms. Of particular importance will be $\mathsf{FA}_{2^{\aleph_0}}$ for proper forcings, which results from the formulation of the Proper Forcing Axiom PFA by demanding that the antichains which are to be met are all of size at most $2^{\aleph_0}$ (cf. Lemma 1.4).

We shall study mouse reflection principles under the hypothesis of bounded forcing axioms. A mouse reflection principle says that if an initial segment of V is closed under a given mouse operator $X \mapsto J(X)$, then so is some longer initial segment of V. (Cf. Definition 2.2 for a precise definition of what we mean by a "mouse operator.") A typical example would be the statement that if H_{ω_2} is

[1]Some of the results of this paper were obtained while the second author was visiting the Institut Mittag–Leffler in Djursholm, Sweden, Sept 14–Oct 10, 2009. He would like to thank the organizers for arranging such a fruitful and enjoyable program. Both authors gratefully acknowledge support through the DFG grant no. SCHI 484/3-1.

closed under the mouse operator $X \mapsto X^\#$, then V is closed under $X \mapsto X^\#$. Let $X \mapsto M_{la}(X)$ be the mouse operator which sends X to the least X–mouse which has an initial segment which is a model of ZFC plus "there is a Woodin cardinal." Woodin [31, Theorem 10.108] essentially showed that if Bounded Martin's Maximum BMM^{++} holds and if H_{ω_2} is closed under the mouse operator $X \mapsto M_{la}(X)$, then V is closed under $X \mapsto M_{la}(X)$. (This gives that the model of BMM^{++} plus $(*)$ constructed in the proof of [31, Theorem 10.99] seems to start from an optimal large cardinal hypothesis.) We here show the following theorem.[2] (Cf. Definition 2.5 on what it means that a mouse operator "does not go beyond $X \mapsto M_{la}(X)$" or "does not go beyond $X \mapsto M_n^\#(X)$" for $n < \omega$.)

Theorem 0.1 *Assume* BPFA *to hold. Let J be a mouse operator which does not go beyond $X \mapsto M_{la}(X)$, and suppose H_{ω_2} to be closed under J. Then V is closed under J.*

Recall that Woodin's axiom $(*)$ is the conjunction of the following two statements (cf. [31, Definition 5.1]):
(a) AD, the Axiom of Determinacy, holds in $L(\mathbb{R})$, and
(b) $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$–generic extension of $L(\mathbb{R})$.

Theorem 0.2 *Assume Woodin's axiom $(*)$ to hold. If* BPFA *holds, then V is closed under $X \mapsto M_{la}(X)$.*

The only further ingredient beyond Theorem 0.1 which is necessary to derive Theorem 0.2 is Lemma 3.1 (to be shown in section 3) which might be part of the folklore and which says that under $(*)$, H_{ω_2} is closed under $M_1^\#$ (and much more). Theorems 0.1 and 0.2 will be shown in section 4.

As a consequence of Theorem 0.2, BPFA does not hold in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$.[3] In contrast to Theorem 0.2, Lemma 5.1 (of section 5) will show that there is a model of $\mathsf{AD}^{L(\mathbb{R})}$ plus BPFA which is not even closed under $X \mapsto X^\#$.

The authors of [14] ask whether their forcing which uses a precipitous ideal on ω_1 to increase $\boldsymbol{\delta}^1_2$ can be iterated. An affirmative answer to this question in the absence of inner models with Woodin cardinals would be particularly interesting. The same question could be asked concerning the forcing of [4]. The paper [5] has a negative result in this direction: it says that the forcings of [14] and [4] are semi–proper iff and only if *all* stationary set preserving forcings are semi–proper, cf. [5, Theorem 5.7].

In this paper, we shall answer the question of [14] in the negative in a strong sense. We shall prove that in the absence of an inner model with a Woodin cardinal, once ω_2 of V is collapsed to ω_1 by a forcing which preserves ω_1, then in the extension there is no forcing whatsoever which does not collapse ω_1 and which resurrects a precipitous ideal on ω_1.

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[2]We thank Boban Velickovic for asking us whether under BPFA the closure of H_{ω_2} under $X \mapsto X^\#$ implies the closure of V under $X \mapsto X^\#$. We also thank Andrés Caicedo and Martin Zeman for very helpful comments on an earlier version of this paper.

[3]This was asked by Stuart Zoble.

Theorem 0.3 *Suppose that there is no inner model with a Woodin cardinal, and let K denote the core model. Assume κ to be such that there is a precipitous ideal on κ. Then $\kappa^{+K} = \kappa^{+V}$.*

In the statement of this theorem, K denotes the core model as constructed in [13] in the theory ZFC plus "there is no inner model with a Woodin cardinal."

By Theorem 0.3, in the absence of an inner model with a Woodin cardinal the forcings of [14] and [4] don't even exist in further forcing extensions which don't collapse ω_1 after having forced with either one of them once.

Theorem 0.3 has the following consequences.

An ideal I on ω_1 is called *strong* iff I is precipitous and

$$\Vdash_{I^+/I} j(\check{\omega}_1) = \check{\omega}_2.$$

(Cf. [2, Definition 5.6].) Clearly, the Club Bounding Principle CBP (for every $f\colon \omega_1 \to \omega_1$ there is a canonical function $f_\alpha\colon \omega_1 \to \omega_1$, $\alpha < \omega_2$, such that $\{\xi < \omega_1 \colon f(\xi) \leq f_\alpha(\xi)\}$ contains a club) implies that every precipitous ideal on ω_1 is strong.

Theorem 0.4 *The following theories are equiconsistent.*
(1) ZFC *plus "there is a strong ideal on ω_1."*
(2) ZFC *plus "there is an ω_2–saturated ideal on ω_1."*
(3) ZFC *plus "there is a Woodin cardinal."*

Con(3) $\Longrightarrow$ Con(2) is due to Shelah, cf. [24]. Con(2) $\Longrightarrow$ Con(3) is due to Steel and Jensen–Steel, cf. [26] and [13]. (2) $\Longrightarrow$ (1) is part of the folklore. Con(1) $\Longrightarrow$ Con(3) is new. The proof of the equiconsistency of (1) and (2) affirmatively answers a question of Baumgartner and Taylor,[4] cf. [2, p. 603 and Question 6.11].

Theorem 0.5 *Suppose that there is a precipitous ideal on ω_1 and $\boldsymbol{\delta}^1_2 = \aleph_2$. There is then an inner model with a Woodin cardinal.*

Woodin has shown that the hypothesis of Theorem 0.5 can be forced over a model of ZFC in which there are $\delta^* < \delta$ such that δ is a Woodin cardinal, δ^* is Woodin in $L(V_{\delta^*})$, and $V_{\delta^*} \prec V_\delta$. (Cf. [31, Theorem 3.25]. In fact, NS_{ω_1} will be saturated in the extension.). Section 6 is devoted to proofs of theorems 0.3, 0.4, and 0.5.

Among other things, the paper [5] discusses the following consequences of BMM plus "NS_{ω_1} is precipitous": acg (admissible club guessing), $\boldsymbol{\delta}^1_2 = \aleph_2$, ψ_{AC}, and CBP (the Club Bounding Principle). By the above results, each one of these four consequences implies, in the presence of a precipitous ideal on ω_1, that there is an inner model with a Woodin cardinal.

We'll in fact prove the following result, to be proven in section 7:

Theorem 0.6 *Suppose* BPFA *holds and that there is a precipitous ideal on ω_1. Then there is an inner model with a Woodin cardinal.*

[4]The authors thank Sean Cox for bringing this to their attention.

We do not know if the theory BMM plus "NS_{ω_1} is precipitous" yields inner models with two Woodin cardinals. In order to be able to significantly strengthen the conclusion of Theorem 0.6 we seem to need to prove mouse reflection results which go beyond Theorem 0.1 and which will be shown in section 4:

Theorem 0.7 *Assume $\mathsf{FA}_{2^{\aleph_0}}$ for proper forcings. Let $n < \omega$, and suppose that V is closed under $M_n^\#$. Let J be a mouse operator which does not go beyond $M_{n+1}^\#$, and suppose H_{ω_2} to be closed under J. Then V is closed under J.*

Theorem 0.7 produces the following result, to be shown in section 7:

Theorem 0.8 *Assume $\mathsf{FA}_{2^{\aleph_0}}$ for proper forcings and there is a precipitous ideal on ω_1. Then Projective Determinacy holds.*

It is currently open whether BMM implies that there must be a precipitous ideal on ω_1.

1 From bounded to unbounded forcing axioms.

We shall be interested in forcing axioms that have the bounded and unbounded versions as special cases.

Definition 1.1 *Let Γ be a class of complete Boolean algebras, and let κ be an uncountable cardinal. Then $\mathsf{FA}(\Gamma)_\kappa$, or FA_κ* for forcings in Γ, *denotes the statement that whenever $\mathbb{B} \in \Gamma$ and $\{A_i : i < \omega_1\}$ is a family of maximal antichains in $\mathbb{B}$ such that A_i has size at most κ for each $i < \omega_1$, then there is a filter G in $\mathbb{P}$ such that $G \cap A_i \neq \emptyset$ for all $i < \omega_1$.*

We thus have that BPFA is $\mathsf{FA}_{\aleph_1}$ for proper forcings, BMM is $\mathsf{FA}_{\aleph_1}$ for stationary set preserving forcings, PFA is FA_κ for proper forcings and all κ, MM is FA_κ for stationary set preserving forcings and all κ, etc. Also, $\mathsf{FA}_{2^{\aleph_0}}$ for stationary set preserving forcings implies MM_c, etc.

The paper [30] also discusses forcing axioms as given by Definition 1.1.[5]

The following definition is just for the purpose of formulating Theorem 1.3.

Definition 1.2 *Let $\mathcal{M} = (M; \in, A_1, \dots, A_k)$ be a transitive structure, and let φ be a formula. Let $\Psi(\mathcal{M}, \varphi)$ be the statement that there is some transitive structure $\bar{\mathcal{M}}$ of size $\aleph_1$, some $\pi \colon \bar{\mathcal{M}} = (\bar{M}; \in, \bar{A}_1, \dots, \bar{A}_k) \to \mathcal{M}$ with $\pi \restriction \omega_1 = \mathrm{id}$, and some transitive $(H; \in)$ such that $\bar{\mathcal{M}} \in H$ and*

$$(H; \in) \models \varphi(\bar{\mathcal{M}}).$$

The following might be part of the folklore.

[5] This was pointed out to us by Andrés Caicedo after the first version of this paper had been submitted.

Theorem 1.3 *Let* Γ *be a class of complete Boolean algebras. The following are equivalent.*

(a) $\mathsf{FA}(\Gamma)_\kappa$.

(b) *For all Boolean algebras* $\mathbb{B} \in \Gamma$*, for all transitive structures* $\mathcal{M}$ *of size at most* κ*, and for all formulae* φ*,*

$$V \models \Psi(\mathcal{M}, \varphi) \iff V^{\mathbb{B}} \models \Psi(\mathcal{M}, \varphi).$$

Notice that for $\kappa = \aleph_1$ this is easily seen to reproduce Bagaria's characterization of the bounded forcing axioms, cf. [1].

Proof of Theorem 1.3. (b) $\Longrightarrow$ (a) is very easy. Let $(A_i : i < \omega_1)$ be a sequence of maximal antichains in $\mathbb{B}$ such that each A_i has size at most κ. Let, in V,

$$\chi \colon \mathcal{M} = (M; \in, (A'_i : i < \omega_1)) \to (H_\theta; \in, (A_i : i < \omega_1)),$$

where θ is sufficiently large, M is transitive and of size κ, and

$$\{\mathbb{B}\} \cup \bigcup_{i<\omega_1} A_i \subset \operatorname{ran}(\chi).$$

Let G be $\mathbb{B}$–generic over V. Then

$$G' = \chi^{-1}{}''G$$

is a filter which meets every A'_i. Let H^* be a transitive model of ZFC^- such that $\mathcal{M}$, $G' \in H^*$. In $V[G]$, we may pick some

$$\tilde{\pi} \colon H' \to H^*$$

with H' transitive, $\mathcal{M}$, $G' \in \operatorname{ran}(\tilde{\pi})$ and such that $\tilde{\pi} \restriction \omega_1 = \mathrm{id}$. Therefore, by (b), in V there is some

$$\pi \colon \bar{\mathcal{M}} = (\bar{M}; \in, (\bar{A}_i : i < \omega_1)) \to \mathcal{M}$$

and some transitive H with $\bar{\mathcal{M}} \in H$ and

$$H \models \text{“there is a filter } \bar{G} \text{ which meets every } \bar{A}_i, i < \omega_1.$$

But then

$$G^* = \{p \in \mathbb{B} \colon \exists q \in \bar{G}\ \chi \circ \pi(q) \leq p\}$$

is a filter which meets every A_i, $i < \omega_1$.

Let us now show (a) $\Longrightarrow$ (b). This is straightforward albeit somewhat tedious. We need to see that if $V^{\mathbb{B}} \models \Psi(\mathcal{M}, \varphi)$, then $V \models \Psi(\mathcal{M}, \varphi)$. So let us suppose that

$$1_{\mathbb{B}} \Vdash \Psi(\mathcal{M}, \varphi).$$

Let us pick $\mathbb{B}$–terms $\dot{m}$, τ, σ, τ^*, $\dot{h}$, ρ, and $\gamma < \omega_2$ such that

$1_{\mathbb{B}} \Vdash$ “ $\dot{m}$ and $\dot{h}$ are transitive models,

$\tau \colon \check{\omega}_1 \to \dot{m}$ is an enumeration,[6] $\sigma \colon \dot{m} \to \check{\mathcal{M}}$ is elementary,

[6]We sometimes confuse a model with its underlying set.

$\tau^* \colon \check{\omega}_1 \to \dot{h}$ is an enumeration, $\rho \colon \dot{h} \to \check{\gamma}$ is a (the) rank function,

$\dot{m} = (\tau^*(\check{0}); \in, \tau^*(\check{1}), \dots, \tau^*(\check{k})) \in \dot{h}$, and $\dot{h} \models \varphi(\dot{m})$. "

Now for formulae θ, *countable* ordinals ξ, ξ', ξ_1, ..., ξ_l, and α, and $s = 1$, ..., k, let us consider the following sets, which are easily be seen to be dense in $\mathbb{B}$.

(1) $\{p \colon p \Vdash \dot{m} \models \theta(\tau(\check{\xi}_1), \dots, \tau(\check{\xi}_l)) \leftrightarrow \check{\mathcal{M}} \models \theta(\sigma(\tau(\check{\xi}_1)), \dots, \sigma(\tau(\check{\xi}_l)))\}$.

(2) $\{p \colon \exists x \in M \; p \Vdash \sigma(\tau(\check{\xi})) = \check{x}\}$.

(3) $\{p \colon \exists \xi < \omega_1 \; p \Vdash \sigma(\tau(\check{\xi})) = \check{\alpha}\}$.

(4) $\{p \colon \exists \rho < \omega_1 \; p \Vdash \check{\mathcal{M}} \models \exists v \; \theta(v, \sigma(\tau(\check{\xi}_1)), \dots, \sigma(\tau(\check{\xi}_k))) \to$
$\dot{m} \models \theta(\tau(\check{\rho}), \tau(\check{\xi}_1), \dots, \tau(\check{\xi}_k))\}$.

(5) $\{p \colon \exists \rho < \omega_1 \; p \Vdash \tau^*(\check{\xi}) \neq \tau^*(\check{\xi}') \to \tau^*(\check{\rho}) \in \tau^*(\check{\xi}) \triangle \tau^*(\check{\xi}')\}$.

(6) $\{p \colon \exists \beta < \gamma \; p \Vdash \rho(\tau^*(\check{\xi})) = \check{\beta}\}$.

(7) $\{p \colon p \Vdash \tau^*(\check{\xi}) \in \tau^*(\check{\xi}') \to \rho(\tau^*(\check{\xi})) < \rho(\tau^*(\check{\xi}'))\}$.

(8) $\{p \colon \exists \xi' < \omega_1 \; p \Vdash \tau(\check{\xi}) = \tau^*(\check{\xi}')\}$.

(9) $\{p \colon \; \exists \rho < \omega_1 \; p \Vdash \tau^*(\check{\xi}) \in \tau(\check{\xi}') \to \tau^*(\check{\xi}) = \tau(\check{\rho})\}\}$.

(10) $\{p \colon \exists \xi' < \omega_1 \; p \Vdash \tau^*(\check{\xi}) \in \tau^*(\check{0}) \to \tau^*(\check{\xi}) = \tau(\check{\xi}')\}$.

(11) $\{p \colon p \Vdash \sigma(\tau(\check{\xi})) \in \check{A}_s \to \tau(\check{\xi}) \in \tau^*(\check{s})\}$.

(12) $\{p \colon \exists \xi' < \omega_1 \; p \Vdash \tau^*(\check{\xi}) \in \tau^*(\check{s}) \to (\tau^*(\check{\xi}) = \tau(\check{\xi}') \wedge \sigma(\tau(\check{\xi}')) \in \check{A}_s)\}$.

(13) $\{p \colon p$ decides $\theta(\tau^*(\check{\xi}_1), \dots, \tau^*(\check{\xi}_l))\}$.

(14) $\{p \colon \exists \xi < \omega_1 \; p \Vdash \dot{h} \models \exists v \; \theta(v, \tau^*(\check{\xi}_1), \dots, \tau^*(\check{\xi}_l)) \to$
$\dot{h} \models \theta(\tau^*(\check{\xi}), \tau^*(\check{\xi}_1), \dots, \tau^*(\check{\xi}_l))\}$.

By $\mathsf{FA}(\Gamma)_\kappa$, there is a filter G meeting every one of these dense sets. Notice that we listed only $\aleph_1$ many dense sets, and that for each of them there is a maximal antichain contained in it which has size at most κ.

It follows from (1) that

$$G \cap \{p \colon \exists \rho < \omega_1 \; p \Vdash \tau(\check{\xi}) \neq \tau(\check{\xi}') \to \tau(\check{\rho}) \in \tau(\check{\xi}) \triangle \tau(\check{\xi}')\} \neq \emptyset$$

for all ξ, $\xi' < \omega_1$. (1) and (2) then give that the relation E on ω_1 defined by

$$\xi E \xi' \iff G \Vdash \tau(\check{\xi}') \in \tau(\check{\xi})^7$$

is well–founded and extensional. We may thus let

$$\tau(\check{\xi})^G = \{\tau(\check{\xi}')^G \colon G \Vdash \tau(\check{\xi}') \in \tau(\check{\xi})\},$$

$$\bar{M} = \{\tau(\check{\xi})^G \colon \xi < \omega_1\},$$

[7] We write $G \Vdash \varphi$ for $\exists p \in G \; p \Vdash \varphi$.

and

$$\bar{A}_s = \{\tau(\check{\xi})^G \colon G \Vdash \sigma(\tau(\check{\xi})) \in \check{A}_s\}$$

for $s = 1, \ldots, k$. Set

$$\bar{\mathcal{M}} = (\bar{M}; \bar{A}_1, \ldots, \bar{A}_k).$$

We may define an embedding

$$\pi \colon \bar{\mathcal{M}} \to \mathcal{M}$$

by

$$\pi(\tau(\check{\xi})^G) = x, \text{ where } x \text{ is unique with } G \Vdash \sigma(\tau(\check{\xi})) = \check{x}.$$

By (1) through (4), σ will be an elementary embedding with $\sigma \restriction \omega_1 = \mathrm{id}$.

By (5), (6), and (7), the relation E^* on ω_1 defined by

$$\xi E^* \xi' \iff G \Vdash \tau^*(\check{\xi}) \in \tau^*(\check{\xi}')$$

is well–founded and extensional. We may thus let

$$\tau^*(\check{\xi})^G = \{\tau^*(\check{\xi}')^G \colon G \Vdash \tau^*(\check{\xi}') \in \tau^*(\check{\xi})\}$$

and

$$H = \{\tau^*(\check{\xi})^G \colon \xi < \omega_1\}.$$

By (8) and (9), for every $\xi < \omega_1$ there is some $\xi' < \omega_1$ such that

$$\tau(\check{\xi})^G = \tau^*(\check{\xi}')^G,$$

and thus $\bar{M} \subset H$. Moreover, by (10) in fact, $\tau^*(\check{0})^G = \bar{M} \in H$. Using (11) and (12), $\tau^*(\check{s})^G = \bar{A}_s$ for $s = 1, \ldots, k$. Thus

$$\dot{m}^G = (\tau^*(\check{0})^G; \in, \tau^*(\check{1})^G, \ldots, \tau^*(\check{k})^G) = \bar{\mathcal{M}} \in H.$$

By (13) and (14), we may prove inductively on the complexity of the formula θ that

$$H \models \theta(\tau^*(\check{\xi}_1)^G, \ldots, \tau^*(\check{\xi}_l)^G) \iff G \Vdash \dot{h} \models \theta(\tau^*(\check{\xi}_1), \ldots, \tau^*(\check{\xi}_l)).$$

In particular,

$$H \models \varphi(\bar{\mathcal{M}}),$$

and we are done. □

Theorem 1.3 may easily be used to show the well–known fact that if $\mathsf{FA}(\Gamma)_\kappa$ holds true for all κ (i.e., if the *un*bounded Forcing Axiom holds for Γ), then for all $\mathbb{B} \in \Gamma$ and for all universally Baire sets $A \subset \mathbb{R}$,

$$(H_{\omega_2}; \in, A) \prec_{\Sigma_1} ((H_{\omega_2})^{V^{\mathbb{B}}}; \in, A^*),$$

where A^* is the version of A in $V^{\mathbb{B}}$ (i.e., $p[T]^{V^{\mathbb{B}}}$, where T, U witness the universal Baireness of $A = p[T]$).

Recall that a predicate $\dot{A}$ *occurs positively* in a formula φ iff $\varphi \equiv x \in \dot{A}$ or $\dot{A}$ doesn't occur in φ at all or else $\varphi \equiv \psi_0 \wedge \psi_1$ or $\psi_0 \vee \psi_1$ and $\dot{A}$ occurs positively in both ψ_0 and ψ_1 or else $\varphi \equiv \forall x \psi$ or $\exists x \psi$ and $\dot{A}$ occurs positively in ψ. If $\dot{A}$ occurs positively in φ and if $\bar{A} \subset A \subset M$, then $(M; \bar{A}) \models \varphi$ implies $(M; A) \models \varphi$. Theorem 1.3 may then easily be used to prove the following.

Corollary 1.4 *Let* Γ *be a class of Boolean algebras, and suppose* $\mathsf{FA}(\Gamma)_{2^{\aleph_0}}$ *to hold. Let* $A_1, \ldots, A_k$ *be sets of reals. Then for all* $\mathbb{B} \in \Gamma$ *and for all* Σ_1 *formulae* φ *in which all of* $A_1, \ldots, A_k$ *occur positively,*

$$(H_{\omega_2}; \in, A_1, \ldots, A_k) \models \varphi(A_1, \ldots, A_k) \iff$$

$$((H_{\omega_2})^{V^{\mathbb{B}}}; \in, A_1, \ldots, A_k) \models \varphi(A_1, \ldots, A_k).[8]$$

2 Mice.

This section defines the concept of "mouse reflection" as it will be used in the present paper. We refer the reader to [32] on inner model theory. We will use standard notation throughout. We use the phrase "mouse" here as being defined in [12, Definition 1.1]: a *mouse* is a premouse such that the transitive collapse of any of its countable (sufficiently elementary) substructures is $\omega_1 + 1$ iterable. If X is a set of ordinals, then an X-*mouse* is an X-premouse such that the transitive collapse of any of its countable (sufficiently elementary) substructures is $\omega_1 + 1$ iterable.

In order to prove Theorem 0.1, we need to verify the following folklore result according to which the "mousehood" of a premouse of size at most $\aleph_1$ is a $\Sigma_1^{H_{\omega_2}}$ property provided that there be no inner model with a Woodin cardinal.

Lemma 2.1 *Suppose that there is no inner model with a Woodin cardinal. Let* U *be a transitive model of* ZFC^- *(i.e., of* ZFC *without the power set axiom) plus "there is no inner model with a Woodin cardinal." Also assume* $\omega_1 \subset U$*. Let* $\mathcal{M} \in U$ *be a premouse (possibly of uncountable size) and with no definable Woodin cardinal.*[9] *Then*

$$\mathcal{M} \textit{ is a mouse } \iff U \models \mathcal{M} \textit{ is a mouse.}$$

Proof. Let $\bar{\mathcal{M}}$ be a countable premouse such that there is a sufficiently elementary embedding of $\bar{\mathcal{M}}$ into $\mathcal{M}$. Because of our hypotheses, $\bar{\mathcal{M}}$ is countably (and hence fully) iterable with respect to normal trees if any only if the following holds true: if $\mathcal{T}$ is any countable putative[10] normal iteration tree on $\bar{\mathcal{M}}$, then either $\mathcal{T}$ has successor length and its last model is well–founded or else $\mathcal{T}$ has limit length and there is a maximal (and hence cofinal) branch b through $\mathcal{T}$ such that $\mathcal{M}_b^{\mathcal{T}}$ has an initial segment which is isomorphic to $J_\alpha(\mathcal{M}(\mathcal{T}))$, where α is least such that $\delta(\mathcal{T})$ is not definably Woodin in $J_\alpha(\mathcal{M}(\mathcal{T}))$. Here, $\mathcal{M}(\mathcal{T})$ is the common part model of $\mathcal{T}$ and $\delta(\mathcal{T})$ is its height; the model $J_\alpha(\mathcal{M}(\mathcal{T}))$ would be called the $\mathcal{Q}$–structure for $\mathcal{T}$ (cf. e.g. [25]).

Now in U there is a tree S of height ω searching for

- a countable premouse $\bar{\mathcal{M}}$ together with a sufficiently elementary embedding of $\bar{\mathcal{M}}$ into $\mathcal{M}$,
- a countable putative normal iteration tree $\mathcal{T}$ on $\bar{\mathcal{M}}$, and either
- a proof that $\mathcal{T}$ has a last ill–founded model, or else

[8] Notice that $A_1, \ldots, A_k$ don't get reinterpreted in $V^{\mathbb{B}}$.

[9] I.e., for each $\delta \leq \mathcal{M} \cap \mathrm{OR}$, either $\rho_\omega(\mathcal{M}) < \delta$ or else there is some $n < \omega$ and some $A \in r\Sigma_n^{\mathcal{M}}$ such that δ is not Woodin with respect to $A \cap \delta$.

[10] i.e., we do not demand that if $\mathcal{T}$ has successor length, then the last model of $\mathcal{T}$ be well–founded

- a proof that $\mathcal{T}$ has limit length but no cofinal branch b such that $\mathcal{M}_b^{\mathcal{T}}$ has an initial segment which is isomorphic to $J_\alpha(\mathcal{M}(\mathcal{T}))$, where α is least such that $\delta(\mathcal{T})$ is not definably Woodin in $J_\alpha(\mathcal{M}(\mathcal{T}))$.

We may let S prove the last statement by having S search for a countable transitive model U' of ZFC^- such that $\{\bar{\mathcal{M}}, \mathcal{T}\} \subset U'$, $\delta(\mathcal{T})$ is not definably Woodin in $J_\alpha(\mathcal{M}(\mathcal{T})) \models$ for some $\alpha < U' \cap \mathrm{OR}$, and if α is the least such, then $U' \models$ "$\mathcal{M}$ and $\mathrm{lh}(\mathcal{M})$ are countable, and $\mathcal{T}$ has no cofinal branch b such that $\mathcal{M}_b^{\mathcal{T}}$ has an initial segment which is isomorphic to the $J_\alpha(\mathcal{M}(\mathcal{T}))$." Notice that the statement that $\mathcal{T}$ has no such cofinal branch b is Π^1_1 in real parameters from U' coding $\mathcal{T}$ and $J_\alpha(\mathcal{M}(\mathcal{T}))$, so that $\mathbf{\Sigma}^1_1$-absoluteness tells us that this works. We need $\omega_1 \subset U$ to allow S to search for a U' as described of arbitrary countable height.

But now $\mathcal{M}$ is not iterable with respect to normal trees if and only if S is ill–founded in V if and only if S is ill–founded in U if and only if in U, $\mathcal{M}$ is not iterable with respect to normal trees. We have that $\mathcal{M}$ is a mouse if and only if $\mathcal{M}$ is iterable with respect to ω stacks of normal iteration trees on $\mathcal{M}$, cf. [12, Definition 1.1]. A straightforward slight variant of the argument given here thus shows that $\mathcal{M}$ is a mouse if and only if $\mathcal{M}$ is a mouse inside U. $\square$ (Lemma 2.1)

We'll need an appropriate generalization of Lemma 2.1 in order to be able to prove Theorems 0.7 and 0.8. To formulate this generalization, we need the concept of a "mouse operator," which for our purposes we formulate as follows.[11]

Definition 2.2 *Let $\varphi \equiv \varphi(v_0, v_1)$ be a Σ_1–formula in the language of boldface premice. Let X be a set of ordinals. An X–premouse $\mathcal{M}$ is called φ–*small *iff*

$$\mathcal{M} \models \neg\, \varphi(X).$$

The mouse operator given by φ *is the unique partial map $X \mapsto J(X) = J_\varphi(X)$ which assigns to any set X of ordinals the unique X–mouse $J(X)$ such that $J(X)$ is sound above X, $J(X)$ is not φ–small, but every proper initial segment of $J(X)$ is φ–small, if it exists (otherwise $J(X) = J_\varphi(X)$ remains undefined). A* mouse operator *is a partial map $X \mapsto J(X)$ for which there is some Σ_1–sentence φ such that $X \mapsto J(X)$ is the mouse operator given by φ.*

Let J be a mouse operator which is given by φ, and let λ be an uncountable cardinal (we allow $\lambda = \infty$). We say that J is total on bounded subsets of λ *(or,* on H_λ*) iff for all sets X of ordinals which are bounded in λ, $J(X)$ exists.*

Notice that if $X \mapsto J(X)$ is a mouse operator, then $\rho_1(J(X)) \leq \sup(X)$ whenever $J(X)$ exists.

Examples of mouse operators we shall be concerned with are $X \mapsto X^\#$ and, more generally, $X \mapsto M_n^\#(X)$ for $n < \omega$. (Cf. [25].) But we shall also need the following mouse operator, cf. Theorems 0.1 and 0.7.

Definition 2.3 *For a set X of ordinals, $M_{la}(X)$ is the least X–mouse $\mathcal{M}$ (if it exists) such that for some $\alpha < \mathcal{M} \cap \mathrm{OR}$,*

$$\mathcal{M} \models \text{“}\mathsf{ZFC} \textit{ plus there is a Woodin cardinal.”}$$

[11] In general, one would have to allow real parameters in the formula φ in definition 2.2.

The following is a condensation result for mouse operators. Its proof is trivial.

Lemma 2.4 *Let $X \mapsto J(X)$ be a mouse operator which is given by φ, z. Suppose that X codes z and $J(X)$ exists. If*

$$\pi \colon \mathcal{M} \to_{\Sigma_1} J(X),$$

where, say, $\pi(\bar{X}) = X$ (and $\pi(z) = z$), then $J(\bar{X})$ exists and in fact $\mathcal{M} = J(\bar{X})$.

Definition 2.5 *Let $n < \omega$. We say that $X \mapsto J(X)$* does not go beyond *$X \mapsto M_{la}(X)$ iff $M_{la}(X)$ is never a* proper *initial segment of $J(X)$. We say that $X \mapsto J(X)$* does not go beyond *$X \mapsto M_n^\#(X)$ iff $M_n^\#(X)$ is never a* proper *initial segment of $J(X)$, and $X \mapsto J(X)$ is n–*small *iff $M_n^\#(X)$ is not a proper or improper initial segment of $J(X)$.*

If $n > 0$, then $X \mapsto M_n^\#(X)$ is both in $L(\mathbb{R})$ as well as "nice" according to the following ad hoc definition. The concept of "niceness" will play a role in Lemma 3.1.

Definition 2.6 *A mouse operator $X \mapsto J(X)$ is said to be in $L(\mathbb{R})$ iff it is total on H_{ω_1}, every $J(X)$ is tame,[12] and the function which assigns to a bounded subset X of ω_1 the unique ω_1 iteration strategy for $J(X)$ with respect to stacks of normal iteration trees on $J(X)$ is an element of $L(\mathbb{R})$.*

Let $X \mapsto J(X)$ be a mouse operator. We call $X \mapsto J(X)$ nice *iff for all X such that $J(X)$ exists, $J(X)$ has at least two measurable cardinals, and if λ is the second smallest measurable cardinal of $J(X)$ and if Y is a set of ordinals in $J(X)^{\mathbb{P}}$, where $\mathbb{P} \in J(X)||\lambda$ is a poset, then $J(Y)$ exists and is Σ_1–definable over $J(X)^{\mathbb{P}}$ from the parameter Y.*

Definition 2.7 *Let κ, λ be infinite cardinals with $\kappa < \lambda$. We also allow $\lambda = \infty$. Then (κ, λ)–*mouse reflection *says that whenever $X \mapsto J(X)$ is a mouse operator which is total on H_κ, then $X \mapsto J(X)$ is total on H_λ.*

In this language, [31, Theorems 9.78 and 9.84] study $(\aleph_2, \aleph_3)$–mouse reflection; cf. also [28]. We shall be concerned with $(\aleph_2, \infty)$–mouse reflection in what follows.

If $X \mapsto J(X)$ is total on all sets of ordinals, then we also say that $X \mapsto J(X)$ is total on V.

Recall that if $n < \omega$ and $X \mapsto M_n^\#(X)$ is total on V, then $X \mapsto M_n^\#(X)$ is also total on $V^{\mathbb{P}}$ and

$$(M_n^\#(X))^V = (M_n^\#(X))^{V^{\mathbb{P}}}$$

for all $X \in V$, whenever $\mathbb{P}$ is a poset in V (cf. [3, Lemma 3.7]).

We shall need the following version of Lemma 2.1. The point is that we allow H_{ω_2} to be closed under complicated mouse operators in Lemma 2.8, whereas Lemma 2.1 assumes that there be no inner model with a Woodin cardinal.

[12] I.e., if $E_\alpha^{J(X)} \neq \emptyset$, then $J(X)|\alpha \models$ "there is no Woodin cardinal $\geq \text{crit}(E_\alpha^{J(X)})$."

Lemma 2.8 *Let U be a transitive model of* ZFC^- *plus "every set is contained in a transitive model of* ZFC*." Also assume $\omega_1 \subset U$. Let $\mathcal{M} \in U$ be an X–premouse (possibly of uncountable size) with no definable Woodin cardinal. Suppose that $M_{la}(X)$ does not exist. Then*

$$\mathcal{M} \textit{ is an } X\textit{–mouse} \iff U \models \mathcal{M} \textit{ is an } X\textit{–mouse.}$$

Proof. The proof is a variant of the proof of Lemma 2.1. We let the tree $S \in U$ of height ω search for

- a countable premouse $\bar{\mathcal{M}}$ together with a sufficiently elementary embedding of $\bar{\mathcal{M}}$ into $\mathcal{M}$,
- a countable putative normal iteration tree $\mathcal{T}$ on $\bar{\mathcal{M}}$, and either
- a proof that $\mathcal{T}$ has a last ill–founded model, or
- a proof that $\mathcal{T}$ has limit length and if α is least such that $J_\alpha(\mathcal{M}(\mathcal{T})) \models \mathsf{ZFC}$, then $J_{\alpha+1}(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is not a Woodin cardinal," but there is no no cofinal branch b such that $\mathcal{M}_b^{\mathcal{T}}$ has an initial segment which is isomorphic to an initial segment of $J_{\alpha+1}(\mathcal{M}(\mathcal{T}))$, or else
- a proof that $\mathcal{T}$ has limit length and if α is least such that $J_\alpha(\mathcal{M}(\mathcal{T})) \models \mathsf{ZFC}$, then $J_{\alpha+1}(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is a Woodin cardinal."

The rest is as in the proof of Lemma 2.1. □ (Lemma 2.8)

Arguments as in the proof of the following lemma will be used in section 7.

Lemma 2.9 *Let $n < \omega$, and suppose that $X \mapsto M_n^\#(X)$ is total on V. Then*

$$\{(x, y) \in \mathbb{R}^2 \colon y \textit{ codes } M_n^\#(x)\}$$

is universally Baire.

Proof. Let θ be an uncountable regular cardinal. Let T be a tree of height ω which searches for x, y, $\mathcal{N}$, H, and π such that

- $x, y \in \mathbb{R}$,
- y codes an x–premouse $\mathcal{M}$,
- $\mathcal{N}$ is an H–premouse, $\pi \colon \mathcal{N} \to M_n^\#(H_\theta)$, and $\pi(H) = H_\theta$,
- there is some $\mathbb{Q} \in H$ and some $\mathbb{Q}$–generic filter g over $\mathcal{N}$ such that $x \in \mathcal{N}[g]$,
- $\mathcal{M}$ is equal to $M_n^\#(x)$, as computed from $\mathcal{N}[g]$[13].

We claim that T projects to the set of all (x, y) such that y codes $M_n^\#(x)$ in any extension $V[G]$, where G is $\mathbb{P}$–generic over V for some $\mathbb{P} \in H_\theta$. In order to verify this, let us first assume that G is $\mathbb{P}$–generic over V for some $\mathbb{P} \in H_\theta$ and that y codes $M_n^\#(x)$ in $V[G]$. Then (x, y) is in the projection of T, as being witnessed by $\mathcal{N}$, H, and π, where for some σ and P^*, $\pi = \sigma \restriction \mathcal{N} \colon \mathcal{N} \to M_n^\#(H_\theta)$, $\sigma \colon P^* \to H_{(2^{<\theta})^+}[G]$, P^* is countable and transitive, $H = \pi^{-1}(H_\theta)$, and $\mathcal{N} = \sigma^{-1}(M_n^\#(H_\theta))$. We may simply put $\mathbb{Q} = \sigma^{-1}(\mathbb{P})$ and $g = \sigma^{-1}(G)$. On the other hand, if (x, y) is in the

[13]I.e., $\mathcal{M}$ is either the least sound x-premouse from the $L[E, x]$-construction inside $\mathcal{N}[g]$ which is not n-small or else $L[E, x]^{\mathcal{N}[g]}$ is n-small and $\mathcal{M}$ results from the $L[E, x]$-construction done inside $\mathcal{N}[g]$ and of height $\mathcal{N} \cap \mathrm{OR}$ by adding the top extender derived from the top extender of $\mathcal{N}[g]$. By [7], the resulting structure is iterable in the latter case.

projection of T, as being witnessed by $\mathcal{N}$, H, π, $\mathbb{Q}$ and g, then $\mathcal{N}[g]$ inherits the iterability from $\mathcal{N}$, which is in turn iterable as being certified by $\pi\colon \mathcal{N} \to M_n^{\#}(H_\theta)$. Therefore, y must code $M_n^{\#}(x)$.

We may now also construct a tree U such that U searches for x, y, y', Q, where (x, y') is in the projection of T and Q is a countable transitive model with y, $y' \in Q$ such that Q knows that y' is not isomorphic to y. Then U projects to the complement of $p[T]$ in any extension $V[G]$, where G is $\mathbb{P}$–generic over V for some $\mathbb{P} \in H_\theta$. □ (Lemma 2.9)

3 $(*)$ and the closure under mouse operators.

We now first prove the lemma which was mentioned in the introduction after the statement of Theorem 0.2.

Lemma 3.1 *Assume $(*)$ to hold. Let $X \mapsto J(X)$ be a nice mouse operator in $L(\mathbb{R})$. Then $X \mapsto J(X)$ is total on H_{ω_2}.*

Proof. For $x \in \mathbb{R}$, we let κ_x denote the least measurable cardinal of $J(x)$, we let U_x denote the unique measure on κ_x in $J(x)$, and we let $\mathbb{P}_x \in J(x)^{\mathrm{Col}(\omega,<\kappa_x)}$ denote the standard c.c.c. forcing for producing Martin's Axiom MA_{ω_1}. Let us consider the set D of all $p \in \mathbb{P}_{\max}$ such that if $p = (M, I, a)$, then there is some $x \in \mathbb{R}$ such that

$$M = J(x)^{\mathrm{Col}(\omega,<\kappa_x)*\mathbb{P}_x},$$

and I is the precipitous ideal of M induced by U_x, i.e.,

$$I = \{X \in \mathcal{P}(\kappa_x) \cap M \colon \exists Y \in U_x\ X \cap Y = \emptyset\}.$$

Standard $\mathbb{P}_{\max}$ arguments show that $D \in L(\mathbb{R})$ and D is dense in $\mathbb{P}_{\max}$ (cf. [31, Lemma 4.36]).

Now let $A \subset \omega_1$. By $(*)$, we may assume without loss of generality that A is $\mathbb{P}_{\max}$–generic over $L(\mathbb{R})$. Let G_A be the $\mathbb{P}_{\max}$–generic filter which is given by A. As $D \in L(\mathbb{R})$ is dense in $\mathbb{P}_{\max}$, we may pick $p \in D \cap G_A$. Let

$$((M_\alpha, I_\alpha, a_\alpha), (\pi_{\bar\alpha,\alpha} \colon \bar\alpha \le \alpha \le \omega_1))$$

be the generic iteration of $p = (M_0, I_0, a_0)$ given by G_A, so that $a_{\omega_1} = A$. Let $M_0 = J(x)^{\mathrm{Col}(\omega,<\kappa_x)*\mathbb{P}_x}$, where $x \in \mathbb{R}$. For all $\alpha \le \omega_1$, the map $\pi_{0,\alpha} \restriction J(x)$ is the map obtained by iterating U_x and its images α times, and

$$M_\alpha = \pi_{0,\alpha}(J(x))^{\mathrm{Col}(\omega,<\pi_{0,\alpha}(\kappa_x))*\pi_{0,\alpha}(\mathbb{P}_x)}.$$

The reason is that the forcing $\mathrm{Col}(\omega, <\kappa_x) * \mathbb{P}_x$ has the κ_x-c.c. from the point of view of $J(x)$.

We claim that M_{ω_1} witnesses that $J(A)$ exists. We first need to see that M_{ω_1} is a mouse, i.e., that transitive collapses of its countable (Σ_1 elementary) substructures are $\omega_1 + 1$ iterable. Let

$$\sigma \colon \mathcal{P} \to M_{\omega_1}$$

be Σ_1 elementary, where $\mathcal{P}$ is countable and transitive. Then there is some $\alpha < \omega_1$ and some

$$\sigma' \colon \mathcal{P} \to M_\alpha$$

such that $\pi_{\alpha,\omega_1} \circ \sigma' = \sigma$. But $M_\alpha = \pi_{0,\alpha}(J(x))^{\mathrm{Col}(\omega,<\pi_{0,\alpha}(\kappa_x))*\pi_{0,\alpha}(\mathbb{P}_x)}$, where $\pi_{0,\alpha}(J(x))$ is the α^{th} iterate of $J(x)$ obtained by hitting the measure U_x and its images α times. This clearly implies that M_α (and hence $\mathcal{P}$) is $\omega_1 + 1$ iterable.

But $a_{\omega_1} = A$ and $X \mapsto J(X)$ is nice. Therefore, $J(A)$ exists and is Σ_1–definable over M_{ω_1} from the parameter A. $\square$ (Lemma 3.1)

4 Mouse reflection at $\aleph_2$.

We now show Theorems 0.1, 0.2, and 0.7. The proofs of Theorems 0.1 and 0.7 use a key idea of Stevo Todorčević to phrase a Σ_2 statement in a Σ_1 way under favorable circumstances (cf. [29], proof of Lemma 4).

Proof of Theorem 0.1. Let J be a mouse operator as in the statement of Theorem 0.1. Let $X \subset \kappa$, where $\kappa \geq \aleph_2$ is a cardinal. Let $S(X)$ denote the stack of all X–mice which are sound above κ and project to κ or below κ, i.e., $\mathcal{P} \lhd S(X)$ iff there is some X–mouse $\mathcal{Q} \trianglerighteq \mathcal{P}$ such that $\mathcal{Q}$ is sound above κ and $\rho_\omega(\mathcal{Q}) \leq \kappa$. Then $S(X)$ is itself an X–mouse, $S(X) \models \mathsf{ZFC}^-$, and κ is the largest cardinal of $S(X)$.

Let us now suppose that $J(X)$ does not exist and work towards a contradiction. Let us suppose that the mouse operator J is defined in terms of the Σ_1–sentence φ. By our hypothesis, $S(X)$ is φ–small.

We first claim that $\mathrm{cf}^V(S(X) \cap \mathrm{OR}) \geq \aleph_2$. In order to prove this, let

$$\pi \colon \bar{S} \to S(X)$$

be fully elementary, where $\mathrm{Card}(\bar{S}) = \aleph_1$. Setting $\bar{X} = \pi^{-1}(X)$ and $\bar{\kappa} = \pi^{-1}(\kappa)$, $\bar{S}$ is a φ–small $\bar{X}$–mouse with largest cardinal $\bar{\kappa}$. Because $J(\bar{X})$ exists, we may let $\mathcal{Q} \trianglelefteq J(\bar{X})$ be least such that $\bar{S} \lhd \mathcal{Q}$ and $\rho_\omega(\mathcal{Q}) \leq \bar{\kappa}$. Let $n < \omega$ be such that $\rho_{n+1}(\mathcal{Q}) \leq \bar{\kappa} < \rho_n(\mathcal{Q})$, and let

$$\mathcal{Q}^* = \mathrm{ult}_n(\mathcal{Q}; E_\pi),$$

where E_π is the extender derived from π. We may and shall assume that π was chosen in such a way that $\mathcal{Q}^*$ is well–founded (i.e., transitive) and in fact is an X–mouse (cf. [15]). Now if $\mathrm{cf}^V(S(X) \cap \mathrm{OR}) < \aleph_2$, then we may and shall also assume that $\mathrm{ran}(\pi) \cap \mathrm{OR}$ is cofinal in $S(X) \cap \mathrm{OR}$. We would then have $\mathcal{Q}^* \rhd S(X)$, $\mathcal{Q}^*$ is an X–mouse which is sound above κ and $\rho_{n+1}(\mathcal{Q}) \leq \kappa$, which contradicts the definition of $S(X)$. We must therefore have $\mathrm{cf}^V(S(X) \cap \mathrm{OR}) \geq \aleph_2$.

Let us now define a tree $T = T_{S(X)}$, derived from $S(X)$, as follows. We put $\mathcal{Q} \in T$ iff $\mathcal{Q} \lhd S(X)$, and setting $\lambda^{\mathcal{Q}} = \kappa^{+\mathcal{Q}}$ we have that $\mathcal{Q}|\lambda^{\mathcal{Q}} \prec_{\Sigma_\omega} S(X)$ and $\rho_\omega(\mathcal{Q}) \leq \kappa$. If $\mathcal{Q} \in T$, then we shall write $n(\mathcal{Q})$ for the unique $n < \omega$ with $\rho_{n+1}(\mathcal{Q}) \leq \kappa < \rho_n(\mathcal{Q})$. If $\bar{\mathcal{Q}}$, $\mathcal{Q} \in T$, then we write $\bar{\mathcal{Q}} \leq_T \mathcal{Q}$ iff $n(\bar{\mathcal{Q}}) = n(\mathcal{Q})$, call it n, and there is a weakly $r\Sigma_n$ elementary embedding

$$\sigma \colon \bar{\mathcal{Q}} \to \mathcal{Q}$$

such that $\sigma \restriction \lambda^{\bar{\mathcal{Q}}} = \text{id}$, $\sigma(p_n(\bar{\mathcal{Q}})) = p_n(\mathcal{Q})$, and if $\lambda^{\bar{\mathcal{Q}}} < \bar{\mathcal{Q}} \cap \text{OR}$, then $\sigma(\lambda^{\bar{\mathcal{Q}}}) = \lambda^{\mathcal{Q}}$. If $\bar{\mathcal{Q}}$, $\mathcal{Q} \in T$, then we shall write $\sigma_{\bar{\mathcal{Q}},\mathcal{Q}}$ for the unique map σ as above. The elements of T and the maps between them are thus as in the usual construction of $\square_\kappa$ inside $S(X)$ (cf. [19]).

Let us write $\lambda = S(X) \cap \text{OR}$. In $V^{\text{Col}(\omega_1,\lambda)}$, T can be shrinked a little bit so as to produce a tree of height and size $\aleph_1$. Namely, letting $C \in V^{\text{Col}(\omega,\lambda)}$ be a club subset of λ of order type ω_1, we may let $\mathcal{Q} \in T^* = T^*_{S(X)}$ iff $\mathcal{Q} \in T$ and $\lambda^{\mathcal{Q}} = \kappa^{+\mathcal{Q}} \in C$, and we let $\leq_{T^*} = \leq_T \restriction T^*$. We claim that T^* does not have a branch of length ω_1 in $V^{\text{Col}(\omega_1,\lambda)}$. Suppose not, and let b be a branch through T^* of length ω_1. It is then easy to verify that

$$\text{dir lim } (\mathcal{Q}, \sigma_{\bar{\mathcal{Q}},\mathcal{Q}} \colon \bar{\mathcal{Q}} \leq_{T^*} \mathcal{Q} \in b)$$

is (isomorphic to) an X–mouse, call it $\mathcal{Q}^*$, with $\kappa^{+\mathcal{Q}^*} = \lambda$ and which is sound above κ and $\rho_\omega(\mathcal{Q}^*) \leq \kappa$. However, there can be only one such X–mouse of the same height as $\mathcal{Q}^*$ in $V^{\text{Col}(\omega_1,\lambda)}$, so that in fact $\mathcal{Q}^* \in V$. As $\mathcal{Q}^*$ is certainly 1–small, we must have that $\mathcal{Q}^*$ is an X–mouse in V as well, so that we have a contradiction with the definition of $S(X)$. Hence in fact T^* does not have a branch of length ω_1 in $V^{\text{Col}(\omega_1,\lambda)}$.

Now let $\mathbb{P}$ denote the natural forcing for specializing T^*, i.e., for partitioning T^* into countably many antichains (cf. [10, p. 274f.]). The following statement, Φ, is then true in $V^{\text{Col}(\omega_1,\lambda)*\mathbb{P}}$, as being witnessed by κ, X, and $S(X)$.

(Φ) "There is some $\bar{\kappa} < \aleph_2$ and some $\bar{X} \subset \bar{\kappa}$ and there is some φ–small $\bar{X}$–mouse $\mathcal{S}$ such that $\mathcal{S} \cap \text{OR} < \aleph_2$ and $\text{cf}(\mathcal{S} \cap \text{OR}) = \omega_1$, $\mathcal{S}$ is a model of ZFC^- with largest cardinal $\bar{\kappa}$ such that the tree $T^*_{\mathcal{S}}$ which is derived from $\mathcal{S}$ is a tree of height and size $\aleph_1$ which does not have any branches of length ω_1."

(Here, by the "tree $T^*_{\mathcal{S}}$ derived from $\mathcal{S}$" we mean a tree which is derived from $\mathcal{S}$ in exactly the same manner as $T^* = T^*_{S(X)}$ was derived from $S(X)$ above.) Now $\text{Col}(\omega_1, \lambda) * \mathbb{P}$ is ω–closed $*$ c.c.c. and hence proper. Moreover, by Lemma 2.8, the fact that $\mathcal{S}$ is a φ–small $\bar{X}$–mouse can be expressed in a Σ_1 fashion over H_{ω_2}. Therefore, by BPFA, the statement Φ holds true in V also, as being witnessed by κ', X', and $\mathcal{S}'$, say.

But because $J(X')$ exists, there is some least $\mathcal{Q} \trianglelefteq J(X')$ such that $\mathcal{Q} \triangleright \mathcal{S}'$ and $\rho_\omega(\mathcal{Q}) \leq \kappa'$. Let m be the unique $n < \omega$ with $\rho_{n+1}(\mathcal{Q}) \leq \kappa' < \rho_n(\mathcal{Q})$. By the usual $\square_{\kappa'}$–type arguments there is a club $C \subset \mathcal{S}' \cap \text{OR}$ such that for every $\lambda \in C$ there is some $\bar{\mathcal{Q}} \triangleleft \mathcal{S}'$ such that $\lambda = (\kappa')^{+\bar{\mathcal{Q}}}$, $\rho_{m+1}(\bar{\mathcal{Q}}) \leq \kappa' < \rho_n(\bar{\mathcal{Q}})$, and there is a weakly $r\Sigma_m$ elementary embedding

$$\sigma \colon \bar{\mathcal{Q}} \to \mathcal{Q}$$

with $\sigma \restriction \lambda = \text{id}$, $\sigma(p_m(\bar{\mathcal{Q}})) = p_m(\mathcal{Q})$, and $\sigma(\lambda) = \mathcal{S}' \cap \text{OR}$. This shows that there is a cofinal branch through the tree $T^*_{\mathcal{S}'}$ which is derived from $\mathcal{S}'$.

We have reached a contradiction! $\square$ (Theorem 0.1)

We do not know if the conjunction of $(*)$ and BPFA implies that V is closed under $X \mapsto M_1^\#(X)$.

Proof of Theorem 0.2. This now readily follows from Lemma 3.1 and Theorem 0.1. $\square$ (Theorem 0.2)

The proof of Theorem 0.7 is an obvious generalization of the proof of Theorem 0.1. The new wrinkle is that the "mousehood" of a given premouse of size at most $\aleph_1$ is no longer expressible in a $\Sigma_1^{H_{\omega_2}}$ fashion in general.

PROOF of Theorem 0.7. Let n and J be as in the statement of Theorem 0.7. Suppose there to be some $X \subset \kappa$, where $\kappa \geq \aleph_2$, such that $J(X)$ does not exist, and suppose that the mouse operator J is defined in terms of the Σ_1–formula φ. Let $S(X)$, λ, and $\mathbb{P}$ be exactly as in the proof of Lemma 0.1. The statement Φ as formulated there will again be true in $V^{\mathrm{Col}(\omega_1,\lambda)*\mathbb{P}}$ as being witnessed by κ, X, and $S(X)$.

Let $g \in V^{\mathrm{Col}(\omega_1,\lambda)}$ be $\mathrm{Col}(\omega_1,\kappa)$–generic over $S(X)$. We my then reorganize $S(X)[g]$ as a φ–small Y–mouse, call it $\mathcal{S}$, where $Y \subset \omega_1$. If $\gamma < \mathcal{S} \cap \mathrm{OR}$ is such that $\rho_\omega(\mathcal{S}||\gamma) \leq \omega_1$, and if $\pi \colon \bar{\mathcal{S}} \to \mathcal{S}$, $\pi \in V^{\mathrm{Col}(\omega_1,\lambda)*\mathbb{P}}$ is such that $\bar{\mathcal{S}}$ is countable and transitive, then $\bar{\mathcal{S}}$ is in $V^{\mathrm{Col}(\omega_1,\lambda)}$ and hence in V, because, setting $\alpha = \mathrm{crit}(\pi)$,

$$\mathrm{ran}(\pi) = \mathrm{Hull}^{\mathcal{S}||\gamma}(\alpha \cup \{p\}),$$

where p is the standard parameter of $\mathcal{S}||\gamma$ and the hull is the appropriate fine structural one. Let us define in V the set of reals A as the set of all real codes for countable x–mice, where $x \subset \alpha$ for some $\alpha < \omega_1$. We have seen that if $\pi \colon \bar{\mathcal{S}} \to \mathcal{S}||\gamma$, where $\rho_\omega(\mathcal{S}||\gamma) \leq \omega_1$ and $\pi \in V^{\mathrm{Col}(\omega_1,\lambda)*\mathbb{P}}$, then $\bar{\mathcal{S}}$ is coded by a real in A.

Moreover, the tree $T^*_{\mathcal{S}}$, which is defined from $\mathcal{S}$ in much the same way as $T^*_{S(X)}$ was defined from $S(X)$ in the proof of Theorem 0.1, also does not have a branch of length ω_1 in $V^{\mathrm{Col}(\omega_1,\lambda)}$. This is by the argument from the proof of Theorem 0.1 and because any end–extension $\mathcal{Q}^*$ of $\mathcal{S}$ which is sound above ω_1 and such that $\rho_\omega(\mathcal{Q}^*) \leq \omega_1$ can be lightened by a *$\mathcal{P}$–construction* to produce an end–extension $\bar{\mathcal{Q}}^*$ of $S(X)$ which is sound above κ and such that $\rho_\omega(\mathcal{Q}^*) \leq \kappa$. The details of the procedure of $\mathcal{P}$–constructions may be found in [23, Section 1].

We thus verified that the following statement, Ψ, holds true in $V^{\mathrm{Col}(\omega_1,\lambda)*\mathbb{P}}$, as being witnessed by Y and $\mathcal{S}$.

(Ψ) "There is some $\bar{X} \subset \omega_1$ and there is some φ–small $\bar{X}$–premouse $\mathcal{S}'$ such that $\mathcal{S}' \cap \mathrm{OR} < \aleph_2$ and $\mathrm{cf}(\mathcal{S}' \cap \mathrm{OR}) = \omega_1$, $\mathcal{S}'$ is a model of ZFC^- with largest cardinal ω_1, if $\pi \colon \bar{\mathcal{S}} \to \mathcal{S}'||\gamma$ is such that $\rho_\omega(\bar{\mathcal{S}}||\gamma) \leq \omega_1$ and $\bar{\mathcal{S}}$ is countable and transitive, then $\bar{\mathcal{S}}$ is coded by a real in A, and the tree $T^*_{\mathcal{S}'}$ which is derived from $\mathcal{S}'$ is a tree of height and size $\aleph_1$ which does not have any branches of length ω_1."

By $\mathsf{FA}_{2^{\aleph_0}}$ for proper forcings, Ψ is true in V, as being witnessed by Y', $\mathcal{S}''$, say.

However, Ψ clearly gives that $\mathcal{S}''$ is a mouse. This gives a contradiction as in the proof of Theorem 0.1. $\square$ (Theorem 0.7)

5 BPFA and $\mathsf{AD}^{L(\mathbb{R})}$.

In order to show that in the presence of just $\mathsf{AD}^{L(\mathbb{R})}$ (rather than Woodin's $(*)$) BPFA doesn't imply any form of $(\aleph_2, \infty)$–mouse reflection, we need the concepts of remarkable and reflecting cardinals which are introduced in [20] and [8], respectively.

Lemma 5.1 *Let $\kappa < \lambda$, where κ is a remarkable limit of Woodin cardinals and λ is reflecting. Let $V = L[A]$, where $A \subset \kappa$. There is then a set–generic extension $V^{\mathbb{Q}}$ of V such that*

$$V^{\mathbb{Q}} \models \mathsf{BPFA} \ + \ \mathsf{AD}^{L(\mathbb{R})}$$

and $\kappa = \omega_1^{V^{\mathbb{Q}}}$.

If $V^{\mathbb{Q}}$ is as in Lemma 5.1, then in $V^{\mathbb{Q}}$ there is a subset of ω_1 (for instance, A) which does not have a sharp. The hypothesis of Lemma 5.1 is consistent by [20, Lemma 1.7].

Proof of Lemma 5.1. If $\mathbb{P} \in V^{\mathrm{Col}(\omega,<\kappa)}$ is proper in $V^{\mathrm{Col}(\omega,<\kappa)}$, then

$$L(\mathbb{R})^{V^{\mathrm{Col}(\omega,<\kappa)*\mathbb{P}}} \equiv L(\mathbb{R})^{V^{\mathrm{Col}(\omega,<\kappa)}},$$

i.e., these two models have the same first order theory (cf. [20, Theorem 2.4]). As AD holds in the $L(\mathbb{R})$ of $V^{\mathrm{Col}(\omega,<\kappa)}$, due to the fact that κ is a limit of Woodin cardinals, AD therefore holds in the $L(\mathbb{R})$ of $V^{\mathrm{Col}(\omega,<\kappa)*\mathbb{P}}$ as well.

We may now let $\mathbb{P} \in V^{\mathrm{Col}(\omega,<\kappa)}$ be the Goldstern–Shelah poset for forcing BPFA, exploiting the fact that λ is still reflecting in $V^{\mathrm{Col}(\omega,<\kappa)}$, and we may set $\mathbb{Q} = \mathrm{Col}(\omega, < \kappa) * \mathbb{P}$. □ (Lemma 5.1)

The hypothesis that κ be a limit of Woodin cardinals is not really necessary as part of the hypothesis of Lemma 5.1, of course. It would just be enough to assume for V_κ to be closed under $X \mapsto M_\omega^\#(X)$ or even slightly less.

6 Precipitous ideals.

Let us now turn to precipitous ideals. In order to prove the main result of this section, Theorem 0.3, we need an abstract criterion for the iterability of a certain phalanx, which is provided by the following lemma.

Lemma 6.1 *Let $\mathcal{M}$ and $\mathcal{N}$ be fully iterable premice, and let $j \colon \mathcal{M} \to \mathcal{N}$ be a non–trivial elementary embedding. Let $\mathcal{T}$ and $\mathcal{U}$ denote the iteration trees on $\mathcal{M}$ and $\mathcal{N}$, respectively, arising from the comparison of $\mathcal{M}$ and $\mathcal{N}$, and let us suppose that there is no drop along the main branch of $\mathcal{U}$, and that the final models $\mathcal{M}_\infty^{\mathcal{T}}$ of $\mathcal{T}$ and $\mathcal{M}_\infty^{\mathcal{U}}$ of $\mathcal{U}$ are the same. Let us assume that $\mathrm{crit}(\pi_\infty^{\mathcal{T}}) \geq \mathrm{crit}(j)$ and $\mathrm{crit}(\pi_\infty^{\mathcal{U}}) \geq \mathrm{crit}(j)$, where we write $\pi_\infty^{\mathcal{T}}$ and $\pi_\infty^{\mathcal{U}}$ for the iteration maps from the main branches of $\mathcal{T}$ and $\mathcal{U}$, respectively.*

Let F be the $(\kappa, j(\kappa))$–extender on $\mathcal{M}$ derived from j, i.e., $X \in F_a$ iff $a \in j(X)$, where $a \in [j(\kappa)]^{<\omega}$ and $X \in \mathcal{P}([\kappa]^{\mathrm{Card}(a)}) \cap \mathcal{M}$. Then F is an extender over $\mathcal{N}$ as well, and in fact the phalanx

$$(\mathcal{N}, \mathrm{Ult}(\mathcal{N}; F), j(\kappa))$$

is fully iterable.

Proof. Let us write $\mathcal{Q} = \mathcal{M}^{\mathcal{T}}_{\infty} = \mathcal{M}^{\mathcal{U}}_{\infty}$. By the Dodd–Jensen Lemma, there is no drop along the main branch of $\mathcal{T}$, so that both $\pi^{\mathcal{T}}_{\infty}$ and $\pi^{\mathcal{U}}_{\infty}$ are indeed well–defined.

Write $\kappa = \mathrm{crit}(j)$. As $\mathrm{crit}(\pi^{\mathcal{T}}_{\infty}) \geq \kappa$ and $\mathrm{crit}(\pi^{\mathcal{U}}_{\infty}) \geq \kappa$, setting $\lambda = \kappa^{+\mathcal{M}}$, we have that $\lambda = \kappa^{+\mathcal{N}}$, $\mathcal{M}|\lambda = \mathcal{N}|\lambda$ and both $\mathcal{T}$ and $\mathcal{U}$ only use extenders with indices larger than λ.

We shall produce an iterate $\mathcal{Q}^*$ of $\mathcal{N}$ obtained by using only extenders with index above $j(\lambda)$ and an embedding

$$\ell \colon \mathrm{Ult}(\mathcal{N}; F) \to \mathcal{Q}^*$$

with $\ell \restriction j(\kappa) = \mathrm{id}$. The phalanx

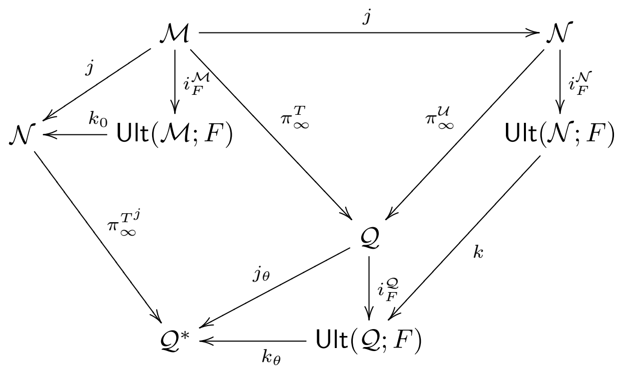
$$(\mathcal{N}, \mathcal{Q}^*, j(\kappa))$$

is certainly iterable, as every iteration of it may be construed as a continuation of the iteration of $\mathcal{N}$ which produces $\mathcal{Q}^*$, and therefore the phalanx

$$(\mathcal{N}, \mathrm{Ult}(\mathcal{N}; F), j(\kappa))$$

is also iterable.

The construction to follow is summarized by figure 1.



The iterability of $(\mathcal{N}, \mathrm{Ult}(\mathcal{N}; F), j(\kappa))$.

Let us first copy the iteration $\mathcal{T}$ onto $\mathcal{N}$ via the map j, producing an iteration tree $\mathcal{T}^j$ on $\mathcal{N}$. As usual, we shall have that if the j_α's are the copy maps,

$$j_\alpha \colon \mathcal{M}^{\mathcal{T}}_{\alpha} \to \mathcal{M}^{\mathcal{T}^j}_{\alpha},$$

where $\alpha < \mathrm{lh}(\mathcal{T})$, then

$$j_\beta \restriction \mathrm{lh}(E^{\mathcal{T}}_{\alpha}) = j_\alpha \restriction \mathrm{lh}(E^{\mathcal{T}}_{\alpha})$$

whenever $\alpha \leq \beta < \mathrm{lh}(\mathcal{T})$. As $\nu(E^{\mathcal{T}}_{\alpha}) > \lambda$ for all $\alpha < \mathrm{lh}(\mathcal{T})$, i.e., all indices of extenders used in $\mathcal{T}$ are larger than λ, this agreement implies that for all $a \in [j(\kappa)]^{<\omega}$, for all $X \in \mathcal{P}([\kappa]^{\mathrm{Card}(a)}) \cap \mathcal{M}$, and for all α, $\beta < \mathrm{lh}(\mathcal{T})$,

$$a \in j_\beta(X) \text{ iff } a \in j_\alpha(X).$$

But $j_0 = j$, so that in fact the $(\kappa, j(\kappa))$–extender derived from any j_α, $\alpha < \mathrm{lh}(\mathcal{T})$, is just F.

In other words, we may factor any of the maps $j_\alpha \colon \mathcal{M}_\alpha^\mathcal{T} \to \mathcal{M}_\alpha^{\mathcal{T}^j}$ as

$$j_\alpha = k_\alpha \circ i_F^{\mathcal{M}_\alpha^\mathcal{T}},$$

where

$$i_F^{\mathcal{M}_\alpha^\mathcal{T}} : \mathcal{M}_\alpha^\mathcal{T} \to \mathrm{Ult}(\mathcal{M}_\alpha^\mathcal{T}; F)$$

is the ultrapower map and

$$k_\alpha \colon \mathrm{Ult}(\mathcal{M}_\alpha^\mathcal{T}; F) \to \mathcal{M}_\alpha^{\mathcal{T}^j}$$

is the factor map which is defined as

$$i_F^{\mathcal{M}_\alpha^\mathcal{T}}(f)(a) \mapsto j_\alpha(f)(a),$$

where $a \in [j(\kappa)]^{<\omega}$ and $f \in \mathcal{M}_\alpha^\mathcal{T}$ are appropriate. Notice that

$$\mathrm{crit}(k_\alpha) \geq j_\alpha(\kappa) = j(\kappa)$$

for every $\alpha < \mathrm{lh}(\mathcal{T})$.

Let $\theta + 1 = \mathrm{lh}(\mathcal{T})$, so that $\mathcal{Q} = \mathcal{M}_\theta^\mathcal{T}$.

Let

$$i_F^\mathcal{N} \colon \mathcal{N} \to \mathrm{Ult}(\mathcal{N}; F)$$

and

$$i_F^\mathcal{Q} \colon \mathcal{Q} \to \mathrm{Ult}(\mathcal{Q}; F)$$

be the ultrapower maps. We may define

$$i \colon \mathrm{Ult}(\mathcal{N}; F) \to \mathrm{Ult}(\mathcal{Q}; F)$$

by

$$i_F^\mathcal{N}(f)(a) \mapsto i_F^\mathcal{Q} \circ \pi_\infty^\mathcal{U}(f)(a),$$

where $a \in [j(\kappa)]^{<\omega}$ and $f \in \mathcal{N}$ are appropriate. Notice that $\mathrm{crit}(i) \geq j(\kappa)$.

Now consider

$$\ell = k_\theta \circ i \colon \mathrm{Ult}(\mathcal{N}; F) \to \mathcal{Q}^*.$$

We have that $\mathrm{crit}(\ell) \geq j(\kappa)$, and $\mathcal{Q}^*$ is an iterate of $\mathcal{N}$ obtained by an iteration which uses only extenders with index larger than $j(\lambda)$. That is, $\mathcal{Q}^*$ and ℓ are as desired. □ (Lemma 6.1)

We now prove Theorem 0.3.

Proof of Theorem 0.3. In order to not get involved into issues which only hide the key idea, let us pretend that there be a large (for instance, measurable) cardinal, Ω, up to which K may be defined. K will be fully iterable.

Let I be a precipitous ideal on κ, let G be I^+/I–generic over V, and let, in $V[G]$,

$$j \colon V \to M \subset V[G]$$

be the generic elementary embedding produced by the ultrapower given by G. Here, M is transitive. Let us assume that $\kappa^{+K} < \kappa^{+V}$ and work towards a contradiction. Let us write $\lambda = \kappa^{+K}$.

Let $f\colon \kappa \to \mathcal{P}(\kappa) \cap K$ be bijective, $f \in V$. Then $f \in M$, as $f(\xi) = j(f)(\xi) \cap \kappa$ for all $\xi < \kappa$, and therefore $j \restriction \mathcal{P}(\kappa) \in M$, as $(j \restriction \mathcal{P}(\kappa))(x) = y$ iff there is some $\xi < \kappa$ with $f(\xi) = x$ and $j(f)(\xi) = y$. (This is "the ancient Kunen argument.") Hence in fact $j \restriction K|\lambda \in M$.

Let us denote by F the $(\kappa, j(\kappa))$–extender on K derived from j. We have seen that $F \in M$. We know that K is still the core model of $V[G]$ (and is still fully iterable there). Let K^M denote the core model from the point of view of M. K^M is fully iterable inside M; by our hypothesis that there be no inner model with a Woodin cardinal, this implies that K^M is also fully iterable in $V[G]$. By the Dodd–Jensen Lemma, K^M is a universal weasel.

In fact, in what follows we will confuse K, K^M with very soundness witnesses (cf. [26, Definition 5.3 and Lemma 8.3] and [15]) for appropriate initial segments thereof.

Claim 1. Let $\mathcal{T}, \mathcal{U}$ denote the iteration trees arising from the comparison of K with K^M, let $\mathcal{Q}$ be the common coiterate of K and K^M, and let $\pi^{\mathcal{T}}_{\infty}\colon K \to Q$ and $\pi^{\mathcal{U}}_{\infty}\colon K^M \to Q$ be the embeddings given by the cofinal branches through $\mathcal{T}$ and $\mathcal{U}$, respectively. Then $\mathrm{crit}(\pi^{\mathcal{T}}_{\infty}) = \kappa$ and $\mathrm{crit}(\pi^{\mathcal{U}}_{\infty}) \geq \kappa$.

PROOF of Claim 1: Standard arguments (playing with the hull and definability property, cf. the proof of [26, Theorem 8.6]) give that $\pi^{\mathcal{U}}_{\infty} \circ j = \pi^{\mathcal{T}}_{\infty}$, so that $\mathrm{crit}(\pi^{\mathcal{U}}_{\infty}) < \kappa$ iff $\mathrm{crit}(\pi^{\mathcal{T}}_{\infty}) < \kappa$. But then if $\mathrm{crit}(\pi^{\mathcal{U}}_{\infty}) < \kappa$, then the first extenders used along the main branches of $\mathcal{T}, \mathcal{U}$, respectively, would be compatible (which is impossible). We get that $\mathrm{crit}(\pi^{\mathcal{U}}_{\infty}) \geq \kappa$, and $\mathrm{crit}(\pi^{\mathcal{T}}_{\infty}) = \kappa$ (as $\mathrm{crit}(j) = \kappa$). □ (Claim 1)

Claim 1 implies that $\lambda = \kappa^{+K}$ and $K|\lambda = K^M|\lambda$. In particular, we may take the ultrapower $\mathrm{Ult}(K^M; F)$ of K^M by F. By our next Claim, $\mathrm{Ult}(K^M; F)$ is well–founded, an in fact more is true.

Claim 2. The phalanx $(K^M, \mathrm{Ult}(K^M; F), j(\kappa))$ is iterable (in $V[G]$ as well as in M).

PROOF of Claim 2: This readily follows from Lemma 6.1 above, by setting $\mathcal{M} = K$ and $\mathcal{N} = K^M$. We may apply 6.1 by Lemma 1. □ (Claim 2)

Notice that $\mathrm{Ult}(K^M; F)|j(\lambda) = K^M|j(\lambda)$.

We may now derive a contradiction by using standard arguments. Let us work in M, and let $\mathcal{T}$ and $\mathcal{U}$ denote the iteration trees on K^M and $(K^M, \mathrm{Ult}(K^M; F), j(\kappa))$, respectively, arising from their comparison. Let $\mathcal{Q}$ be the common coiterate. We cannot have that $\mathcal{Q}$ is above K^M on the phalanx–side of the comparison (cf. the proof of [26, Theorem 8.6]). Let $\pi^{\mathcal{T}}_{\infty}\colon K^M \to \mathcal{Q}$ and $\pi^{\mathcal{U}}_{\infty}\colon \mathrm{Ult}(K^M; F) \to \mathcal{Q}$ be the embeddings given by the cofinal branches through $\mathcal{T}$ and $\mathcal{U}$, respectively. Let us write $i_F\colon K^M \to \mathrm{Ult}(K^M; F)$ for the ultrapower map.

We have that $\pi^{\mathcal{U}}_\infty \circ i_F = \pi^{\mathcal{T}}_\infty$, $\mathrm{crit}(\pi^{\mathcal{T}}_\infty) = \mathrm{crit}(i_F) = \kappa$, and $\mathrm{crit}(\pi^{\mathcal{U}}_\infty) \geq j(\kappa)$. The first extender used along the main branch of $\mathcal{T}$ is therefore compatible with F, and in fact cannot be shorter than F. Therefore, F must be on the sequence of K^M, which is nonsense, as K^M does not have superstrong cardinals. □ (Theorem 0.3)

The conclusion of Theorem 0.3 holds under more liberal hypotheses. We basically need to assume that there is a good reason for the existence of K which also implies that K^M is fully iterable. We leave it to the reader to formulate useful generalizations. We shall use such generalizations in the proofs of Theorems 0.6 and 0.8.

We now turn to proofs of Theorems 0.4 and 0.5.

Proof of Theorem 0.4: The implication Con(3) $\Longrightarrow$ Con(2) is due to Shelah who showed that if there is a Woodin cardinal, then there is a forcing extension (obtained by a semi–proper forcing) in which there is a saturated ideal on ω_1. (Cf. [31, Theorem 2.64].) The implication Con(2) $\Longrightarrow$ Con(3) is due to Steel and Jensen–Steel (cf. [26] and [13]). (2) $\Longrightarrow$ (1) is easy.

Let us now prove Con(1) $\Longrightarrow$ Con(3). Suppose (1) holds, but there is no inner model with a Woodin cardinal. Let K denote the core model. (Cf. [13].) By Theorem 0.3, $\omega_1^{+K} = \omega_2$. It is easy to see that ω_1 must be a limit cardinal in K, so that ω_1 is inaccessible in K. (It is in fact measurable in K.)

Let us define $f \colon \omega_1 \to \omega_1$ by $f(\xi) = \xi^{+K}$ for $\xi < \omega_1$. As I is strong, there is some $\alpha < \omega_2$ such that $S = \{\xi < \omega_1 \colon f(\xi) < f_\alpha(\xi)\}$ is stationary. We may pick

$$\sigma \colon (H; \in, \bar{K}) \to (H_{\omega_2}; \in, K|\omega_2)$$

such that H is countable and transitive, $\xi = \mathrm{crit}(\sigma) \in S$, $\sigma(\xi) = \omega_1$, and, setting $\tau = f_\alpha(\xi)$, $\tau \in H$ and $\sigma(\tau) = \alpha$. By the Condensation Lemma, $\bar{K}|\tau = K|\tau$. So $\tau \leq f(\xi)$ by the definition of f. I.e., $f_\alpha(\xi) \leq f(\xi)$. Contradiction! □ (Theorem 0.4

Proof of Theorem 0.5. Let us suppose that there is no inner model with a Woodin cardinal, and let again K denote the core model. Let us first verify the following.

Claim 1. If $(\omega_1^V)^{+K} = \omega_2$, then $K|\omega_1^V$ is universal with respect to countable mice with no definable Woodin cardinals.

Proof. Let $\mathcal{M}$ be a countable mouse. Let us assume that $\mathcal{M}$ does not have a definable Woodin cardinal. As $K|\omega_2^V$ is universal with respect to countable mice (cf. [17]), there must in fact be some $\delta < \omega_2^V$ such that $K||\delta$ wins the comparison against $\mathcal{M}$. Say $\rho_1(K||\delta) = \omega_1^V$. Let $\mathcal{T}$ and $\mathcal{U}$ denote the normal iteration trees on $\mathcal{M}$ and $K||\delta$, respectively, arising from the comparison of $\mathcal{M}$ with $K||\delta$. Notice that both $\mathcal{M}$ and $K||\delta$ have unique iteration strategies.

Let $f \colon \omega_1^V \to K||\delta$ be bijective, where $f \in K$. Let us pick

$$\pi \colon H \to H_\theta$$

such that H is countable and transitive, θ is large enough, and

$$\{\mathcal{M}, K||\delta, \mathcal{T}, \mathcal{U}, f\} \subset \mathrm{ran}(\pi).$$

Set $\bar{K} = \pi^{-1}(K||\delta)$, $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$, and $\bar{\mathcal{U}} = \pi^{-1}(\mathcal{U})$. By our hypotheses, the iteration trees $\bar{\mathcal{T}}$ and $\bar{\mathcal{U}}$ are according to the unique iteration strategies for $\mathcal{M}$ and $\bar{K}$, respectively, and they witness that $\bar{K}$ wins the comparison against $\mathcal{M}$.

But $\bar{K}$ is the transitive collapse of $\mathrm{ran}(f \restriction \mathrm{crit}(\pi))$, and therefore $\bar{K} \in K$ and has size $< \omega_1^V$ in K. Inside K, $K|\omega_1^V$ is certainly universal with respect to mice of size $< \omega_1^V$, and therefore the fact that $\bar{K} \in K$ wins the comparison against $\mathcal{M}$ implies that $K|\omega_1$ wins the comparison against $\mathcal{M}$, too. □ (Claim 1)

In the light of Theorem 0.3, the proof of the following Claim, which is shown in [9], finishes the proof of Theorem 0.5.

Claim 2. Suppose that $x^{\#}$ exists for every $x \in \mathbb{R}$, and $\boldsymbol{\delta}^1_2 = \aleph_2$. Then $K|\omega_1^V$ is not universal with respect to countable mice, and in fact the mouse order on the set of all countable mice has length ω_2.

Proof. Jensen has shown that the hypothesis of this Claim implies that $x^\dagger$ exists for every real x (cf. [9]).

Let us fix $x \in \mathbb{R}$ for a while, and let $\kappa = \kappa_x < \Omega = \Omega_x$ denote the two measurable cardinals of $x^\dagger$. Let K_x denote the (lightface) core model of $x^\dagger$ of height Ω. By absoluteness, K_x is a mouse in V. Let

$$(\mathcal{N}_i^x, \pi_{ij}^x \colon i \leq j \leq \omega_1)$$

denote the linear iteration of $\mathcal{N}_0^x = x^\dagger$ obtained by iterating the unique measure on κ and its images ω_1 times. By [22], $\pi^x_{ii+1} \restriction \pi^x_{0i}(K_x)$ is an iteration of $\pi^x_{0i}(K_x)$, and there is hence a (not necessarily normal) iteration tree $\mathcal{T}$ on K_x of length $\omega_1 + 1$ such that

$$\mathcal{M}^{\mathcal{T}}_{\omega_1} = \pi^x_{0\omega_1}(K_x).$$

By [26],

$$\kappa^{+x^\dagger} = \kappa^{+K_x},$$

so that

$$\omega_1^{+\mathcal{N}^x_{\omega_1}} = \omega_1^{+\pi^x_{0\omega_1}(K_x)}.$$

Now by $\boldsymbol{\delta}^1_2 = \aleph_2$,

$$\sup(\{\omega_1^{\mathcal{N}^x_{\omega_1}} : x \in \mathbb{R}\}) = \aleph_2,$$

and therefore the supremum of all $\mathcal{P} \cap \mathrm{OR}$ such that there is some countable mouse $\mathcal{M}$ (with no definable Woodin cardinal) and some iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length $\omega_1 + 1$ such that $\mathcal{P} = \mathcal{M}^{\mathcal{T}}_{\omega_1}$ is equal to $\aleph_2$. On the other hand, a boundedness argument shows that for a fixed countable mouse $\mathcal{M}$, the supremum of all $\mathcal{P} \cap \mathrm{OR}$ such that there is some iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length $\omega_1 + 1$ such that $\mathcal{P} = \mathcal{M}^{\mathcal{T}}_{\omega_1}$ is smaller that $\omega_1^{+L[\mathcal{M}]}$ (cf. [31, p. 56f.]).

This shows that the mouse order on the set of all countable mice has length ω_2. This readily implies that $K|\omega_1$ cannot be universal with respect to countable mice (with no definable Woodin cardinals), as otherwise $\{K|\delta \colon \delta < \omega_1\}$ would be cofinal in the mouse order on the set of all countable mice. □ (Claim 2)

□ (Theorem 0.5)

7 A core model induction.

In this final section we prove Theorems 0.6 and 0.8.

PROOF of Theorem 0.6. In the light of Theorem 0.3, it obviously suffices to verify the following lemma.

Lemma 7.1 *Assume* BPFA. *Suppose that there is no inner model with a Woodin cardinal. Then* $\omega_1^{+K} < \omega_2$.

PROOF. We just need to take another look at the proof of Theorem 0.1. Let us work towards a contradiction, i.e., let us assume that there is no inner model with a Woodin cardinal and that $\omega_1^{+K} = \omega_2$. We may then construe $K||\omega_2$ as $S(K||\omega_1)$, where, as in the proof of Theorem 0.1, $S(K||\omega_1)$ is the stack of all mice $\mathcal{P} \trianglerighteq K||\omega_1$ such that $\mathcal{P}$ is sound and $\rho_\omega(\mathcal{P}) \leq \omega_1$. (Notice that by Claim 1 in the proof of Theorem 0.5, in fact $\rho_\omega(\mathcal{P}) = \omega_1$ for all such $\mathcal{P}$.)

We may then proceed exactly as in the proof of Theorem 0.1, where only the application of Lemma 2.8 is to be replaced by an application of Lemma 2.1. As $K||\omega_1 \in H_{\omega_2}$ (in contrast to X before), $K||\omega_1$ may be used as a parameter, so that the argument for Theorem 0.1 produces some mouse $\mathcal{S}' \triangleright K||\omega_1$ with largest cardinal ω_1 such that $\mathcal{S}' \cap \mathrm{OR} < \omega_2$ and the tree $T^*_{\mathcal{S}'}$ derived from $\mathcal{S}'$ does not have a cofinal branch.

However, a simple condensation argument shows that $\mathcal{S}' \triangleleft K||\omega_2$. We may then let $\alpha > \mathcal{S}' \cap \mathrm{OR}$ be least such that $\rho_\omega(K||\alpha) = \omega_1$. The mouse $K||\alpha$ can then be used to produce a cofinal branch through $T^*_{\mathcal{S}'}$. Contradiction! □ (Lemma 7.1)

□ (Theorem 0.6)

The rest of the paper is devoted to a

PROOF of Theorem 0.8. We'll prove inductively that for all $n < \omega$, V is closed under $X \mapsto M_n^\#(X)$. Let us fix the following statements.

1. $(1)_n \equiv$ "H_{ω_1} is closed under $\mathcal{M}_n^\#$"
2. $(2)_n \equiv$ "H_{ω_2} is closed under $\mathcal{M}_n^\#$"
3. $(3)_n \equiv$ "V is closed under $\mathcal{M}_n^\#$"

Let us ad hoc write $(3)_{-1}$ for "$0 = 0$." Assuming $\mathsf{FA}_{2^{\aleph_0}}$ for proper forcings and the existence of a precipitous ideal on ω_1, we will now go ahead and show that for an arbitray $n < \omega$

$$(3)_{n-1} \Longrightarrow (3)_{n-1} + (1)_n \Longrightarrow (3)_{n-1} + (2)_n \Longrightarrow (3)_n.$$

Let us start with showing that V is closed under $X \mapsto M_0^\#(X) = X^\#$. Lemmas 7.2 and 7.3 are well known.

Lemma 7.2 *Suppose there is a precipitous ideal on* ω_1. *Then* $(1)_0$ *holds true.*

Proof. Let I be a precipitous ideal on ω_1 and let G be a $\mathcal{P}(\omega_1) \setminus I$-generic over V. Let $j : V \to \text{Ult}(V, G)$ be the associated ultrapower map. For any $x \in \mathbb{R} \cap M$ the restriction of j to $L[x]$ gives a nontrivial elementary embedding from $L[x]$ into itself. Hence $x^{\#}$ exists in $V[G]$. But sharps can't be added by forcing, hence $x^{\#} \in V$.

□ (Lemma 7.2)

Lemma 7.3 *Suppose there is a precipitous ideal on ω_1. Then $(2)_0$ holds true.*

Proof. Let I be a precipitous ideal on ω_1 and let G be a $\mathcal{P}(\omega_1) \setminus I$-generic over V. Let $j : V \to \text{Ult}(V, G) = M$ be the associated ultrapower map. Let $A \subseteq \omega_1$, $A \in V$. By Lemma 7.2 and elementarity, $(H_{\omega_1})^M$ is closed under sharps in M, and also $A = j(A) \cap \omega_1^V \in (H_{\omega_1})^M$. Hence $A^{\#}$ exists in $V[G]$, so that $A^{\#} \in V$, since forcing can't add sharps. □ (Lemma 7.3)

Lemma 7.4 *Suppose* BPFA *holds and that there is a precipitous ideal on ω_1. Then $(3)_0$ holds.*

Proof. This follows from Lemma 7.3 together with Theorem 0.1.

□ (Lemma 7.4)

In order to proceed further, we shall need the following lemma.

Lemma 7.5 *Let $n < \omega$. Suppose $(3)_n$ holds and that I is a precipitous ideal on ω_1. Let G be I^+/I-generic over V, and let $M = \text{Ult}(V; G)$ be the generic ultrapower, where M is transitive. Let X be a set of ordinals in M, let $\mathcal{P} \in M$, and suppose that either*

$$M \models \mathcal{P} \text{ is an } n\text{–small mouse,}$$

or else

$$M \models \mathcal{P} = M_n^{\#}(X).$$

Then $\mathcal{P}$ is also a mouse in $V[G]$.

Proof. We proceed by induction. Let Φ_n be the following statement.

"There is tree $T \in M$ such that

$$V[G] \models p[T] = \left\{(x, y) \in \mathbb{R}^2 : y \text{ is a code for } M_n^{\#}(x)\right\}."$$

Also, let Ψ_n be the following statement, i.e., the conclusion of Lemma 7.5.

"Let X be a set of ordinals in M, let $\mathcal{P} \in M$, and suppose that either

$$M \models \mathcal{P} \text{ is an } n\text{–small mouse,}$$

or else

$$M \models \mathcal{P} = M_n^{\#}(X).$$

Then $\mathcal{P}$ is also a mouse in $V[G]$."

Writing $\Phi_{-1} \equiv 0 = 0$," we will show that for every $n < \omega$,

$$(3)_n + \Psi_n \Longrightarrow \Phi_n \text{ and } (3)_n + \Phi_{n-1} \Longrightarrow \Psi_n.$$

As Ψ_0 is trivial, this will do it.

So let $n < \omega$ be arbitrary, and let us first prove that $(3)_n + \Psi_n$ implies Φ_n. Let us assume $(3)_n$ and Ψ_n to hold.

Let $\theta = (2^{\aleph_1})^+$. Let $T \in M$ be a tree of height ω searching for x, y, $\mathcal{N}$, H and π such that

- $x, y \in \mathbb{R}$,
- y codes an x–premouse $\mathcal{M}$,
- $\mathcal{N}$ is an H–premouse, $\pi \colon \mathcal{N} \to j(M_n^\#(H_\theta)) = (M_n^\#((H_{j(\theta)})^M))^M$, and $\pi(H) = H_\theta$,
- there is some $\mathbb{Q} \in H$ and some $\mathbb{Q}$–generic filter g over $\mathcal{N}$ such that $x \in \mathcal{N}[g]$,
- $\mathcal{M}$ is equal to $M_n^\#(x)$, as computed from $\mathcal{N}[g]$.[14]

We aim to verify that T witnesses that Φ_n holds true.

First let $x, y \in \mathbb{R}^{V[G]}$ be such that $V[G] \models$ "y codes $M_n^\#(x)$". Let τ be a name for x. Notice that (without of loss of generality) $\tau \in H_\theta$. In $V[G]$, we may pick some elementary

$$\bar{\sigma} \colon P^* \to M_n^\#(H_\theta)[G]$$

such that P^* is countable and H_θ, I, τ, $x \in \text{ran}(\bar{\sigma})$. By [3, Lemma 3.7],

$$(M_n^\#(H_\theta))^V = (M_n^\#(H_\theta))^{V[G]},$$

so that $M_n^\#(H_\theta)[G]$ is equal to (or may be construed as) $(M_n^\#(H_\theta[G]))^{V[G]}$ and $M_n^\#(x)$, as computed from P^*, is equal to $(M_n^\#(x))^{V[G]}$. Let $\mathcal{N}$ be the transitive collapse of $\text{ran}(\bar{\sigma}) \cap M_n^\#(H_\theta)$, so that

$$\bar{\sigma} \restriction \mathcal{N} \colon \mathcal{N} \to M_n^\#(H_\theta).$$

We get that $(x, y) \in p[T]$, as being witnessed by $\mathcal{N}$, $\bar{\sigma}^{-1}(H_\theta)$, and

$$(j \restriction M_n^\#(H_\theta)) \circ (\bar{\sigma} \restriction \mathcal{N}).$$

It is now easy to verify that we also have that if $(x, y) \in p[T] \cap V[G]$, then y codes $M_n^\#(x)$. We have thus verified Φ_n.

Now let again $n < \omega$ be arbitrary and let us prove that $(3)_n + \Phi_{n-1}$ implies Ψ_n. Let us assume $(3)_n$ and Φ_{n-1}. Suppose T to be the tree given by Φ_{n-1}.

Let $\mathcal{P} \in M$ be as in Ψ_n. As Ψ_n is trivial for $n = 0$, we may and shall as well assume that $n > 0$. Notice that the relevant $\mathcal{Q}$–structures for iterating (a countable substructure of) $\mathcal{P}$ are initial segments of $M_{n-1}^\#$ of the common part model; moreover, $T \in M$ projects to the (code for) such potential $\mathcal{Q}$–structures (cf. [25]). We may therefore let $U \in M$ be a tree of height ω searching for

- a countable premouse $\bar{\mathcal{P}}$ together with a sufficiently elementary embedding of $\bar{\mathcal{P}}$ into $\mathcal{P}$,

[14] cf. the footnote in the proof of Lemma 2.9

• a countable putative normal iteration tree $\mathcal{T}$ on $\bar{\mathcal{P}}$ such that for all limit ordinals $\lambda < \mathrm{lh}(\mathcal{T})$, $\mathcal{M}_\lambda^\mathcal{T}$ has an initial segment which is a $\mathcal{Q}$–structure and also an initial segment of $M_{n-1}^\#(\mathcal{M}(\mathcal{T} \restriction \lambda))$, and either
• a proof that $\mathcal{T}$ has a last ill–founded model, or else
• a proof that $\mathcal{T}$ has limit length but no cofinal branch b such that $\mathcal{M}_b^\mathcal{T}$ has an initial segment which is isomorphic to a $\mathcal{Q}$–structure which is provided by an initial segment of $M_{n-1}^\#(\mathcal{M}(\mathcal{T}))$.

By the choice of T it is clear that inside M as well as $V[G]$, $\mathcal{M}$ is iterable if and only if U is well–founded. This proves Ψ_n. □ (Lemma 7.5)

Lemma 7.6 *Let $n < \omega$. Assume $\mathsf{FA}_{2^{\aleph_0}}$ for proper forcings and that there is a precipitous ideal on ω_1. Assume also that $(3)_n$ holds true. Then $(1)_{n+1}$ hold true.*

Proof. The proof is almost identical to the proof of Theorem 0.6, with only two wrinkles. Let $x \in \mathbb{R}$, and suppose that $M_{n+1}^\#(x)$ does not exist. Let us assume that $x = \emptyset$, as the argument is easily seen to relativize. So as $M_{n+1}^\#$ does not exist, K^c does not have a Woodin cardinal, is $(n+1)$–small, and is fully iterable, and the true core model K exists.

Let I be a precipitous ideal on ω_1, let G be I^+/I generic over V, and let $j \colon V \to M = \mathrm{Ult}(V;G)$ be the associated embedding, where M is transitive. By the forcing absoluteness of K, $K^{V[G]} = K$ and K is still fully iterable in $V[G]$. By Lemma 7.5, K^M is also fully iterable in $V[G]$. We may therefore run the argument for Theorem 0.3 and show that $\omega_1^{+K} = \omega_2$.

Using $\omega_1^{+K} = \omega_2$, we get a contradiction by an amalgamation of the arguments from the proofs of Theorem 0.6 and Lemma 7.1. □ (Lemma 7.6)

Lemma 7.7 *Let $n < \omega$. Assume that there is a precipitous ideal on ω_1. If $(3)_n$ and $(1)_{n+1}$ hold true, then $(2)_{n+1}$ holds true.*

Proof. Let $A \subset \omega_1$, $A \in V$. Let I be a precipitous ideal on ω_1, let G be I^+/I-generic over V, and let $j : V \to \mathrm{Ult}(V;G)$ be the associated ultrapower map, where M is transitive. By elementarity, $(1)_{n+1}$ holds true in M. As A is (coded by) a real in M, $M_{n+1}^\#(A)$ exists in M and is iterable in M. As $(1)_n$ holds, we have by 7.5 that $(M_{n+1}^\#(X))^M$ is iterable in $V[G]$ as well, i.e.,

$$(M_{n+1}^\#(X))^M = (M_{n+1}^\#(X))^{V[G]}.$$

By standard arguments (cf. [3]), we then have that $M_{n+1}^\#(A)$ exists in V and

$$(M_{n+1}^\#(X))^V = (M_{n+1}^\#(X))^{V[G]}.$$

□ (Lemma 7.7)

The following is provided by Theorem 0.7.

Lemma 7.8 *Let $n < \omega$. Assume $\mathsf{FA}_{2^{\aleph_0}}$ for proper forcings. If $(3)_n$ and $(2)_{n+1}$ hold true, then $(3)_{n+1}$ holds true.*

This finishes the proof of Theorem 0.8. □ (Theorem 0.8)

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