

Ax 1. $\bullet \forall x \{ [\phi(x) \rightarrow \psi(x)] \wedge P(\phi) \} \rightarrow P(\Psi)$

Ax 2. $P(\neg\phi) \leftrightarrow \neg P(\phi)$

Th 1. $P(\phi) \rightarrow \diamond \exists x [\phi(x)]$

Df 1. $G(x) \leftrightarrow \forall \phi [P(\phi) \rightarrow \phi(x)]$

Ax 3. $P(G)$

Th 2. $\diamond \exists x G(x)$

Df 2. $\phi \text{ ess } x \leftrightarrow \phi(x) \wedge \forall \psi (\psi(x) \rightarrow \bullet \forall \alpha [\phi(\alpha) \rightarrow \psi(\alpha)])$

Ax 4. $P(\phi) \rightarrow \bullet P(\phi)$

Th 3. $G(x) \rightarrow G \text{ ess } x$

Df 3. $E(x) \leftrightarrow \forall \phi [\phi \text{ ess } x \rightarrow \bullet \exists x \phi(x)]$

Ax 5. $P(E)$

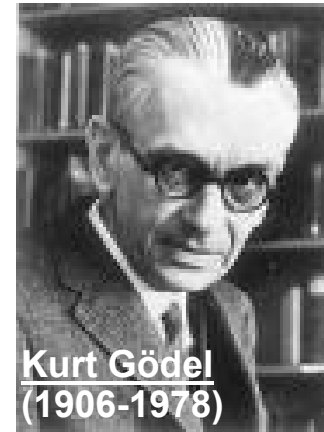
Th 4. $\bullet \exists x G(x)$

Gödel's Ontological Proof of the Existence of God



Gödel's Ontological Proof of the Existence of God

- A first sketch of the proof can already be found in Gödel's notebook dated around the year 1941.
- In 1970 Gödel showed this proof to his student Dana Scott.
- Scott made a note of the proof and presented it in his seminar at Princeton University in the fall of 1970.
- From then on, Gödel's proof has become widely circulated.



Gödel's Ontological Proof of the Existence of God

Anselm's Ontological Proof of the Existence of God



**St. Anselm of Canterbury
(1033-1109)**

“Therefore, Lord, [...] I understand that you are as we believe and you are that which we believe. Now we believe that you are something than which nothing greater can be imagined. Then is there no such nature, since the fool has said in his heart: God is not? But certainly this same fool, when he hears this very thing that I am saying - something than which nothing greater can be imagined - understands what he hears; and what he understands is in his understanding, even if he does not understand that it is. For it is one thing for a thing to be in the understanding and another to understand that a thing is.

[...]

Therefore even the fool is bound to agree that there is at least in the understanding something than which nothing greater can be imagined, because when he hears this he understands it, and whatever is understood is in the understanding.

And certainly that than which a greater cannot be imagined cannot be in the understanding alone. For if it is at least in the understanding alone, it can be imagined to be in reality too, which is greater. Therefore if that than which a greater cannot be imagined is in the understanding alone, that very thing than which a greater cannot be imagined is something than which a greater can be imagined. But certainly this cannot be. There exists, therefore, beyond doubt something than which a greater cannot be imagined, both in the understanding and in reality.”

(Anselm of Canterbury: *Proslogion*, Translation by Jonathan Barnes)

Gödel's Ontological Proof of the Existence of God

Descartes' Ontological Proof of the Existence of God



René Descartes
(1596-1650)

“It is certain that I... find the idea of God in me, that is to say, the idea of a supremely perfect being... And I know no less clearly and distinctly that an actual and eternal existence belongs to his nature... existence can no more be separated from the essence of God... than the idea of a mountain can be separated from the idea of a valley; so that there is no less contradiction in conceiving a God, that is to say, a supremely perfect being, who lacks some particular perfection, than in conceiving a mountain without a valley.”

(René Descartes: Fifth Meditation)

Gödel's Ontological Proof of the Existence of God



Immanuel Kant
(1724-1804)

Kant's Refutation of an Ontological Proof of The Existence of God

“Being' is obviously not a real predicate; that is, it is not a concept of something which could be added to the concept of a thing. It is merely the positing of a thing, or of certain determinations, as existing in themselves. Logically, it is merely the copula of a judgment. [...] If, now, we take the subject (God) with all its predicates (among which is omnipotence), and say 'God is', or 'There is a God', we attach no new predicate to the concept of God, but only posit the subject in itself with all its predicates, and indeed posit it as being an object that stands in relation to my concept.”

(Immanuel Kant: *Critique of Pure Reasoning*, Book II, Chapter III, Sec. 4)

Gödel's Ontological Proof of the Existence of God

Leibniz' Objection to an Ontological Proof of the Existence of God



Gottfried Wilhelm Leibniz
(1646-1714)

„The point is that the argument silently assumes that this idea of a wholly great or wholly perfect being is *possible* and doesn't imply a contradiction. Even without that assumption Descartes's argument enables us to prove something, namely that *If God is possible he exists* – a privilege that no other being possesses!“

(G.W. Leibniz: *New Essays IV, X: Knowledge of God's Existence*)

Gödel's Ontological Proof of the Existence of God

S5-System

Axioms:

(Prop) All instantiations of propositional tautologies

(Dist) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ Axiom of distribution

(S5) $\Diamond\Box\varphi \rightarrow \Box\varphi$ Becker's principle

Rules:

All rules of classical propositional logic

(Nec) From $\vdash \varphi$, infer $\vdash \Box\varphi$ Rule of necessitation

Gödel's Ontological Proof of the Existence of God

Ax1 $P(\neg\varphi) \leftrightarrow \neg P(\varphi)$

A property is either positive or its negation (its complement) is positive.

Ax2 $P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)] \rightarrow P(\psi)$

Any property strictly implied by a positive property is positive.

D1 $G(x) \leftrightarrow \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$

x is God-like if and only if x “incorporates” all positive properties.

Ax3 $P(G)$

The property of being God-like is positive.

Ax4 $P(\varphi) \rightarrow \Box P(\varphi)$

Positive properties are necessarily positive properties.

D2 $\varphi \text{ Ess } x \leftrightarrow \varphi(x) \wedge \forall\psi[\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y))]$

φ is an essence of x if and only if φ is a property of x and every property ψ that x has is strictly implied by φ .

D3 $E(x) \leftrightarrow \forall\varphi[\varphi \text{ Ess } x \rightarrow \Box\exists y \varphi(y)]$

x necessarily exists if and only if every essence of x is necessarily exemplified.

Ax5 $P(E)$

Necessary existence is positive.

Gödel's Ontological Proof of the Existence of God

TH1 $P(\varphi) \rightarrow \diamond\exists x \varphi(x)$

Proof: Assumption: 1. $P(\varphi)$ und 2. $\Box\forall x \neg\varphi(x)$.

From 2. we get with *ex falso quodlibet*:

$\Box\forall x (\varphi(x) \rightarrow x \neq x)$.

With 1. and Ax2 we can derive: $P(\lambda x(x \neq x))$.

Because $\Box\forall x (\varphi(x) \rightarrow x = x)$, it holds that $P(\lambda x(x = x))$ – which contradicts Ax1.

Corollary to TH1: $\diamond\exists x G(x)$

Proof: with Ax3 and modus ponens

Gödel's Ontological Proof of the Existence of God

TH2 $G(x) \rightarrow G \text{ Ess } x$

If x is a God-like being, then the property of being God-like is the essence of x .

Proof: Assumption: 1. $G(x)$ und 2. $\psi(x)$ [for an arbitrary ψ].

Assumption: $\neg P(\psi)$. With Ax1, 1. and D1: $\neg\psi(x)$ – which contradicts 2.

Therefore, $P(\psi)$ und hence with Ax4: (*) $\Box P(\psi)$.

It holds that $\Box[P(\psi) \rightarrow \forall x[G(x) \rightarrow \psi(x)]]$ (because of D1 and Nec) und therefore (with DIST):

$\Box P(\psi) \rightarrow \Box \forall x[G(x) \rightarrow \psi(x)]$. Together with (*) we get: $\Box \forall x[G(x) \rightarrow \psi(x)]$ – and therefore

With 1., 2. and D2: $G \text{ Ess } x$.

Corollary to TH2: $G(x) \rightarrow \Box \exists y G(y)$

A God-like being is necessarily exemplified.

Proof: From $G(x)$ we get with D1 and Ax5: $E(x)$ – and therefore with D3 because of TH2:

$\Box \exists y G(y)$.

Gödel's Ontological Proof of the Existence of God

TH3 $\Box\exists yG(y)$

Proof: $\forall x[G(x) \rightarrow \Box\exists yG(y)]$ (because of corollary to TH2). From this we get:

$\exists xG(x) \rightarrow \Box\exists yG(y)$ (by quantified logic) and therefore with Nec:

$\Box[\exists xG(x) \rightarrow \Box\exists yG(y)]$.

Because of the modal logic theorem in K: $\Box(\phi \rightarrow \psi) \rightarrow (\Diamond\phi \rightarrow \Diamond\psi)$, we get:

$\Diamond\exists xG(x) \rightarrow \Diamond\Box\exists yG(y)$.

Because of *Becker's principle S5* ($\Diamond\Box\phi \rightarrow \Box\phi$) it holds that:

$\Diamond\Box\exists yG(y) \rightarrow \Box\exists yG(y)$, and therefore (with hypothetical syllogism):

$\Diamond\exists xG(x) \rightarrow \Box\exists yG(y)$ ("Anselm's principle") – and therefore with corollary to TH1 and *modus ponens*: $\Box\exists yG(y)$.

Gödel's Ontological Proof of the Existence of God

Extensional Interpretation of Gödel's Argument

TH1* $P(\varphi) \rightarrow \exists x\varphi(x)$.

Proof: If φ were positive and had no elements, then the empty set $\emptyset = \varphi$, would be positive.

Since necessarily $\emptyset \subseteq U$ (where U is the union set), it follows with Ax2: $P(U)$.

Since $\emptyset = \neg U$ it also follows with Ax1: $\neg P(U)$ – in contradiction to $P(U)$. Therefore, φ cannot be an empty set, i.e., $\exists x\varphi(x)$.

Together with Ax3 we now get: $\exists xG(x)$.

Corollary: There is exactly one God-like being.

Proof: If there were two different x, y with $G(x)$ and $G(y)$ and if $\neg P(\{x\})$, then we would derive with Ax1 $P(\neg\{x\})$ – and therefore because of $G(x)$ we would get the contradiction:

$\neg\{x\}(x)$, i.e., $x \notin \{x\}$.

Therefore: $P(\{x\})$ – and since $G(y)$, i.e., since y is an element of the intersection of all positive sets: $\{x\}y$, i.e., $x=y$.

Gödel's Ontological Proof of the Existence of God

With $\exists xG(x)$ we can now immediately and without using Becker's principle S5: $\Box\exists yG(y)$.

Proof: From Ax5 and D1 we get: $G \subseteq E$. Since $\exists xG(x)$, we get: (1) $E(x)$, and with TH2 we get: (2) $G \text{ Ess } x$ (for a x with $G(x)$).

From (1) we can derive with D3: $G \text{ Ess } x \rightarrow \Box\exists yG(y)$, and with (2): $\Box\exists yG(y)$.

Gödel's Ontological Proof of the Existence of God

D2 is extensionally equivalent with:

$$\varphi \text{ Ess } x \leftrightarrow \varphi = \{x\}.$$

From this and D3 it follows that the predicate E of necessary existence is, in an extensional reading, no genuine existence predicate, since necessary existence cannot extensionally be distinguished from the necessary exemplification of an object, i.e.;

$$E(x) \leftrightarrow \Box \exists y(y = x).$$

- ☞ **It is not at all clear what additional insight into the nature and existence of „God“ can be gained by a modal-logic version with the concept of an essence of an object and the concept of necessary existence in comparison to a simply extensional version.**

Gödel's Ontological Proof of the Existence of God

- Ax1: A property is either negative or its negation (its complement) is negative.
- Ax2: Any property strictly implied by a negative property is negative.
- D1 „G“ is the intersection of all negative sets, i.e., a *summum malum*.
- Ax3 Being such a *summum malum*, is itself a negative property.
- Ax4 Negative properties are necessarily negative.
- Ax5 Since being a *summum malum* is the essence of a *summum malum*, necessary existence (as necessary exemplification of all essences of a being) is negative.



Gödel's axioms and definitions also provide a proof of the necessary existence of a *summum malum*.

Gödel's Ontological Proof of the Existence of God

Summary

- The application of modal logic seems to be superfluous since an extensional interpretation of the axioms can be motivated.
- The original modal-logic proof in which the necessary existence of God is derived from the possible existence of God is question-begging, since the proof uses the strong Becker's principle of a S5 system of modal logic.
- The central basic notions of the proof, i.e., "G" and "P", remain underdetermined such that anti-theistic interpretations are possible.
- The main question of an ontological proof of the existence of God remains unanswered, namely the question whether the properties of omnipotence and omniscience are exemplified in a being.