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# Ideals, ideal extenders and forcing axioms

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# Ideals, ideal extenders and forcing axioms

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# Contents

# Introduction

While set theory started with Cantor's question, whose positive assertion is called $\mathsf{CH}$ nowadays, whether every subset of the reals can be mapped bijectively either to the real line or to the natural numbers, it is Gödel who started modern set theory with two results. First the incompleteness theorems in 1931, which made logicians and mathematicians realize, that every mathematical statement wasn't decidable. Suddenly questions didn't necessarily ask whether something was true, but if it was consistent *relative* to another statement.

**Definition.** Let $\varphi$ and $\psi$ be statements in the language of set theory and $T$ a theory in the language of set theory. We say that $T + \varphi$ has *at least consistency strength of* $\psi$ if one can construct a model of $T + \psi$ from any model of $T + \varphi$.

We say that $T + \psi$ is *equiconsistent to* $T + \varphi$ if $T + \varphi$ has at least consistency strength of $\psi$ and vice-versa.

We will often stop referring to the theory $T$ in case $T = \mathsf{ZFC}$.

The real kick-off though was his second breaking result in 1938, the relative consistency proof that $\mathsf{CH}$, Cantor's old question, holds true in $L$. His proof introduced one of the two main strategies of modern set theory when it comes to relative consistency: building so called *core models*. That is, starting with a model $V$ of $\mathsf{ZF}$, Gödel constructed a second model $L \subseteq V$, called the class of all constructible sets. He showed that in $L$ the generalized continuum hypothesis, $\mathsf{GCH}$, holds as well as the axiom of choice. Thus those two statement are consistent relative to $\mathsf{ZF}$.

The second method was established in 1963 by Cohen for his proof of the independency of $\mathsf{CH}$: *forcing*. While the inner model methods constructed "thinner" models which are contained in the previous one, the forcing method adds a new set, a so called generic set, $G$, such that the initial model is contained in the constructed model. With this method, he added $\omega_2$ many reals to a model of $\mathsf{CH}$ without changing any cardinalities, thus constructing a model in which $\mathsf{CH}$ was false. While it was already known that there were some statements that weren't decidable in $\mathsf{ZFC}$, this put an end to the hope that every "interesting" statement was in fact decidable in $\mathsf{ZFC}$. But that way opened up many interesting possibilities. The question of the relative consistency of various statements not decidable in $\mathsf{ZFC}$ became one of the central objects of set theory.

Going back to Cantor's question, Easton later proved that the generalized continuum hypothesis was vastly independent of anything else. But Cantor's initial approach, proving that sets of increasing complexity were either countable or of the cardinality of the continuum, proved interesting on its own. The first interesting theorem in this line of research is due to Cantor and Bendixson:

**Theorem (Cantor-Bendixson, 1883).** *Let $C$ be a closed set, then $C$ has the perfect set property (i.e. is either countable, or contains a subset with no isolated points).*

This was then strengthened in 1916 by Hausdorff and Aleksandrov to Borel sets and finally by Suslin to analytical sets. This was the end of the hope of an internal proof of $\mathsf{CH}$ as the first twist appeared at the coanalytical level:

**Theorem (Luzin and Sierpiński, 1923).** *Every coanalytical set is the union of $\omega_1$ many Borel sets.*

This result was sharp in the sense of:

**Theorem (Kondô, 1939).** *If $V = L$ then there exists an uncountable $\Pi^1_1$- (i.e. a coanalytical-)set without a perfect subset.*

As the representation of sets of reals as the union of a perfect set and a "small" set seemed to break down at this stage, by an "inner model argument" by the way, hence one went to some smaller class of subset of $\mathbb{R}$: $\utilde{\Sigma}^1_2$ prewellorders, which seemed to bear more structure.

**Theorem (Martin, 1969/70).** *Every $\utilde{\Sigma}^1_2$ well-founded relation has length less than $\omega_2$.*

Setting $\utilde{\delta}^1_2$ as the sup of the lengths of those $\utilde{\Sigma}^1_2$ well-founded relations one can restate the theorem simply as $\utilde{\delta}^1_2 \leq \omega_2$. The question about the possible values of $\utilde{\delta}^1_2$ haunted set theory since then. Notice that $\utilde{\delta}^1_2$ is absolute between $\mathsf{ZF}$-models having the same reals, hence one could study it in $L(\mathbb{R})$ as well. Under the axiom of determinacy, the length of $\utilde{\delta}^1_n$ was computed, and it turned out that $\utilde{\delta}^1_2 = \omega_2$. Obviously $\utilde{\delta}^1_2 < \omega_2$ under $\mathsf{CH}$, and the question whether the equality was possible was first answered in 1982:

**Theorem (Steel-Van Wesep, 1982).** *Suppose $\mathsf{ZF}+\mathsf{AD}+\mathsf{AC}_{\mathbb{R}}$. There is a forcing such that, the forcing extension is a model of $\mathsf{ZFC} + \utilde{\delta}^1_2 = \omega_2 +$ " the non-stationary ideal on $\omega_1$ is $\omega_2$-saturated".*

Obtaining $\utilde{\delta}^1_2 = \omega_2$ is since then considered by most set theorists as a "natural negation" of the continuum hypothesis, as it implies that a much more palpable set of reals is of large cardinality. After Steel and Van Wesep's result, Woodin showed that the equality was implied by $\mathsf{NS}_{\omega_1}$ is $\omega_2$-saturated and $\mathcal{P}(\omega_1)^\#$ exists. With Foreman, Magidor and Shelah's result that MM implies that $\mathsf{NS}_{\omega_1}$ is $\omega_2$-saturated and every $X$ has a sharp, we have that MM implies $\utilde{\delta}^1_2 = \omega_2$. Woodin then reduced the hypothesis to BMM and a measurable cardinal. It is in this light that we give our modest contribution to this vast research effort of understanding the structure of the real line:

**Theorem (Corollary 2.19).** *Suppose* BMM *and "there exist a precipitous ideal on $\omega_1$" then $\utilde{\delta}^1_2 = \omega_2$.*

In the last few theorems, we have seen a new type of axiom emerging, so called *forcing axioms*. Basically forcing axioms are the assertion that, for some types of forcing and for a given collection of dense sets of such a given forcing, there is a set that behaves like a generic object for that collection.

**Definition.** Let $\Gamma$ be a class of partial orders. The Forcing Axiom for $\Gamma$, $\mathsf{FA}(\Gamma)$, is the following principle: let $\mathbb{P} \in \Gamma$ and let $\langle D_i;\ i \in \omega_1\rangle$ denote a collection of sets dense in $\mathbb{P}$. Then there is a filter $F \subseteq \mathbb{P}$ meeting every $D_i, i < \omega_1$.

In this case, BMM is the a bounded version, that is, we limit ourselves to dense sets of cardinality at most $\omega_1$ of the boolean algebra of regular open sets of $\mathbb{P}$, where $\mathbb{P}$ is a forcing in $\Gamma$. BMM can be formulated as $\mathsf{BFA}(\Gamma)$, the bounded forcing axiom for the class $\Gamma$ of all stationary set preserving forcings. The other forcing axiom we want to study is BPFA = $\mathsf{BFA}(\Delta)$, where $\Delta$ is the class of proper forcings. One particular thing that has stroke our interest is that, while BPFA alone has relatively low consistency strength[1], and a precipitous ideal is equiconsistent to a measurable cardinal, the conjunction of both explodes in strength:

**Theorem (Theorem 3.1).** *Suppose* BPFA *holds and that there is a precipitous ideal on $\omega_1$. Then there is an inner model with a Woodin cardinal.*

Somehow this and the next result of that section, concerning a similar axiom, $\mathrm{BPFA}^{\mathrm{uB}}$, indicates that forcing absoluteness on $H_{\omega_2}$ together with the existence of generic embeddings with critical point $\omega_1$ could lead to interesting results. This made us wonder if one could define generic embeddings with stronger properties, while still staying below $\omega_2$-saturation, which would directly lead to some inner model with a Woodin cardinal. If one took the analogy of measurable cardinal and precipitous ideals seriously, the next natural step would be to find embeddings who mimic strong cardinals. This is what we did in the last chapter of this thesis. We have two suggestion for the definition of such embeddings, one taking the combinatorial approach, *ideally strong cardinals* and mimicking the behavior of extenders, the other approach going directly for the generic embeddings, *generically strong cardinals*. Our first concern was to prove, much in the spirit of relative consistency highlighted at the beginning of this introduction, that these two definition are indeed in the realm of strong cardinals. In Theorem 4.10 and Theorem 4.22 we constructed forcing extensions containing ideally strong cardinals and generically strong cardinals respectively, starting with the same amount of strong cardinals. In Lemma 4.13 we show the other direction of equiconsistency, that is, starting with a model of generically strong cardinal, we show that these are strong in a core model.

## Overview

In Chapter One we introduce the concepts of precipitous ideals, extender, generic embeddings and forcing axiom and review some main results about them. After that we give a quick overview of the inner model theory we will be working with. There are three situation we will be working in, one below a strong cardinal, where we will use the theory as outlined in Jensen's manuscripts [Jenc] or Zeman's book [Zem02]. The second context we will be working in is below one Woodin cardinal. We define premice

[1] it is consistent with $V = L$

in that context and give a thorough definition of iteration tree and iteration strategies. The exposition mainly follows [MS94] and [Ste96]. Our analysis of BPFA and of ideal extender will be in that situation. Then we will develop what happens if one allows mice to bear finitely many Woodin cardinals. We will use the notes of [Ste] and the recent book [SS]. In this context we will have to work with mice relativized to a set. After giving some specific results to that type of mice, we will explain how iteration strategies can behave in that context, and give some results about mouse operators. As in the previous subsection, we will close with the exposition of basic properties of $K$.

Chapter two deals with one particular forcing $\mathbb{P}(I,\theta)$, where $I$ is a precipitous ideal on $\omega_1$. The forcing is a variant of Jensens $\mathcal{L}$-forcing which was developed in [Jena] and [Jend]. The forcing adds a generic iteration of length $\omega_1$ such that the last model is $\langle H_\theta^V, \in, I\rangle$. In the first section, we will define the forcing and show why it is stationary set preserving in case $I = \mathsf{NS}_{\omega_1}$. In the next section, we will show why a slight variant of it increases $\underset{\sim}{\delta}^1_2$. We will use that fact to prove that BMM and precipitousness of $\mathsf{NS}_{\omega_1}$ implies that $\underset{\sim}{\delta}^1_2 = \omega_2$. We will then state some other implications easily gained from that fact.

Chapter three is solely devoted to the analysis of the consistency strength of BPFA + " there is a precipitous ideal on $\omega_1$". After reviewing some facts about the $\square_\kappa$ sequence in the first section, we show in section two that BPFA implies that the cardinal successor in $K$ of $\omega_1$ is computed badly. That is

$$\omega_1^{+K} < \omega_2,$$

We first prove that statement under the hypothesis that there are "no inner models with a strong cardinal". This "warm-up" shows the main ideas of the proof. We then show that if one looks carefully at the complexity of the statement "$\mathcal{M}$ is iterable", the same result holds true if we assume that there are no inner models with a Woodin cardinal. In the next section, we will show that the existence of a precipitous ideal implies the contradictory inner model theoretic situation, namely:

$$\omega_1^{+K} = \omega_2$$

This will be done under the hypothesis that there is "no inner model with a strong cardinal". Finally in the last section, we will show that if one assumes $\text{BPFA}^{\text{uB}}$ instead of BPFA, we can show the closure of the universe under the $\mathcal{M}_n^\#$-operator for all $n$. Specifically this implies that all projective sets of reals are determined.

Chapter four introduces a new concept: ideal extenders. These are a generalization of precipitous ideals to extenders. In the first section we review some forcing techniques in the cases of measurable cardinals and precipitous ideals. The second section defines ideal extender and constructs some assuming the existence of a strong cardinal. Then we show that if we assume that for every set $A$, there is an ideal extenders on $\kappa$ that conserves $A$ in the ultrapower, $\kappa$ must be a strong cardinal in the core model. In the next section, we construct two ideally strong cardinals, from an $A$-strong cardinal and a strong above it. We will then we show that if we don't require generic embeddings to be given by an ideal extender, we can construct almost arbitrarily many generically strong cardinals. In turn, we show that generically strong cardinals must be strong in the core model. We apply the techniques we developed so far to supercompact cardinals in the next section, where we show that from $\omega$ supercompact cardinals, we can build a model where every $\aleph_n$ is generically strong.

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# 1 Definitions

We will now introduce some of the main concept we will work with in this thesis.

## 1.1 Ideals and filters

**Definition 1.1.** Let $X$ be a set, we call $I \subseteq \mathcal{P}(X)$ an *ideal on* $X$ if

i. $\varnothing \in I$,

ii. $I$ is closed downward under inclusion, i.e. $x \subseteq y \wedge y \in I \Rightarrow x \in I$

iii. $I$ is closed under finite union, i.e. $x, y \in I \Rightarrow x \cup y \in I$.

We say that $F \subseteq \mathcal{P}(X)$ is a *filter on* $X$ if

i. $X \in F$,

ii. $F$ is closed upward under inclusion, i.e. $x \subseteq y \wedge x \in F \Rightarrow y \in F$,

iii. $F$ is closed under finite intersection, i.e. $x, y \in F \Rightarrow x \cap y \in F$.

In the following we will further assume that if $I$ (respectively $F$) is an ideal (respectively a filter) on $X$, they are not equal to $\mathcal{P}(X)$.

The concept of filter has been first used in measure theory, which explains the second definition:

**Definition 1.2.** Let $I$ be an ideal on $X$. $x \subseteq X$ is an *$I$-measure one set* if there is a $y \in I$ such that $x = X \setminus y$. We denote the set of all $I$-measure one sets by $I^{c}$.

$x \subseteq X$ is an *$I$-positive set* if for all $y \in I$ we have that $x \cap (X \setminus y) \neq \varnothing$. We denote the set of all $I$-positive sets by $I^{+}$.

In the same spirit, for a filter $F$ on $X$, the set of the $F$ *positive set* is

$$F^{+} = \{x \subseteq X; \forall y \in F \; x \cap y \neq \varnothing\}.$$

The set of the *$F$-nullset* is denoted by

$$F^{-} = \{x \subseteq X; \exists y \in F \; x \cap y = \varnothing\},$$

if $x \in F^{-}$ we say that x has *$F$-measure zero.*

**Remark 1.3.** Let $I$ and $F$ be as in the definition. $I^c$ is a filter, the so called *dual filter to* $I$ and $I^+ = \mathcal{P}(X) \setminus I$. Dually $F^-$ is an ideal, called the *dual ideal to* $F$. Further $F^+ = \mathcal{P}(X) \setminus F^-$, in words: the $F$-positive sets are the non-zero sets.

**Definition 1.4.** We call a filter $F$ on $X$ an *ultrafilter* if for all $x \subseteq X$, either $x \in U$ or $X \setminus x \in U$.

Remark that if $F$ is an ultrafilter, $F^+ = F$.

**Definition 1.5.** Let $\kappa$ be a regular cardinal. A filter $U$ is $<\kappa$*-complete* if for all $\lambda < \kappa$ and for all sequence $\langle x_i; i < \lambda \rangle$ of subsets of $X$, $\bigcap_{i<\lambda} x_i \in U$.

An ultrafilter $U$ is called *non-trivial* if it is not generated by a point, that is for all $a \in X$ $\{x \subseteq X; a \in x\} \neq U$.

Remark that, by the axiom of choice, there are non-trivial ultrafilters. We generally will not mention that an ultrafilter is non-trivial, but we will assume it throughout this thesis.

By $\mathsf{NS}_{\omega_1}$ we shall denote the nonstationary ideal on $\omega_1$, that is the ideal generated by all non-stationary subsets of $\omega_1$

## 1.2 Large cardinals

Let us introduce three large cardinals that will give the consistency strength frame of this work: measurable cardinals, strong cardinals and Woodin cardinals.

**Definition 1.6.** We call a cardinal $\kappa$ a *measurable cardinal* if there is a fully elementary embedding $j : V \to M \subseteq V$, where $M$ is transitive and $j$ has critical point $\kappa$.

One core observation about measurable cardinals is that there is a combinatorial property that describes measurability:

**Lemma 1.7.** *$V \models$ "$U$ is a $<\kappa$-complete non-trivial ultrafilter" $\iff$ there is a non-trivial embedding $\pi : V \to M \subseteq V$, where $M$ is transitive and $j$ has critical point $\kappa$.*

By strengthening the requirements on the embedding, one gets stronger large cardinal properties.

**Definition 1.8.** A cardinal $\mu$ is $\lambda$*-strong* if there is an fully elementary embedding $j : V \to M \subseteq V$, where $M$ is transitive and $j$ has critical point $\kappa$ such that $H_\lambda \subseteq M$. A cardinal $\mu$ is *strong* if it is strong for all $\lambda$.

**Definition 1.9.** Let $\varphi(u)$ be a first order statement in the language of set theory in one free variable $u$. Let $A$ be the class of all $x$ such that $\varphi(x)$. We say that an ordinal is $\varphi$*-strong* or $A$*-strong*, if and only if for all $\alpha$ there is a $j : V \to M \supsetneq V_\alpha$, where $M$ is transitive such that $j(A) \cap V_\alpha = A \cap V_\alpha$.

**Definition 1.10.** A cardinal $\delta$ is a *Woodin cardinal* if for all $A \subseteq V_\delta$ there are arbitrarily large $\kappa < \delta$ such that for all $\lambda < \delta$ there exists an elementary embedding $j : V \to M$ , where $M$ is transitive and $j$ has critical point $\kappa$, such that $j(\kappa) > \lambda$, $V_\lambda \subseteq M$ and $j(A) \cap V_\lambda = A \cap V_\lambda$.

## 1.3 Extenders

**Definition 1.11.** Let $M$ be a model of some large enough fragment of ZFC. Let $\kappa$ be a cardinal and let $\lambda > \kappa$ be an ordinal.

i. A $\langle\kappa,\lambda\rangle$-*system of filters* over $M$ is a set

$$F \subseteq \{\langle a,x\rangle;\ x \in \mathcal{P}({}^{a}\kappa) \cap M \wedge a \in [\lambda]^{<\omega}\},$$

such that for all $a \in [\lambda]^{<\omega}$, $F_a = \{x; \langle a,x\rangle \in F\}$ is a filter and there is an $x \in \mathcal{P}({}^{a}\kappa) \cap M$ with $x \notin F_a$. We set $\mathrm{supp}(F) = \{a \in [\lambda]^{<\omega}; F_a \neq \varnothing\}$. Notice that $F$ is not necessarily in $M$, we will drop $M$ in the rest of the definition, as everything is to be read as relativized to it.

ii. We say that $F$ is a $\langle\kappa,\lambda\rangle$-*system of ultrafilters* if it is a $\langle\kappa,\lambda\rangle$-*system of filter* such that each $F_a$ is an ultrafilter for $a \in \mathrm{supp}(F)$.

iii. let $F$ be a $\langle\kappa,\lambda\rangle$-system of filters. Let $a,b \in \mathrm{supp}(F)$, such that $a \subseteq b$. Let $s_{a,b} : a \to b$ be the identity. For a set $x \in \mathcal{P}({}^{a}\kappa)$, let

$$x_{a,b} = \left\{u \in {}^{b}\kappa;\ u \circ s_{a,b} \in x\right\},$$

and for a function $f : {}^{a}\kappa \to M$, we let $f_{a,b} : {}^{b}\kappa \to M$ be such that

$$f_{a,b}(u) = f(u \circ s_{a,b}).$$

iv. A $\langle\kappa,\lambda\rangle$-system of filters $F$ is called *compatible* if for all $a \subseteq b \in \mathrm{supp}(F)$

$$x \in F_a \leftrightarrow x_{a,b} \in F_b.$$

v. A $\langle\kappa,\lambda\rangle$-system of filters $F$ is called *normal* if for every $a \in X$ and for every function $f : {}^{a}\kappa \to M$ such that there is a $b \in a$ with

$$\{u \in {}^{a}\kappa :\ f(u) \in u(b)\} \in F_a$$

then there is a $c \in b$ with $a \cup \{c\} \in \mathrm{supp}(F)$ such that

$$\left\{u \in {}^{a\cup\{c\}}\kappa;\ f_{a,a\cup\{c\}}(u) = u(c)\right\} \in F_{a\cup\{c\}}.$$

vi. We call an $\langle\kappa,\lambda\rangle$-system of ultrafilters $F$ a $\langle\kappa,\lambda\rangle$-*extender* if it is compatible and normal and if $\mathrm{supp}(F) = [\lambda]^{<\omega}$.

**Definition 1.12.** Let $E$ be a $\langle\kappa,\lambda\rangle$-extender. Let $a,b \in \lambda$ and $f : {}^{a}\kappa \to M$, $g : {}^{b}\kappa \to M$. We define the equivalence relation $\sim_E$ for $f$ and $g$ by:

$$f \sim_E g \iff \left\{u \in {}^{a\cup b}\kappa;\ f_{a,a\cup b}(u) = g_{b,a\cup b}(u)\right\} \in E_{a\cup b}.$$

Similarly define $\in_E$ by:

$$f \in_E g \iff \left\{u \in {}^{a\cup b}\kappa;\ f_{a,a\cup b}(u) \in g_{b,a\cup b}(u)\right\} \in E_{a\cup b}.$$

For $a \in [\lambda]^{<\omega}$ and $f : {}^a\kappa \to M$, let $\alpha$ be minimal such that there is an $h \in (V_\alpha)^M$ with $h \sim_E f$ and let us fix

$$[f]_E = \left\{g;\ g \in (V_\alpha)^M \wedge g \sim_E f\right\}$$

If $\langle \{[f]_E;\ a \in [\lambda]^{<\omega} \wedge f : {}^a\kappa \to M\}\ ;\ \in_E\rangle$[1] is well-founded, we write $\mathrm{Ult}(M,E)$ for the transitive collaps:

$$\mathrm{Ult}(M,E) \cong \langle \{[f]_E;\ a \in [\lambda]^{<\omega} \wedge f : {}^a\kappa \to M\}\ ;\ \in_E\rangle.$$

We will encounter another type of extender in the last chapter of this thesis. It is possible to take the whole $V_\lambda$ instead of just $\lambda$ as the underlying index set. One then has to consider functions with domain $V_\kappa$ instead of just $\kappa$. The fundamental gain is that, if $V_\lambda$ is closed under sequence of length $\kappa$, and $E$ is a $\langle\kappa, V_\lambda\rangle$-extender, then is $\mathrm{Ult}(V,E)$ is closed under $\kappa$-sequences. For more on that type of extender see [MS89].

## 1.4 Generic embeddings

We shall write $X \leq_I Y$ if and only if $X \setminus Y \in I$. Forcing with $\langle I^+, \leq_I\rangle$ adds a $V$-measure $G$ and thereby a generic embedding $\pi{:}V \to \mathrm{Ult}(V;G)$. The ideal $I$ is precipitous if and only if $\mathrm{Ult}(V;G)$ is well-founded for any generic $G$. (Cf. [Jec03].) For optimality reasons, we will work in some fragment of $\mathsf{ZFC}^-$, called $\mathsf{ZFC}^*$, that is strong enough to take generic ultrapower. For more on $\mathsf{ZFC}^*$ and the exact definition, see [Woo99].

**Definition 1.13.** Let $M$ be a transitive model of $\mathsf{ZFC}^*$ + "$\omega_1$ exists" and let $I \subseteq \mathcal{P}(\omega_1^M)$ be such that $\langle M; \in, I\rangle \models$ "$I$ is a uniform and normal ideal on $\omega_1^M$." Let $\gamma \leq \omega_1$. Then

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \gamma\rangle, \langle G_i; i < \gamma\rangle\rangle$$

is called a *putative generic iteration of* $\langle M; \in, I\rangle$ (*of length* $\gamma+1$) if and only if the following hold true.

i. $M_0 = M$ and $I_0 = I$.

ii. For all $i \leq j \leq \gamma$, $\pi_{i,j}{:}\langle M_i; \in, I_i\rangle \to \langle M_j; \in, I_j\rangle$ is elementary,

iii. $I_i = \pi_{0,i}(I)$, and $\kappa_i = \pi_{0,i}(\omega_1^M) = \omega_1^{M_i}$.

iv. For all $i < \gamma$, $M_i$ is transitive and $G_i$ is $\langle I_i, \leq_{I_i}\rangle$-generic over $M_i$.

v. For all $i + 1 \leq \gamma$, $M_{i+1} = \mathrm{Ult}(M_i; G_i)$ and $\pi_{i,i+1}$ is the associated ultrapower map.

vi. $\pi_{i,j} \circ \pi_{j,k} = \pi_{i,k}$ for $i \leqslant j \leqslant k$.

[1] we used scott's trick in order to make each $[f]_{\sim_E}$ a set.

vii. If $\lambda \leq \gamma$ is a limit ordinal, then $\langle M_\lambda, \pi_{i,\lambda}, i < \lambda\rangle$ is the direct limit of $\langle M_i, \pi_{i,j}, i \leqslant j < \lambda\rangle$.

We call

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \gamma\rangle, \langle G_i; i < \gamma\rangle\rangle$$

a *generic iteration of* $\langle M; \in, I\rangle$ (*of length* $\gamma + 1$) if and only if it is a putative generic iteration of $\langle M; \in, I\rangle$ and $M_\gamma$ is transitive. $\langle M; \in, I\rangle$ is *generically* $\gamma + 1$ *iterable* iff every putative generic iteration of $\langle M; \in, I\rangle$ of length $\gamma + 1$ is an iteration.

Notice that we want (putative) iterations of a given model $\langle M; \in, I\rangle$ to exist in $V$, which amounts to requiring that the relevant generics $G_i$ may be found in $V$. The following lemma is therefore only interesting in situations in which $M$ (or a large enough initial segment thereof) is countable so that we may actually find generics in $V$.

**Lemma 1.14 (Woodin).** *Let $M$ be a transitive model of* ZFC*, and let $I \subseteq \mathcal{P}(\omega_1^M)$ be such that $\langle M; \in, I\rangle \vDash$ "$I$ is a uniform and normal precipitous ideal on $\omega_1^M$." Then $\langle M; \in, I\rangle$ is generically $\gamma + 1$ iterable whenever $\gamma < \min(M \cap \mathrm{OR}, \omega_1^V + 1)$.*

Proof. The proof is taken from [Woo99, Lemma 3.10, Remark 3.11]. By absoluteness, if $\langle M; \in, I\rangle$ is not generically $\gamma + 1$ iterable, then $\langle M; \in, I\rangle$ is not generically $\gamma + 1$ iterable inside $M^{\mathrm{Col}(\omega,\delta)}$ for some $\delta$. Let $\langle \kappa_0, \eta_0, \gamma_0\rangle$ be the least triple in the lexicographical order such that:

i. $\kappa_0 > \omega_1^M$ is regular in $M$,

ii. $\eta_0 < \kappa_0$, and

iii. for some $\delta$, inside $M^{\mathrm{Col}(\omega,\delta)}$, there is a putative iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \gamma_0\rangle, \langle G_i; i < \gamma_0\rangle\rangle$$

of $\langle H^M_{\kappa_0}; \in, I\rangle$ such that $\pi_{0,\gamma_0}(\eta_0)$ is ill-founded.

As $I$ is precipitous in $M$, $\gamma_0$ and $\eta_0$ are limit ordinals. Choose some $i^* < \gamma_0$ and $\eta^* < \pi_{0,i^*}(\eta_0)$ such that $\pi_{i^*,\gamma_0}(\eta^*)$ is ill-founded. We may construe

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i^* \leqslant i \leqslant j \leqslant \gamma_0\rangle, \langle G_i; i^* \leqslant i < \gamma_0\rangle\rangle$$

as a putative generic iteration of $H^{M_{i^*}}_{\pi_{0,i^*}(\kappa_0)}$. By elementarity, the triple

$$\langle \pi_{0,i^*}(\kappa_0), \pi_{0,i^*}(\eta_0), \pi_{0,i^*}(\gamma_0)\rangle$$

is the least triple $\langle \kappa, \eta, \gamma\rangle$ such that

i. $\kappa > \omega_1^{M_{i^*}}$ is regular in $M_{i^*}$,

ii. $\eta < \kappa$, and

iii. for some $\delta$, inside $M_{i^*}^{\mathrm{Col}(\omega,\delta)}$, there is a putative iteration

$$\langle\langle M'_i, \pi'_{i,j}, I'_i, \kappa'_i; i \leqslant j \leqslant \gamma\rangle, \langle G'_i; i < \gamma\rangle\rangle$$

of $\langle H^{M_{i^*}}_{\pi_{0,i^*}(\kappa)}; \in, I_{i^*}\rangle$ such that $\pi'_{0,\gamma}(\eta)$ is ill-founded.

However, by the existence of

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i^* \leqslant i \leqslant j \leqslant \gamma_0\rangle, \langle G_i; i^* \leqslant i < \gamma_0\rangle\rangle$$

and by absoluteness, the triple $\langle \pi_{0,i^*}(\kappa_0), \eta^*, \gamma_0 - i^*\rangle$ contradicts the alleged characterization of the triple $\langle \pi_{0,i^*}(\kappa_0), \pi_{0,i^*}(\eta_0), \pi_{0,i^*}(\gamma_0)\rangle$ inside $M_{i^*}$. $\dashv$

**Lemma 1.15.** *Let $j : V \to M$ be an elementary embedding between two* ZFC *models. Let $G$ be $\mathbb{P}$-generic over $V$ and $G'$ be $j(\mathbb{P})$-generic over $M$ such that $j''G \subseteq G'$. Then $j$ can be lifted to $j \subseteq \tilde{j} : V[G] \to M[G']$.*

The proof is part of the folklore and not too difficult to prove.

**Remark 1.16.** We can weaken the hypothesis on the models of the previous lemma to some fraction of ZFC as long as the fragment is strong enough to fulfill the forcing theorem.

## 1.5 Forcing axioms

As we have said in the introduction, forcing axioms have a general definition:

**Definition 1.17.** Let $\Gamma$ be a class of partial orders. The *forcing axiom for* $\Gamma$, $\mathsf{FA}(\Gamma)$, is the following principle:

> let $\mathbb{P} \in \Gamma$ and let $\langle D_i;\ i \in \omega_1\rangle$ denote a collection of sets dense in $\mathbb{P}$. Then there is a filter $F \subseteq \mathbb{P}$ meeting every $D_i, i < \omega_1$.

The *bounded forcing axiom for* $\Gamma$, $\mathsf{BFA}(\Gamma)$ is the following principle:

> $\mathbb{P} \in \Gamma$ and $\mathbb{Q} = \mathrm{ro}(\mathbb{P})$, the boolean algebra of regular open sets of $\mathbb{P}$. let $\langle D_i;\ i \in \omega_1\rangle$ denote a collection of sets dense of cardinality at most $\omega_1$ in $\mathbb{Q}$. Then there is a filter $F \subseteq \mathbb{Q}$ meeting every $D_i, i < \omega_1$.

**Definition 1.18.** Let $\Gamma_p$ be the collection of all proper forcing notions, $\Gamma_{sp}$ the collection of all semi-proper forcing notions and $\Gamma_s$ the collection of all stationary set preserving forcing notions. Then:

i. PFA = $\mathsf{FA}(\Gamma_p)$ and BPFA = $\mathsf{BFA}(\Gamma_p)$,

ii. SPFA = $\mathsf{FA}(\Gamma_{sp})$ and BSPFA = $\mathsf{BFA}(\Gamma_{sp})$,

iii. MM = $\mathsf{FA}(\Gamma_s)$ and BMM = $\mathsf{BFA}(\Gamma_s)$, Notice that, as SPFA implies that $\Gamma_{sp} = \Gamma s$, we have that SPFA $\iff$ MM. It turn out that these characterizations are often not as useful as an equivalent one who state the axioms in terms of forcing absoluteness. This characterization is due to Bagaria and is found in [Bag00].

**Lemma 1.19.** *The axiom* Bounded Martin's Maximum*, which we denote by* BMM*, is equivalent to:*

> *For any stationary set preserving partial order* $\mathbb{P}$, $\langle H_{\omega_2}, \in\rangle^V \prec_{\underset{\sim}{\Sigma}_1} \langle H_{\omega_2}, \in\rangle^{V^{\mathbb{P}}}$.

**Lemma 1.20.** *The* Bounded Proper Forcing Axiom*, which we denote by* BPFA*, is equivalent to:*

> *For any proper partial order* $\mathbb{P}$, $\langle H_{\omega_2}, \in\rangle^V \prec_{\underset{\sim}{\Sigma}_1} \langle H_{\omega_2}, \in\rangle^{V^{\mathbb{P}}}$.

It is in this new sense that we want to introduce the last forcing axiom we will consider in this thesis.

**Definition 1.21.** A set $A \subseteq \mathbb{R}$ *universally Baire set* if and only if for every notion of forcing $\mathbb{P}$ there exist trees $T$ and $U$ on $\omega \times \lambda$, where $\lambda = 2^{\mathrm{card}(\mathbb{P})}$, such that

$$A = p[T], \mathbb{R} \smallsetminus A = p[U]$$

and for every generic filter $G$ on $\mathbb{P}$,

$$V[G] \vDash p[T] \cup p[S] = \mathbb{R} \text{ and } p[T] \cap p[S] = \varnothing.$$

we say that $T$ and $U$ are *the trees representing the universally Baireness of $A$ for $\mathbb{P}$.*

**Definition 1.22.** The *Bounded Proper Forcing Axiom for universally Baire sets*, which we denote by $\mathrm{BPFA}^{\mathrm{uB}}$, is the following statement:

> For any proper partial order $\mathbb{P}$ and for any universally Baire set $A \subseteq \mathbb{R}$, if $T$ and $U$ are trees representing the universally Baireness of $A$ for $\mathbb{P}$,
>
> $$\langle H_{\omega_2}, \in, p[T]\rangle^V \prec_{\underset{\sim}{\Sigma}_1} \langle H_{\omega_2}, \in, p[T]\rangle^{V^{\mathbb{P}}}.$$

## 1.6 Inner model theory

We will use three types of inner model theory, depending on the anti large cardinal context we will work in.

### 1.6.1 Below one strong cardinal

We expose the framework we will work in parts of section 3.2 and throughout section 3.3. This is the context in which we will be in most of section 4.2 as well. Below a strong cardinals iteration are still linear, which simplify the theory by a great deal. One of the main consequences for us is that every embedding from $K$ into some model $M$ is already an iteration map. This won't stay true when larger cardinals are present. We will use the theory as developed in [Jenc], for an introduction on the theory of mice and many basic concept we will refer to [Zem02]. Let us give a quick overview of some of the main definitions and theorems.

While [Jenc] use the fine structure developed in [Jenb], we will use the notation introduced in [SZ], the two being equivalent. The central definitions which we will state again here for completeness are those of iteration and mouse.

**Definition 1.23.** We call a structure $N = \langle J_\alpha^{\vec{E}}, E_{\omega\alpha}\rangle$ a *premouse* if and only if the following holds:

i. $\langle J_\alpha^{\vec{E}}, E_{\omega\alpha}\rangle$ is an acceptable $J$-structure (cf. [SZ, definition 1.20 p. 16]).

ii. $\vec{E} \subseteq (\omega\alpha + 1) \times N^2$ such that $E_\xi = \varnothing$ if $\xi$ is not a limit, where
$$E_\xi = \{\langle a, X\rangle;\ \langle \xi, a, X\rangle \in E\}\,.$$

iii. For $\nu \leq \alpha$ set
$$N\|\nu = \langle J_\nu^{\vec{E}}, E_{\omega\nu}\rangle.$$
Either $E_{\omega\nu} = \varnothing$ or $N\|\nu$ is an acceptable $J$-structure such that
   a) $E_{\omega\nu}$ is a $\langle\kappa, \omega\nu\rangle$-extender on $J_\nu^{\vec{E}}$ for some $\kappa$,
   b) $E_{\omega\nu}$ is *weakly amenable* on $J_\nu^{\vec{E}}$ ( i.e. if $\langle X_\xi;\ \xi < \kappa\rangle \in N$, then, for $a \in [\omega\alpha]^{<\omega}$, $\{\xi;\ X_\xi \in E_{\omega\nu,a}\} \in N$).

iv. if $\nu < \alpha$, then $\widehat{N\|\nu}$ exists and is sound (cf. [Jenc, Appendix to §1 p. 2]).

v. Let $\nu \leq \alpha$ and $\pi : J_\nu^{\vec{E}} \to_{E_{\omega\nu}} Q = \langle J_\beta^{\vec{E}^Q}, E^Q_{\omega\beta}\rangle$ $E_{\omega\nu}$, where $E_{\omega\nu} \neq \varnothing$ is a $\langle\kappa, \omega\nu\rangle$-extender.
   a) $J_\nu^{\vec{E}} = J_\nu^{\vec{E}^Q}$ (i.e. $N\|\nu$ is *coherent*),
   b) $\omega\nu \in \mathrm{wfcore}(Q)$,
   c) $\vec{E}^Q_{\omega\nu} = \varnothing$

vi. Let $E_{\omega\nu}$ be an extender on $\kappa$. Let $\kappa^{+N\|\nu} \leq \bar{\nu} < \nu$ such that $\langle J_{\bar{\nu}}^{\vec{E}}, E_{\omega\nu} \cap J_{\bar{\nu}}^{\vec{E}}\rangle$ satisfies the previous condition i to v. Then $E_{\omega\bar{\nu}} = \varnothing$.

**Fact 1.24.** *There is a $Q$-formula $\varphi$ such that*
$$J_\alpha^{\vec{E}} \models \varphi \iff \langle J_\alpha^{\vec{E}}, \varnothing\rangle \text{ is a premouse.}$$

**Definition 1.25.** A *putative iteration $\mathcal{T}$ with indices $\langle\langle\nu_i, \alpha_i\rangle;\ i+1 \leq \theta\rangle$ of a premouse $\mathcal{M}$* is a sequence $\langle\mathcal{M}_i^{\mathcal{T}};\ i < \theta\rangle$ of premice with *iteration maps* $\langle\pi_{i,j};\ i \leq j < \theta\rangle$ such that

i. $\mathcal{M}_0^{\mathcal{T}} = \mathcal{M}$,

ii. the $\pi_{i,j}$ commute,

iii. $\omega\nu_i \leq \alpha_i \leq \mathrm{OR} \cap \mathcal{M}_i^{\mathcal{T}}$,

iv. if $E_{\omega\nu_i}^{\mathcal{M}_i^{\mathcal{T}}} = \varnothing$, then $\mathcal{M}_{i+1}^{\mathcal{T}} = \mathcal{M}_i^{\mathcal{T}} \| \alpha_i$ and $\pi_{i,i+1} = \mathrm{id} \upharpoonright \mathcal{M}_i^{\mathcal{T}} \| \alpha_i$,

v. if $E_{\omega\nu_i}^{\mathcal{M}_i^{\mathcal{T}}} \neq \varnothing$, then $E_{\omega\nu_i}^{\mathcal{M}_i^{\mathcal{T}}}$ is an extender in $\mathcal{M}_i^{\mathcal{T}} \| \alpha_i$ and
$$\pi_{i,i+1} : \mathcal{M}_i^{\mathcal{T}} \| \alpha_i \to^*_{E_{\omega\nu_i}^{\mathcal{M}_i^{\mathcal{T}}}} \mathcal{M}_{i+1},$$

vi. $\{i;\ \omega\alpha_i \in \mathcal{M}_i\}$ is finite,

vii. if $\lambda$ is a limit ordinal, then $\mathcal{M}_\lambda^{\mathcal{T}}$ is the direct limit of the system $\langle\mathcal{M}_i^{\mathcal{T}}, \pi_{i,j};\ i \leq j < \lambda\rangle$,

viii. if $\theta = \mu + 1$, $E_{\omega\nu_\mu}^{\mathcal{M}_\mu^{\mathcal{T}}}$ is an extender in $\mathcal{M}_\mu^{\mathcal{T}} \| \alpha_\mu$ and the ultrapower is eventually ill-founded.

We call a putative iteration $\mathcal{T}$ an *iteration* if all models $\mathcal{M}_i^{\mathcal{T}}$ are well-founded. $\mathcal{M}_i^{\mathcal{T}}$ is called the *$i$-th iterate* and $\theta$ is called the *length of $\mathcal{T}$*.

**Definition 1.26.** We call a (putative) iteration $\mathcal{T}$ *standard* if

i. $\omega\alpha_i = \mathrm{OR} \cap \mathcal{M}_i^{\mathcal{T}}$ if $E_{\omega\nu_i}^{\mathcal{M}_i^{\mathcal{T}}} = \varnothing$,

ii. $\alpha_i$ is the largest $\alpha \leq \mathrm{OR} \cap \mathcal{M}_i^{\mathcal{T}}$ such that $E_{\omega\nu_i}^{\mathcal{M}_i^{\mathcal{T}}}$ is a total extender on $\mathcal{M}_i^{\mathcal{T}} \| \alpha$ if $E_{\omega\nu_i}^{\mathcal{M}_i^{\mathcal{T}}} \neq \varnothing$.

We call a (putative) iteration $\mathcal{T}$

i. *simple* if $\alpha_i = \mathrm{OR} \cap \mathcal{M}_i^{\mathcal{T}}$ for all $i$,

ii. *beyond* $\lambda$, if $\lambda$ is an ordinal such that $\nu_i \geqslant \lambda$ for all $i$ and

iii. *normal* if the $\langle\nu_i;\ i < \mathrm{lh}(\mathcal{T})\rangle$ are a strictly increasing sequence.

**Definition 1.27.** A premouse $\mathcal{M}$ is called *iterable* if and only if every putative iteration is an iteration. An iterable premouse is also called a *mouse*.

Similarly we call a premouse $\mathcal{M}$ *iterable beyond* $\lambda$, $\lambda \in \mathrm{OR}$, if every putative iteration beyond $\lambda$ is an iteration.

As we will only deal with standard iterations, we will not mention that an iteration is standard anymore. Jensen showed (cf. [Jenc, Lemma 3 §2.2 p. 4]) that in this context the Dodd-Jensen lemma holds:

**Lemma 1.28 (Dodd-Jensen Lemma).** *Let $\mathcal{M}$ be an iterate of a mouse $\mathcal{N}$ with iteration map $\pi$. Let $\sigma : \mathcal{N} \to_{\Sigma^*} \mathcal{M}$. Then the iteration is simple and $\hat{\pi}(\xi) \leq \hat{\sigma}(\xi)$ for all $\xi \in \hat{\mathcal{N}}$.*

Hence iteration maps are unique and if $\mathcal{M}$ is an iterate of $\mathcal{N}$ we will denote the unique iteration map by $\pi_{\mathcal{M},\mathcal{N}}$.

One of the main class of iteration we will be studying are the iterations arising from the comparison between two mice.

**Definition 1.29.** Let $\mathcal{M}$ and $\mathcal{N}$ be two premice. We call a the tuple $\langle \mathcal{T}, \mathcal{Q} \rangle$ *the coiteration of $\mathcal{M}$ and $\mathcal{N}$* if

i. $\mathcal{T}$ is an iteration of $\mathcal{M}$ and $\mathcal{Q}$ an iteration of $\mathcal{N}$,

ii. for all $i$, $\nu_i$ is the minimal index such that $E_{\nu_i}^{\mathcal{M}_i^{\mathcal{T}}} \neq E_{\nu_i}^{\mathcal{M}_i^{\mathcal{Q}}}$,

iii. if $\theta + 1$ is the length of $\mathcal{T}$ it is also the length of $\mathcal{Q}$ and $\theta$ is minimal such that either $\mathcal{M}_\theta^{\mathcal{T}} \trianglelefteq \mathcal{M}_\theta^{\mathcal{Q}}$ or $\mathcal{M}_\theta^{\mathcal{Q}} \trianglelefteq \mathcal{M}_\theta^{\mathcal{T}}$.

If $\mathcal{M}_\theta^{\mathcal{T}} \trianglelefteq \mathcal{M}_\theta^{\mathcal{Q}}$, we say that $\mathcal{N}$ *wins the coiteration between $\mathcal{M}$ and $\mathcal{N}$.*

Remark that, by definition, every coiteration is a normal iteration.

One very important result is that all mice are coiterable:

**Lemma 1.30 (Jensen).** *Let $\mathcal{M}$ and $\mathcal{N}$ be two mice of cardinality at most $\theta$. Then $\langle \mathcal{T}, \mathcal{Q} \rangle$ the coiteration of $\mathcal{M}$ and $\mathcal{N}$ exists and is of length less than $\theta^+$. Moreover at most one side is non simple and if the $\mathcal{M}$ side is non-simple, then $\mathcal{M}_{\mathrm{lh}(\mathcal{Q})}^{\mathcal{Q}} \trianglelefteq \mathcal{M}_{\mathrm{lh}(\mathcal{T})}^{\mathcal{T}}$.*

Let us now define p-mouse, a generalization of the mouse definition. We will need it in order to define $0^{\P}$. The exact formulation of "below a strong cardinal" is actually that such mice don't exist.

**Definition 1.31.** We call a model $\mathcal{N} = \langle J_\alpha^{\vec{E}}, E_{\omega\alpha} \rangle$ a *p-premouse* if and only if the following holds:

i. $\langle J_\alpha^{\vec{E}}, \varnothing \rangle$ is a premouse,

ii. $\mathcal{N}$ satisfies all condition of a premouse in Definition 1.23 but v.c) and vi.,

iii. $J_\alpha^{\vec{E}}$ has a largest cardinal $\tau$,

iv. $E_{\omega\alpha}$ is a $\langle \tau, \omega\alpha \rangle$-extender,

v. $J_\alpha^{\vec{E}} \models$ "$o(\kappa) = \alpha$" for some $\kappa < \tau$.

There is a straight forward generalization of iterations of premice to iteration of p-premice. We call a p-premouse $\mathcal{M}$ *iterable* or a *p-mouse* if every putative iteration of $\mathcal{M}$ is an iteration. Most of the results on iterability cary over, especially the Dodd-Jensen lemma.

**Definition 1.32.** We call the least $\omega$-sound p-mouse $0^{\P}$, if it exists.

**Definition 1.33.** $J_\alpha^{\vec{E}}$ is *strong* if and only if $\langle J_\alpha^{\vec{E}}, \varnothing\rangle$ is a premouse and whenever $\mathcal{M}$ is a premouse such that $J_\alpha^{\vec{E}^{\mathcal{M}}} = J_\alpha^{\vec{E}}$ and $\mathcal{M}$ is iterable beyond $\omega\alpha$, then $M$ is a mouse and $J_\alpha^{\vec{E}^{\mathcal{M}}} = J_\alpha^{\vec{E}^{\bar{\mathcal{M}}}}$, where $\bar{\mathcal{M}} = \mathfrak{C}(\mathcal{M})$ (cf. [Jenc, §2.3 p.4]).

Now let us define the core model $K$ as:

**Definition 1.34.** Let $\langle K_\nu, \nu < \mathrm{OR}\rangle$ be the sequence such that $K_\nu = \langle J_\nu^{\vec{E}}, E_{\omega\nu}\rangle$, where $E_{\omega\nu}$ is either

i. The unique extender $F$, such that $\langle J_\nu^{\vec{E}}, F\rangle$ is a strong mouse, if it exists or

ii. $\varnothing$, if $\langle J_\nu^{\vec{E}}, \varnothing\rangle$ is a strong mouse and there are no $F$ such that $\langle J_\nu^{\vec{E}}, F\rangle$ is strong.

if $K_\nu$ exists for $\nu \in \mathrm{OR}$, we set $K = J_\infty^{\vec{E}} = \bigcup_\nu J_\nu^{\vec{E}}$. $K$ is then called the *core model*.

Assuming $0^{\P}$ does not exists, one can show the existence of $K$ as well as many of its properties:

**Theorem 1.35 (Jensen).** *Suppose $0^{\P}$ does not exist. Then the following holds:*

*i. $K$ exists and is iterable.*

*ii. $K$ is rigid, i.e. there are no non-trivial embedding $j : K \to K$.*

*iii. Weak covering holds for $K$, i.e. if $\beta \geqslant \omega_2^V$ is a cardinal in $K$, then*

$$\mathrm{cf}(\beta^{+K}) \geqslant \mathrm{card}(\beta).$$

*iv. If $G$ is generic over $V$ then $K = K^{V[G]}$.*

*v. Every universal weasel $W$ is a simple iterate of $K$.*

*vi. Suppose $j : K \to W$ is a $\Sigma_1$-elementary embedding, then $W$ is a simple iterate of $K$ and $j = \pi_{K,W}$.*

### 1.6.2 Below one Woodin cardinal

We will mainly use this theory in the later stages of Chapter 3, especially in Lemma 3.13 as well as in section 4.3. The theory we expose below is mostly taken from [MS94] and [Ste96]. While the results all suppose that there is no inner model with a Woodin cardinal, the definition cary over to a setting where many Woodin cardinals are allowed. One key feature is that this is the last moment where we have a fully iterable core model $K$ that is rigid, forcing absolute and who satisfy weak covering. Let us restate the central definitions and theorems as well as some tools we will need later. We will omit all proofs as we give references on where to find them.

**Definition 1.36.** A sequence $\vec{E} = \langle E_\beta;\ \beta \in S\rangle$ is *fine at* $\alpha$ if $J_\alpha^{\vec{E}}$ is strongly acceptable and if $\alpha \in S$ it satisfies the following clauses:

i. $E_\alpha$ is a $\langle\kappa,\lambda\rangle$-extender for some $\kappa$ such that $J_\alpha^{\vec{E}\restriction\alpha} \vDash$ "$\kappa^+$ exists",

ii. (bounded generators) $E_\alpha$ is the trivial completion of $E_\alpha \restriction \nu$, where $\nu$ is the natural length of $E_\alpha$, and $E_\alpha$ is not of type $Z$.

iii. (coherence) Let $i : J_\alpha^{\vec{E}\restriction\alpha} \to \mathrm{Ult}(J_\alpha^{\vec{E}\restriction\alpha}, E_\alpha)$ be the canonical embedding,

$$i(\vec{E} \restriction \kappa) \restriction \alpha = \vec{E} \restriction \alpha$$

and $i(\vec{E} \restriction \kappa)_\alpha = \varnothing$, and

iv. (closure under initial segment) let $\nu$ be the natural length of $E_\alpha$. If $\eta$ is an ordinal such that $(\kappa^+)^{J_\alpha^{\vec{E}}} \le \eta \le \nu$ and $\eta$ is the natural length of $E_\alpha \restriction \eta$ and $E_\alpha \restriction \eta$ is not of type $Z$ , then one of the two condition below holds:

   a) there is a $\gamma$ such that $E_\gamma$ is the trivial completion of $E_\alpha \restriction \eta$ or

   b) $\eta \in S$, let $i : J_\eta^{\vec{E}\restriction\eta} \to \mathrm{Ult}(J_\eta^{\vec{E}\restriction\eta}, E_\eta)$ be the ultrapower map, there is a $\gamma < \alpha$ such that $\pi(\vec{E} \restriction \eta)_\gamma$ is the trivial completion of $E_\alpha \restriction \eta$.

**Definition 1.37.** We call a structure $\langle J_\alpha^{\vec{E}}, E_\alpha\rangle$ a *potential premouse* if $\vec{E}$ is fine at all $\nu \le \alpha$. We call a potential premouse $\mathcal{N}$ a *premouse* if $\mathcal{N}$ is a potential premouse of which all proper initial segments are sound.

**Definition 1.38.** A premouse $\mathcal{M}$ is $(n+1)$-small if for every $\langle\kappa,\nu\rangle$-extender on the $\mathcal{M}$-sequence, $\mathcal{M}|\kappa \vDash$ "there are strictly less than $n+1$ Woodin cardinals".

**Definition 1.39.** Let $\mathcal{M}$ be a potential premouse. We call $\delta \in \mathrm{OR} \cap \mathcal{M}$ a *cutpoint* of $\mathcal{M}$ if for every $\nu > \delta$, $E_\nu^{\mathcal{M}} \neq \varnothing$ implies that $\mathrm{cp}(E_\nu^{\mathcal{M}}) > \delta$.

We write $\mathcal{N} \lhd^* \mathcal{M}$ if $\mathcal{N} \lhd \mathcal{M}$ and $\mathcal{N} \cap \mathrm{OR}$ is a cutpoint of $\mathcal{M}$.

The result we will sate in this section restrict themselves to 1-small mice, but the concepts can be used later on, that is why we prefer to state them in full generality.

One important feature of normal iteration and their related maps is that, whenever $\nu$ is the length of the $i$-th extender, then for every $i < \alpha < \beta$ we have that the iteration map $\pi_{\alpha,\beta}$ restricted to $\nu$ is the identity. This was possible below a strong cardinal[2], because, if $\alpha < \beta$ both indexed extender, we had that $j(\mathrm{cp}(E_\beta)) > \alpha$, where $j$ is the canonical ultrapower map by $E_\alpha$. This no longer holds true in the presence of strong cardinals. In order to maintain this property, we will look at iteration trees, where we apply an extender to the largest model such that the previous consideration still holds true.

[2] we use that rather sloppy expression in the sense of the non-existence of $0^\P$

**Definition 1.40.** let $\mathcal{M} = \langle J_\alpha^{\vec{E}}, \in \vec{E} \restriction \alpha, E_\alpha \rangle$ be a premouse. $\mathcal{T}$ is an *iteration tree on* $\mathcal{M}$ *of length* $\theta$ if

$$\mathcal{T} = \langle T, \deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1};\ \alpha+1 < \theta \rangle \rangle,$$

where $T$ is a tree order, which satisfies the condition below. We write $\rho_\alpha$ for the natural length of $E_\alpha$. We will also define potential premice $\mathcal{M}_\alpha$ and embeddings $i_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for ordinals $\alpha$ and $\beta$ less than $\theta$ such that $\alpha T \beta$ and $D \cap ]\alpha, \beta]_T \neq \varnothing$.

i. $\mathcal{M}_0 = \mathcal{M}$, and each $\mathcal{M}_\alpha$ is a potential premouse.

ii. $E_\alpha$ is the extender coded by $\dot{F}^{\mathcal{N}}$, for some active potential premouse $\mathcal{N}$ which is an initial segment of $\mathcal{M}_\alpha$, where $\dot{F}$ is the predicate for the top extender in the language of mice.

iii. $\alpha < \beta \implies \mathrm{lh}(E_\alpha) < \mathrm{lh}(\beta)$.

iv. If $\beta$ is the $T$-predecessor of $\alpha$, then $\kappa = \mathrm{cp}(E_\alpha) < \rho_\beta$, and $\mathcal{M}^*_{\alpha+1}$ is an initial segment of $J_\gamma^{\mathcal{M}_\beta}$ of $\mathcal{M}_\beta$ such that $\mathcal{P}(\kappa) \cap J_\gamma^{\mathcal{M}_\beta} = \mathcal{P}(\kappa) \cap \mathcal{N}$. Moreover

$$\alpha + 1 \in D \iff J_\gamma^{\mathcal{M}_\beta} \text{ is a proper initial segment of } \mathcal{M}_\beta.$$

If we take $n = \deg(\alpha+1)$, then $n$ is maximal such that $\kappa < \rho_n(\mathcal{M}^*_{\alpha+1})$ and we set

$$\mathcal{M}_{\alpha+1} = \mathrm{Ult}_n(\mathcal{M}^*_{\alpha+1}, E_\alpha)$$

and if $\alpha + 1 \notin D$, then

$$i_{\beta,\alpha+1} = \text{canonical embedding of } \mathcal{M}_\beta \text{ into } \mathrm{Ult}_n(\mathcal{M}_\beta, E_\alpha),$$

and $i_{\gamma,\alpha+1} = i_{\beta,\alpha+1} \circ i_{\gamma,\beta}$ for all $\gamma T \beta$ such that $]\gamma, \beta]_T \cap D = \varnothing$.

v. If $\lambda < \theta$ is a limit, then $D \cap [0, \lambda[_T$ is finite, and letting $\gamma$ be the largest element of $D \cap [0, \lambda[_T$,

$\mathcal{M}_\lambda$ is the direct limit of $\mathcal{M}_\alpha$, $\alpha \in [\gamma, \lambda[_T$, under the $i_{\alpha,\beta}$'s
$i_{i,\lambda}$ = canonical embedding of $\mathcal{M}_\eta$ into $\mathcal{M}_\lambda$, for $\eta \in [\gamma, \lambda[_T$.

vi. $\mathcal{M}^*_{\alpha+1}$ is $\deg(\alpha+1)$-sound.

vii. If $\gamma + 1 T \alpha + 1$ and $D \cap ]\gamma+1, \alpha+1]_T = \varnothing$, then $\deg(\gamma+1) \geqslant \deg(\alpha+1)$.

viii. For $\lambda \geqslant \theta$ a limit, $\deg(\lambda) = \deg(\alpha+1)$, for all sufficiently large $\alpha + 1 T \lambda$.

For technical reason we want to introduce *padded iterations*, who are like iteration, but we allow successor steps to be trivial, that is, no extender were picked and we set

$$\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha = \mathcal{M}^*_{\alpha+1}.$$

The importance of padded iterations will become clear when we start talking about the comparison process. All further definition carry on the very same way with padded iteration, therefore we won't mention them specifically.

**Definition 1.41.** let $\mathcal{M}$ be a premouse and $\mathcal{T}$ an iteration tree on $\mathcal{M}$ of limit length. We write

$$\delta(\mathcal{T}) = \sup\left\{\mathrm{lh}(E_\alpha^\mathcal{T});\ \alpha < \mathrm{lh}(\mathcal{T})\right\}$$

and $\mathcal{M}(\mathcal{T})$, the *common part model of* $\mathcal{T}$, the unique passive $\mathcal{P}$ such that $\mathrm{OR}\cap\mathcal{P} = \delta(\mathcal{T})$ and $\forall\alpha < \delta(\mathcal{T})\ \mathcal{M}_\alpha^\mathcal{T}|\mathrm{lh}(E_\alpha^\mathcal{T}) \lhd \mathcal{P}$.

**Definition 1.42.** let $\mathcal{M}$ be a premouse, $\mathcal{T}$ an iteration tree on $\mathcal{M}$ of limit length and $b$ a cofinal branch through the tree. Let $\gamma$ be the least ordinal, if there is one, such that either

i. $\omega\gamma < \mathrm{OR}\cap\mathcal{M}_b^\mathcal{T}$ and $\mathcal{J}_\gamma^{\mathcal{M}_b^\mathcal{T}} \vDash$ "$\delta(\mathcal{T})$ is not Woodin",

ii. $\omega\gamma = \mathrm{OR}\cap\mathcal{M}_b^\mathcal{T}$ and $\rho_\omega(\mathcal{M}_b^\mathcal{T}) < \delta(\mathcal{T})$.

We set

$$\mathcal{Q}(b,\mathcal{T}) = \mathcal{J}_\gamma(\mathcal{M}_b^\mathcal{T}),$$

if such a $\gamma$ exists. If $\mathcal{Q}(b,\mathcal{T})$ exists and is iterable above $\delta(\mathcal{T})$, we call it the *Q-structure* of $b$.

There are two very different steps in the process of building an iteration tree. At the successor step, we choose an extender on the branch of the last model and at limit steps, one chooses a cofinal branch in the tree and build the direct limit. Everything else is already given by these two choices. Similarly to the linear case, we want to be able to consider taking every extender at the successor steps. As for the limit step, we are mostly interested in an easily definable branch, as unique as possible. We will see that in our context, one branch will be singularized as being the "right" one. In order to modelize this dichotomy in levels of freedom, we will use the game concept and consider the construction of an iteration tree to be a play in $\mathscr{G}_k(\mathcal{M},\theta)$.

**Definition 1.43.** Let $\mathcal{M}$ be a premouse and $\theta$ an ordinal. The *iteration game* $\mathscr{G}_k(\mathcal{M},\theta)$ is a two player game of length $\theta$ played as follow:

Suppose we are at stage $\alpha < \theta$: we have an iteration tree

$$\mathcal{T}_\alpha = \langle T_\alpha, \deg_\alpha, D_\alpha, \langle E_\nu, \mathcal{M}_{\nu+1}^*;\ \nu+1 < \alpha\rangle\rangle$$

of length $\alpha$. Player I chooses an extender $E_{\alpha+1}$ and take an ultrapower with the appropriate model $\mathcal{M}_\beta$ such that

$$\mathcal{T}_{\alpha+1} = \langle T_{\alpha+1}, \deg_{\alpha+1}, D_{\alpha+1}, \langle E_\nu, \mathcal{M}_{\nu+1}^*;\ \nu+1 < \alpha+1\rangle\rangle$$

is an iteration, where $T_\alpha$ is a tree such that:

i. $T_{\alpha+1} \upharpoonright \alpha = T_\alpha$,

ii. $\beta T_{\alpha+1} \alpha + 1$,

iii. $\deg_{\alpha+1}(\alpha+1) \leq k$ if it exists.

Suppose we are at stage $\lambda$, where $\lambda$ is a limit ordinal, then player II chooses an $\in$-cofinal branch $b$ through $\bigcup_{\alpha<\lambda} T_\alpha$ such that the direct limit of the models

$$\mathcal{M}^\lambda_\lambda = \lim \left\{ \mathcal{M}^{T_\alpha}_\alpha ; \alpha \in b \right\}$$

is well-founded. And we set the iteration

$$\mathcal{T}_\lambda = \langle T_\lambda, \deg_\lambda, D_\lambda, \langle E_\nu, \mathcal{M}^*_{\nu+1};\ \nu+1 < \lambda \rangle \rangle$$

such that $\alpha T_\lambda \lambda$ for all $\alpha \in b$ and for all $\alpha < \lambda$ $T_\lambda \upharpoonright \alpha = T_\alpha$. Player II looses if one ultrapower or a limit model is ill-founded. If II wins the game if it does not loose for $\theta$ many steps.

As for every $\alpha < \beta$, $\mathcal{T}_\beta \upharpoonright \alpha = \mathcal{T}_\alpha$, we will drop the indices and only speak of *the game* $\mathcal{T}$, where $\mathcal{T}$ the union of all $\mathcal{T}_\alpha$.

**Definition 1.44.** Let $\mathcal{M}$ be a premouse. We call $\Sigma$ an *iteration strategy for II in* $\mathscr{G}_k(\mathcal{M},\theta)$ if $\Sigma$ maps partial plays $\mathcal{T}$ of $\mathscr{G}_k(\mathcal{M},\theta)$ of limit length to some $\in$-cofinal branch $b$ through $T$.

We call an iteration game $\mathcal{T}$ *played according to* $\Sigma$ if for every limit stage $\lambda$, the branch player II chose was $\Sigma(\mathcal{T} \upharpoonright \lambda)$.

We call $\Sigma$ a *winning strategy* if II never looses a game played according to $\Sigma$.

**Definition 1.45.** Let $\theta$ be an ordinal and $k \leq \omega$. We call a premouse $\mathcal{M}$ *$\langle\theta, k\rangle$-iterable* if it has a winning strategy for $\mathscr{G}_k(\mathcal{M},\theta)$.

One of the main tools, when working in inner model theory is the comparison process:

**Definition 1.46.** Let $\mathcal{M}$ and $\mathcal{N}$ be two premice of cardinality at most $\theta$, $\Sigma^{\mathcal{M}}$ a winning strategy for $\mathscr{G}_\omega(\mathcal{M},\theta^+)$ and $\Sigma^{\mathcal{N}}$ a winning strategy for $\mathscr{G}_\omega(\mathcal{N},\theta^+)$ . We call a pair of padded iterations $\langle\mathcal{T},\mathcal{U}\rangle$ a *comparison* if the following conditions are met:

i. $\mathcal{T}$ is a play in $\mathscr{G}_\omega(\mathcal{M},\theta^+)$ played according to $\Sigma^{\mathcal{M}}$ and $\mathcal{U}$ is a play in $\mathscr{G}_\omega(\mathcal{N},\theta^+)$ played according to $\Sigma^{\mathcal{N}}$.

ii. $\mathcal{T}$ and $\mathcal{U}$ have the same length say $\lambda$.

iii. at successor stage $\alpha < \lambda$, let $\nu$ be the minimal ordinal such that

$$E^{\mathcal{M}^{\mathcal{T}}_\alpha}_\nu \neq E^{\mathcal{N}^{\mathcal{U}}_\alpha}_\nu,$$

then $E^{\mathcal{T}}_\alpha = E^{\mathcal{M}^{\mathcal{T}}_\alpha}_\nu$ and $E^{\mathcal{U}}_\alpha = E^{\mathcal{N}^{\mathcal{U}}_\alpha}_\nu$.

Notice that $E_\nu^{\mathcal{M}_\alpha^\mathcal{T}}$ or $E_\nu^{\mathcal{N}_\alpha^\mathcal{U}}$ might very well be empty, that is why we produce padded iterations. We call $\lambda$ the *length of* $\langle\mathcal{T},\mathcal{U}\rangle$.

Notice that once is given two iteration strategies, a comparison is uniquely determined.

**Definition 1.47.** Let $\mathcal{M}$ and $\mathcal{N}$ be two premice. We call $\mathcal{M}$ and $\mathcal{N}$ *coiterable* if there is a comparison $\langle\mathcal{T},\mathcal{U}\rangle$ such that if $\lambda$ is the length of the comparison:

$$\mathcal{M}_\lambda^\mathcal{T} \trianglelefteq \mathcal{M}_\lambda^\mathcal{U} \text{ or } \mathcal{M}_\lambda^\mathcal{U} \trianglelefteq \mathcal{M}_\lambda^\mathcal{T}$$

**Definition 1.48.** We call two extender $E$ and $F$ *compatible* if for some $\eta$, $E$ is the trivial completion of $F \restriction \eta$ or $F$ is the trivial completion of $E \restriction \eta$.

**Lemma 1.49.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two coiterable premouse and $\langle\mathcal{T},\mathcal{U}\rangle$ their coiteration. If $E$ is an extender used in $\mathcal{T}$ and $F$ an extender used in $\mathcal{U}$, then $E$ and $F$ are not compatible.*

Proof. Suppose $E = E_\alpha^\mathcal{T}$ and $F = E_\beta^\mathcal{U}$, $\lambda$ is such that $E$ is the trivial completion of $F \restriction \lambda$. Let further $\gamma$ be the $\mathcal{T}$ predecessor of $\alpha+1$ and $\eta$ be the $\mathcal{U}$ predecessor of $\beta+1$. Since $\mathrm{lh}(E) \le \mathrm{lh}(F)$, we have that $\alpha \le \beta$. $\alpha \ne \beta$, else $\mathrm{lh}(E) = \mathrm{lh}(F)$ and thus $E = F$ and we would not use this extender in the coiteration. Hence $\alpha < \beta$ and $\mathrm{lh}(E)$ is a cardinal in $\mathcal{M}_\eta^\mathcal{U}$. On the other hand the initial condition implies that $E$ is on the extender sequence of $\mathcal{M}_\eta^\mathcal{U}$, but $E$ collapses $\mathrm{lh}(E)$ to the natural length of $E$, a contradiction! ⊣

Let us now briefly state the most important results below one Woodin cardinal from [MS94, 6.1 p. 58]:

**Theorem 1.50 (Uniqueness Theorem).** *Let $\mathcal{T}$ be an iteration tree of limit length $\theta$, and $b$ and $c$ be distinct cofinal well-founded branches of $\mathcal{T}$. Let $\alpha = \mathrm{OR} \cap \mathcal{M}_b^\mathcal{T} \cap \mathcal{M}_c^\mathcal{T}$, so that $\alpha \ge \delta(\mathcal{T})$, and suppose that $\alpha > \delta(\mathcal{T})$. Then*

$$\mathcal{J}_\alpha(\mathcal{M}(\mathcal{T})) \vDash \text{“}\delta(\mathcal{T}) \textit{ is Woodin”}.$$

Hence if we suppose that there are no iterable mice who are not 1-small, for every iteration $\mathcal{T}$ of limit length there is at most one cofinal branch $b$ such that

$$\mathcal{M}_b^\mathcal{T} \vDash \text{“}\delta(\mathcal{T}) \text{ is not Woodin”}.$$

Thus choosing exactly that branch gives an iteration strategy. We call a mice iterable just in case the strategy we just defined is a winning strategy. Notice that if there is an iteration strategy, it must be that one. Next we want to show that we can compare two iterable mice, the result can be found in [MS94, 7.1 p. 69]:

**Theorem 1.51 (The comparison lemma).** *Let $\mathcal{M}$ and $\mathcal{N}$ be $n$-sound, 1-small, $n$-iterable premice, where $n \le \omega$. Then there is a comparison $\langle\mathcal{T},\mathcal{U}\rangle$ of $\mathcal{M}$ and $\mathcal{N}$ such that either*

i. *$\langle \mathcal{T}, \mathcal{U} \rangle$ has successor length $\theta + 1$ and either*

   a) *$\mathcal{M}^{\mathcal{T}}_{\theta}$ is a initial segment of $\mathcal{N}^{\mathcal{U}}_{\theta}$ and $D^{\mathcal{T}} \cap [0,\theta]_{\mathcal{T}} = \varnothing$ and $\deg(\alpha+1) = n$ for all $\alpha+1 \in [0,\theta]_{\mathcal{T}}$, or*

   b) *$\mathcal{N}^{\mathcal{U}}_{\theta}$ is a initial segment of $\mathcal{M}^{\mathcal{T}}_{\theta}$ and $D^{\mathcal{U}} \cap [0,\theta]_{\mathcal{U}} = \varnothing$ and $\deg(\alpha+1) = n$ for all $\alpha+1 \in [0,\theta]_{\mathcal{U}}$,*

   *or,*

ii. *$\langle \mathcal{T}, \mathcal{U} \rangle$ has limit length, one of the two is not simple, and in some $V^{\mathrm{col}(\omega,\kappa)}$ there are well-founded branches $b$ of $\mathcal{T}$ and $c$ of $\mathcal{U}$ such that either*

   a) *$\mathcal{M}^{\mathcal{T}}_{b}$ is an initial segment of $\mathcal{N}^{\mathcal{U}}_{b}$, $D^{\mathcal{T}} \cap b = \varnothing$ and $\deg(\alpha+1) = n$ for all $\alpha+1 \in b$, or*

   b) *$\mathcal{N}^{\mathcal{U}}_{c}$ is an initial segment of $\mathcal{M}^{\mathcal{T}}_{c}$, $D^{\mathcal{U}} \cap c = \varnothing$ and $\deg(\alpha+1) = n$ for all $\alpha+1 \in c$.*

When building the core model around one or more Woodin cardinals, it becomes very convenient to assume that there is a measurable $\Omega$ and to work in $V_{\Omega}$. Again following Steel in [Ste96] one can define $K^c$ and ultimately $K$:

**Definition 1.52.** Suppose $\Omega$ is $S$-thick in $W$. Then we put

$$x \in \mathrm{Def}(W,S) \iff \forall \Gamma \big( \Gamma \text{ is } S\text{-thick in } W \implies x \in \mathrm{Hull}^{W}(\Gamma) \big)$$

**Remark 1.53.** Let $\Omega$ be $S$-thick in $W$, and let $i : W \to Q$ be the iteration map coming from an iteration tree on $W$; then $i'' \mathrm{Def}(W,S) = \mathrm{Def}(Q,S)$.

For the proof see [Ste96, Lemma 5.6].

**Definition 1.54.** Suppose $K^c \vDash$"There are no Woodin cardinals". We set $K$ as the common transitive collaps of $\mathrm{Def}(W,S)$ for all $\Omega+1$-iterable weasel such that $\Omega$ is $A_0$-thick in $W$. We call $K$ *the Mitchell-Steel core model.*

Notice that by results of Steel, if $K^c$ has no Woodin cardinals, it is a $\Omega+1$-iterable universal weasel such that $\Omega$ is $A_0$-thick in $K^c$.

We get that $K$ has the usual properties:

**Theorem 1.55 (Steel).** *Suppose $K^c \vDash$"There are no Woodin cardinals". Then the following holds:*

i. *$K$ exists and is $\Omega+1$-iterable.*

ii. *$K$ is rigid, i.e. there are no non-trivial embedding $j : K \to K$.*

iii. *Weak covering holds for $K$, i.e. if $\beta \geqslant \omega_2^V$ is a cardinal in $K$, then*

$$\mathrm{cf}(\beta^{+K}) \geqslant \mathrm{card}(\beta).$$

iv. *If $G$ is generic over $V$ then $K = K^{V[G]}$.*

v. *$K$ is the unique universal weasel which elementarily embeds into all universal weasels.*

### 1.6.3 Below PD

We can think of this theory as being a continuation of the work drafted above. We will use the theory as presented in [Ste] and more recently in [SS]. The main use of this subsection is to outline the theory we will use for the proof of the core model induction.

We shall consider mice relativized to some transitive set $x$, we will assume that those sets are coded as subsets of the ordinals. We will restrict ourself to the study of *self-wellordered sets*, short *swo*'s, that is to sets that code a wellorder of themselves. Recall that the Jensen hierarchy relativized to a swo $x$ starts with

$$\mathcal{J}_0(x) = \mathrm{TC}(x).$$

**Definition 1.56.** A *potential $X$-premice* $\mathcal{M}$ is a structure of the form:

$$\mathcal{M} = \langle \mathcal{J}_\alpha^{\vec{E}}(X), \in, X, \vec{E} \upharpoonright \alpha, E_\alpha \rangle,$$

where $\vec{E}$ is a fine extender sequence relativized to $X$. We call $\mathcal{M}$ an *$r$-premouse* in case $\mathcal{M}$ is an $X$-premouse for some swo $X$.

Iteration are defined just as in the previous section, as are the concept of strategy, iterable mice and so on.

**Definition 1.57.** Let $\mathcal{M}$ be a $r$-premouse. We say that $x$ is *small generic over $\mathcal{M}$* if $x$ is generic over $\mathcal{J}_0^{\mathcal{M}}$.

**Definition 1.58.** We say that a $r$-premouse $\mathcal{M}$ has the *strong condensation property* if every countable substructure of $\mathcal{M}$ is an initial segment of $\mathcal{M}$.

**Definition 1.59.** A mouse operator on $Z$ is a function $\mathcal{N}$ assigning to each swo $x \in Z$ a countably iterable $x$-premouse $\mathcal{N}(x)$ such that $\mathcal{N}(x)$ is pointwise definable from members of $x \cup \{x\}$. We say that $\mathcal{N}$ is first order just in case there is a theory $T$ in the language of $r$-premice (so having a symbol $x$ for $x$) such that for all $x \in Z$, $\mathcal{N}(x)$ is the least countably iterable $x$-premouse satisfying $T$.

**Lemma 1.60.** *Let $\mathcal{N}$ be a first order mouse operator on $Z$, and suppose $\pi : \mathcal{P} \to \mathcal{N}(x)$ is fully elementary in the language of relativised mice; then $P = \mathcal{N}(\pi^{-1}(x))$.*

Hence if $\mathcal{N}$ is a first order mouse operator on $Y$ and $x \in Y$ is countable, $\mathcal{N}(x)$ has the strong condensation property. Actually every countable substructure is already equal to $\mathcal{N}(x)$.

Let us define two important classes of mouse operators we will discuss in the future. We actually already met different instances of these in the previous subsection.

**Definition 1.61.** Let $\mathcal{M}$ be an $r$-premouse. A $Q$-structure for $\mathcal{M}$ is an $r$-premouse $\mathcal{Q}$ such that

i. $\mathcal{M} \triangleleft^* \mathcal{Q}$,

ii. either $\rho_\omega(\mathcal{Q}) < o(\mathcal{M})$ or there is some $r\Sigma_n^{\mathcal{Q}}$ definable counterexample to the woodiness of $o(\mathcal{M})$,

iii. $\mathcal{Q}$ is countably iterable above $o(\mathcal{M})$.

There is at most one $Q$-structure for any given $r$-premouse $\mathcal{M}$, hence this defines a first-order mouse operator on its domain. We are most interested on the existence of $\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(\mathcal{M}(\mathcal{T}))$, for $\mathcal{T}$ an iteration tree, in this case this definition and the definition in the previous subsection agree. Using this operator and $\mathcal{Q}(b, \mathcal{T})$ for a cofinal branch $b$ of $\mathcal{T}$, we can define an iteration strategy:

**Definition 1.62.** $\Sigma^t$ is the following iteration strategy:

$$\Sigma^t(\mathcal{T}) = \text{the unique branch } b \text{ such that } \mathcal{Q}(\mathcal{T}) = \mathcal{Q}(b, \mathcal{T}).$$

We write $\Sigma^t_{\mathcal{M}}$ for the strategy restricted to iteration trees on $\mathcal{M}$. Let us give the main result that connects $Q$-structures, Woodin cardinals and iteration strategies, taken from [Ste, Theorem 6.10]:

**Theorem 1.63 (Branch Uniqueness Theorem).** *Let $\mathcal{T}$ be an iteration tree such that there are two distinct cofinal branches $b$ and $c$. Let $\delta = \delta(\mathcal{T})$, and suppose that $A \subseteq \delta$ is such that $A \in \mathrm{wfp}(\mathcal{M}_b^{\mathcal{T}}) \cap \mathrm{wfp}(\mathcal{M}_c^{\mathcal{T}})$; then*

$$\mathcal{M}_b^{\mathcal{T}} \vDash \exists \kappa < \delta(\kappa \text{ is } A\text{-reflecting in } \delta)$$

and with its corollary:

**Corollary 1.64.** *Let $\mathcal{T}$ be a $k$-maximal iteration tree; then there is at most one cofinal, wellfounded branch $b$ of $\mathcal{T}$ such that*

*i. $\mathcal{Q}(b, \mathcal{T})$ exists,*

*ii. $\delta(\mathcal{T})$ is a cutpoint of $\mathcal{Q}(b, \mathcal{T})$, and*

*iii. $\mathcal{Q}(b, \mathcal{T})$ is $\delta(\mathcal{T}) + 1$-iterable.*

This makes sure the iteration strategy defined above is indeed well defined.

Let us state a few properties of this iteration strategy as found in [SS, p.5]:

**Lemma 1.65.** *Let $\mathcal{M}$ be a tame $r$-premouse,*

*i. If $\mathcal{T}$ is an iteration tree on $\mathcal{M}$, then there is at most one cofinal $b$ such that $\mathcal{Q}(\mathcal{T})$ and $\mathcal{Q}(b, \mathcal{T})$ are defined, and $\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(b, \mathcal{T})$, moreover*

*ii. if in addition $\mathcal{M} \vDash$ "There are no Woodin cardinals", or even just $\mathcal{M}$ projects below its bottom Woodin (above the set over which it is built, in each case), then for any cofinal $b$ of $\mathcal{T}$ , $\mathcal{Q}(b, \mathcal{T})$ exists, so that*

*iii. if in addition, $\mathcal{M}$ is $\omega_1 + 1$-iterable, then $\Sigma^t$ is its unique $\omega_1 + 1$-iteration strategy.*

The second class of mouse operator we already met are minimal mice that witness the existence of some large cardinal:

**Definition 1.66.** Let $\varphi \equiv \varphi(\nu_0, \nu_1)$ be a $\Sigma_1$-formula in the language of boldface premice, and let $z \in \mathbb{R}$. Let $X$ be a set of ordinals which codes $z$. An $X$-premouse $\mathcal{M}$ is called $(\varphi, z)$-small if and only if

$$\mathcal{M} \vDash \neg\varphi(z, X).$$

The mouse operator given by $\varphi, z$ is the unique partial map $X \to \mathcal{M}(X) = \mathcal{M}_{\varphi,z}(X)$ which assigns to any set $X$ of ordinals the unique $X$-mouse $\mathcal{M}(X)$ such that $\mathcal{M}(X)$ is sound above $X$, $\mathcal{M}(X)$ is not $\varphi, z$-small, but every proper initial segment of $\mathcal{M}(X)$ is $\varphi, z$-small, if it exists.

**Definition 1.67.** Let $X$ be a swo, $n \in \omega$ and let $\dot{F}$ be the predicate for the top extender in the language of $r$-premice.

i. let $\varphi \equiv$ "$\dot{F} \neq \varnothing$", then we write $J_\varphi(X) = X^\#$ and call it the *sharp* of $X$.

ii. Let $\varphi \equiv$ "$\dot{F} \neq \varnothing$ and there is a cardinal that is strong up to $\mathrm{cp}(\dot{F})$", then $J_\varphi(\varnothing) = 0^\P$.

iii. Let $\varphi \equiv$ "$\dot{F} \neq \varnothing$ and there are $n$ Woodin cardinals below $\mathrm{cp}(\dot{F})$", then we write $\mathcal{M}_n^\#(X) = J_\varphi(X)$.

**Remark 1.68.** Let $x \in H_{\omega_1}$, then for every $n < \omega$, $\mathcal{M}_n^\#(x)$ has the strong condensation property, if it exists.

**Definition 1.69.** Let $\mathcal{N}$ be a mouse operator, and let $\mathcal{M}$ be a tame premouse. We say that $\Sigma^t_{\mathcal{M}}$ is $\mathcal{N}$*-guided* on $Y$ just in case whenever $\mathcal{T} \in Y$ is a tree of limit length played by $\Sigma^t_{\mathcal{M}}$, then $\Sigma^t_{\mathcal{M}}(\mathcal{T})$ is defined, and letting $b = \Sigma^t_{\mathcal{M}}$, we have $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{N}(\mathcal{M}(\mathcal{T}))$.

It will be critical in order to show that complexity of iterability is "low", to show that the iterations of $n+1$-small mice are actually $\mathcal{M}_n^\#$-guided. Hence the importance of the existence of $\mathcal{M}_n^\#(X)$. Let us fix as convenient notation:

$$A_n \equiv \text{"for every set } X,\ \mathcal{M}_n^\#(X)\text{" exists and is iterable.}$$

When $A_n$ holds, we say that $V$ *is closed under the* $\mathcal{M}_n^\#$*-operator*. Similarly the closure of many subsets of the universe under the $\mathcal{M}_n^\#$-operator can have important consequences. The most famous one being, when the reals are closed under $\mathcal{M}_n^\#$:

**Theorem 1.70 (Harrington, Martin, Steel, Woodin, Neeman).** *For $n < \omega$ the following are equivalent:*

*i.* $\Pi^1_{n+1}$*-determinacy,*

*ii. for every* $x \in \mathbb{R}$, $\mathcal{M}_n^\#(x)$ *exists,*

*iii. for every* $x \in \mathbb{R}$, $\mathcal{M}_n^\#(x)$ *exists and is unique.*

as an immediate corollary, we get that

**Corollary 1.71.** *Suppose the reals are closed under the $\mathcal{M}_n^\#$-operator for all $n$, then projective determinacy holds.*

Let us state a few result one gets when assuming the closure of the universe under the $\mathcal{M}_n^\#$-operator.

**Lemma 1.72.** *Suppose $A_n$. Let $\mathbb{P}$ be a forcing and $G$ be $\mathbb{P}$-generic over $V$. The following holds:*

*i. Let $X \subseteq \mathrm{OR}$, and $V \models P = \mathcal{M}_n^\#(X)$. Then also $V[G] \models P = \mathcal{M}_n^\#(X)$.*

*ii. For all sets of ordinals $X \in V[G]$, $V[G] \models \mathcal{M}_n^\#(X)$ exists.*

*iii. Let $V \models$ "$H$ is countable and elementarily embeddable into $V_\Omega$", where $\Omega$ is a large limit ordinal. Let further $\mathbb{Q}$ be a forcing in $H$ and $h \in V$ $\mathbb{Q}$-generic over $H$. Then $H[h]$ is closed under $\mathcal{M}_n^\#(X)$.*

For the proof see [BS09, Lemma 3.7]

**Lemma 1.73.** *Suppose $A_n$ holds. Let $x \in \mathbb{R}$ and suppose that $\mathcal{M}_{n+1}^\#(x)$ does not exist. Let $\mathcal{M}$ be a $(n+1)$-small $x$-premouse with no definably Woodin cardinal. $\mathcal{M}$ is iterable if and only if for every countable substructure $\mathcal{P} \prec \mathcal{M}$, $\mathcal{P}$ is $\omega_1$-iterable.*

The proof for the case below one Woodin is [Ste96, lemma 2.4].

The central theorem we will use in the proof of Theorem 3.17 is the so called $K$-existence dichotomy, taken from [SS, Theorem 1.4.3]:

**Theorem 1.74 (*K*-existence Dichotomy).** *Let $\Omega$ be measurable. Suppose that for all $x \in V_\Omega$, $\mathcal{M}_n^\#(x)$ exists and is $(\omega, \Omega+1)$-iterable. Then exactly one of the following holds:*

*i. for all $x \in V_\Omega$, $\mathcal{M}_{n+1}^\#(x)$ exists and is $(\omega, \Omega+1)$-iterable,*

*ii. for some $x \in V_\Omega$, $K^c(x)$ is $(n+1)$-small, has no Woodin cardinals, and is $(\omega, \Omega+1)$-iterable. (Hence $K(x)$ exists, is $(n+1)$-small, and has no Woodin cardinals.)*

# 2 Increasing $u_2$

This chapter has orginaly been published together with Ralf Schindler in the Journal of Symbolic Logic as [CS09]. In this chapter we modify Jensen's $\mathcal{L}$-forcing (cf. [Jena] and [Jend]) and apply this to the theory of precipitous ideals and the question about the size of $u_2$. Forcings which increase the size of $u_2$ were already presented in the past. After Steel and van Wesep had shown that $u_2 = \omega_2$ is consistent in the presence of large cardinal hypotheses (cf. [SVW82]), Woodin proved that if the nonstationary ideal on $\omega_1$ is $\omega_2$-saturated and $\mathcal{P}(\omega_1)^{\#}$ exists, then $u_2 = \omega_2$ (cf. [Woo99, Theorem 3.17]; in particular, $u_2 = \omega_2$ follows from Martin's Maximum by work of Foreman, Magidor and Shelah, cf. [FMS88].) More recently, Ketchersid, Larson, and Zapletal also constructed forcings which increase $u_2$ (cf. [KLZ07]).

Recall that $\underset{\sim}{\delta}^1_2$ is the supremum of the lengths of all $\underset{\sim}{\Delta}^1_2$ well-orderings of the reals, and that if the reals are closed under sharps, then $u_2$, the second uniform indiscernible, is defined to be the least ordinal above $\omega_1$ which is an $x$-indiscernible for every $x \in \mathbb{R}$. By the Kunen-Martin Theorem (cf. [Mos09, Theorem 2G.2]), if $\leq$ is a $\Delta^1_2(x)$ prewellordering of $\mathbb{R}$, then the length of $\leq$ is less than $\omega_1^{+L[x]}$. Moreover, if $x^{\#}$ exists, then there is a $\Delta^1_2(x^{\#})$-prewellordering of $\mathbb{R}$ of length $\omega_1^{+L[x]}$, which implies $\omega_1^{+L[x]} < \underset{\sim}{\delta}^1_2$. Also, $\omega_1^{+L[x^{\#}]} < u_2^x$, the least $x$-indiscernible above $\omega_1$. Therefore, if the reals are closed under sharps, then

$$u_2 = \sup\left\{\omega_1^{+L[x]}; x \in \mathbb{R}\right\} = \underset{\sim}{\delta}^1_2.$$

In this chapter we'll consider generic iterations of structures of the form $\langle M; \in, I\rangle$, where $M$ is a transitive model of $\mathsf{ZFC}^*$ + "$\omega_1$ exists" and inside $M$, $I$ is a uniform and normal ideal on $\omega_1^M$. Here, $\mathsf{ZFC}^*$ is a reasonable weak fragment of $\mathsf{ZFC}$ such that $\mathsf{ZFC}^*$ + "$\omega_1$ exists" is suitable for taking generic ultrapowers by ideals on $\omega_1$ (cf. [Woo99]).

## 2.1 The forcing

We may now state and prove our main result.

**Theorem 2.1.** *Let $I$ be a precipitous ideal on $\omega_1$, and let $\theta > \omega_1$ be a regular cardinal. There is a poset $\mathbb{P}(I,\theta)$, preserving the stationarity of all sets in $I^+$, such that if $G$ is $\mathbb{P}(I,\theta)$-generic over $V$, then in $V[G]$ there is a generic iteration*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

*such that if $i < \omega_1$, then $M_i$ is countable and $M_{\omega_1} = \langle H_\theta; \in, I\rangle$. If $I = \mathsf{NS}_{\omega_1}$, then $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ is stationary set preserving.*

It is easy to see that every set in $I^+$ has to be stationary in $V$. The most difficult part of the construction is to arrange that every set in $I^+$ will remain stationary in the forcing extension.

### 2.1.1 The definition of $\mathbb{P}(I, \theta)$

The proof of Theorem 2.1 stretches over several lemmas and builds upon Jensen's [Jena] and [Jend]. Fixing $I$ and $\theta$, let us pick a regular cardinal $\rho$ such that $2^{2^{<\theta}} < \rho$. Therefore, $H_\theta \in H_\rho$, and in fact every subset of $\mathcal{P}(H_\theta)$ is in $H_\rho$ as well. In particular, the forcing $\mathbb{P}(I, \theta)$ we are about to define will be an element of $H_\rho$. It is easy to verify that if a forcing $\mathbb{Q} \in V$ is $\omega_1$-distributive, then $I$ is still precipitous in $V^{\mathbb{Q}}$. We may and shall therefore assume that $2^{<\theta} = \theta$ and $2^{<\rho} = \rho$, i.e., that $\mathrm{card}(H_\theta) = \theta$ and $\mathrm{card}(H_\rho) = \rho$, because if this were not true in $V$, then we may first force with $\mathbb{Q} = \mathrm{col}(\rho, \rho) \times \mathrm{col}(\theta, \theta)$ and work with $V^{\mathbb{Q}}$ rather than $V$ as our ground model in what follows.

Our starting point is thus that in $V$, $I$ is a precipitous ideal on $\omega_1$ and $\theta$ and $\rho$ are regular cardinals such that $\omega_2 \le \theta = 2^{<\theta} < 2^\theta < \rho = 2^{<\rho}$. Let us fix a well-order, denoted by $<$, of $H_\rho$ of order type $\rho$ such that $<\restriction H_\theta$ is an initial segment of $<$ of order type $\theta$. (In what follows, we shall also write $<$ for $<\restriction H_\theta$.) We shall write

$$\mathcal{H} = \langle H_\rho; \in, H_\theta, I, < \rangle,$$

and we shall also write

$$\mathcal{M} = \langle H_\theta; \in, I, < \rangle.$$

In what follows, models will always be models of the language of set theory. We shall tacitly assume that if $\mathfrak{A}$ is a model, then the well-founded part $\mathrm{wfp}(\mathfrak{A})$ of $\mathfrak{A}$ is transitive.

Let us now define our forcing $\mathbb{P}(I, \theta)$.

**Definition 2.2.** Conditions $p$ in $\mathbb{P}(I, \theta)$ are triples

$$p = \langle \langle \kappa_i^p; i \in \mathrm{dom}(p) \rangle, \langle \pi_i^p; i \in \mathrm{dom}(p) \rangle, \langle \tau_i^p; i \in \mathrm{dom}_-(p) \rangle \rangle$$

such that the following hold true.

i. Both $\mathrm{dom}(p)$ and $\mathrm{dom}_-(p)$ are finite, and $\mathrm{dom}_-(p) \subseteq \mathrm{dom}(p) \subseteq \omega_1$.

ii. $\langle \kappa_i^p; i \in \mathrm{dom}(p) \rangle$ is a sequence of countable ordinals.

iii. $\langle \pi_i^p; i \in \mathrm{dom}(p) \rangle$ is a sequence of finite partial maps from $\omega_1$ to $\theta$.

iv. $\langle \tau_i^p; i \in \mathrm{dom}_-(p) \rangle$ is a sequence of complete $\mathcal{H}$-types over $H_\theta$, i.e., for each $i \in \mathrm{dom}_-(p)$ there is some $x \in H_\rho$ such that, having $\varphi$ range over $\mathcal{H}$-formulae with free variables $u, \vec{v}$,

$$\tau_i^p = \{ \langle \ulcorner \varphi \urcorner, \vec{z} \rangle \ ; \ \vec{z} \in H_\theta \wedge \mathcal{H} \models \varphi[x, \vec{z}] \}.$$

v. If $i$, $j \in \mathrm{dom}_-(p)$, where $i < j$, then there is some $n < \omega$ and some $\vec{u} \in \mathrm{ran}(\pi_j^p)$ such that

$$\tau_i^p = \{ (m, \vec{z}) \ ; \ (n, \vec{u}^\frown m^\frown \vec{z}) \in \tau_j^p \}.$$

vi. In $V^{\mathrm{col}(\omega,2^\theta)}$, there is a *model which certifies p with respect to* $\mathcal{M}$, by which we mean a model $\mathfrak{A}$ such that $\theta+1 \subset \mathrm{wfp}(\mathfrak{A})$, in fact $H_{\theta^+} \in \mathfrak{A}$, $\mathfrak{A} \models \mathsf{ZFC}^-$ ($= \mathsf{ZFC} \setminus$ Power Set), for all $S \in I^+$, $\mathfrak{A} \models$ "$S$ is stationary," and inside $\mathfrak{A}$, there is a generic iteration

$$\langle\langle M_i^{\mathfrak{A}}, \pi_{i,j}^{\mathfrak{A}}, I_i^{\mathfrak{A}}, \kappa_i^{\mathfrak{A}}; i \leqslant j \leqslant \omega_1\rangle, \langle G_i^{\mathfrak{A}}; i < \omega_1\rangle\rangle$$

such that

a) if $i < \omega_1$, then $M_i^{\mathfrak{A}}$ is countable,

b) if $i < \omega_1$ and if $\xi < \theta$ is definable over $\mathcal{M}$ from parameters in $\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$, then $\xi \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$,

c) $M_{\omega_1}^{\mathfrak{A}} = \langle H_\theta; \in, I\rangle$,

d) if $i \in \mathrm{dom}(p)$, then $\kappa_i^p = \kappa_i^{\mathfrak{A}}$ and $\pi_i^p \subseteq \pi_{i,\omega_1}^{\mathfrak{A}}$,

e) if $i \in \mathrm{dom}_-(p)$, then for all $n < \omega$ and for all $\vec{z} \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$,

$$\exists y \in H_\theta\ (n, y^\frown\vec{z}) \in \tau_i^p \Longrightarrow \exists y \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})\ (n, y^\frown\vec{z}) \in \tau_i^p.$$

If $p$, $q \in \mathbb{P}$, then we write $p \leqslant q$ iff $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$, $\mathrm{dom}_-(q) \subseteq \mathrm{dom}_-(p)$, for all $i \in \mathrm{dom}(q)$, $\kappa_i^p = \kappa_i^q$ and $\pi_i^q \subseteq \pi_i^p$, and for all $i \in \mathrm{dom}_-(q)$, $\tau_i^q = \tau_i^p$.

Conditions $p$ should be seen as finite attempts to describe the iteration leading to $\langle H_\theta; \in, I\rangle$, the first component being finitely many critical points $\kappa_i^p$ of the iteration, and the second component being finite attempts $\pi_i^p$ to describe the iteration maps restricted to the ordinals. The presence of $<$ will guarantee that knowing the action of these maps on the ordinals means knowing the maps themselves. The third components $\tau_i^p$ will guarantee that the iteration maps extend to elementary maps into $\mathcal{H}$ with some $x \in H_\rho$ of interest in their range (cf. Lemma 2.14 below), which will be relevant in the verification that $\mathbb{P}(I,\theta)$ preserves the stationarity of all sets in $I^+$.

It should be stressed that $\omega_1^V \in I^+$, so that if $\mathfrak{A}$ certifies any condition $p$ with respect to $\mathcal{M}$, then $\omega_1^{\mathfrak{A}} = \omega_1^V$. It is also clear that

$$\mathfrak{A} \models \mathrm{card}(H_\theta) = \aleph_1.$$

## 2.1.2 Some basic considerations

Let us start the discussion of $\mathbb{P}(I,\theta)$. Let us write $\mathbb{P} = \mathbb{P}(I,\theta)$ from now on.

**Lemma 2.3.** *$\mathbb{P} \neq \varnothing$.*

Proof. We need to verify that in $V^{\mathrm{col}(\omega,2^\theta)}$ there is a model which certifies the trivial condition $\langle\langle\rangle,\langle\rangle,\langle\rangle\rangle$ with respect to $\mathcal{M}$.

Let $g$ be $\mathrm{col}(\omega,<\rho)$-generic over $V$. Notice that inside $V[g]$, $\langle V; \in, I\rangle$ is generically $\rho+1$ iterable by Lemma 1.14. Let us work inside $V[g]$ until further notice.

Let us choose a bijection $\varphi : [\rho]^{<\rho} \to \rho$, and let $\langle S_\nu ; \nu < \rho \rangle$ be a partition of $\rho$ into pairwise disjoint stationary subsets of $\rho$. Define $f : \rho \to [\rho]^{<\rho}$ by

$$f(i) = s \iff i \in S_{\varphi(s)}.$$

In other words, $f"S_{\varphi(s)} = \{s\}$ for every $s \in [\rho]^{<\rho}$.

Let us recursively construct a generic iteration

$$\langle \langle M_i, \pi_{i,j}, I_i, \kappa_i ; i \leqslant j \leqslant \rho \rangle, \langle G_i ; i < \rho \rangle \rangle$$

of $M_0 = \langle H_\theta ; \in, I \rangle$. Suppose $\langle \langle M_k, \pi_{k,j}, I_k, \kappa_k ; k \leqslant j \leqslant i \rangle, \langle G_k ; k < i \rangle \rangle$ has already been constructed, where $i < \rho$. If there is a (unique) $j \leq i$ such that $f(i) \in I_j^+$, i.e., $\pi_{j,i}(f(i)) \in I_i^+$, then let us choose $G_i$ such that $\pi_{j,i}(f(i)) \in G_i$. If there is no such $j \leq i$, then we choose $G_i$ arbitrarily. This defines the generic iteration.

Now let $S \in I_\rho^+$. Let $j < \rho$ and $s \in M_j$ be such that $\pi_{j,\rho}(s) = S$. Whenever $j \leq i < \rho$ and $f(i) = s$, then $\pi_{j,i}(s) \in G_i$, i.e., $\kappa_i \in \pi_{i,i+1}(\pi_{j,i}(s)) = \pi_{j,i+1}(s) \subseteq \pi_{j,\rho}(s) = S$. This shows that

$$S_{\varphi(s)} \setminus j \subseteq \{ i < \rho \; ; \; \kappa_i \in S \},$$

so that $S$ is in fact stationary.

The map $\pi_{0,\rho} : H_\theta \to M_\rho$ admits a canonical extension $\pi : V \to N$, where $N$ is transitive and $\pi(H_\theta) = M_\rho$. Let us now leave $V[g]$ and pick some $h$ which is $\mathrm{col}(\omega, \pi(2^\theta))$-generic over $V[g]$. Of course, $h$ is also $\mathrm{col}(\omega, \pi(2^\theta))$-generic over $N$. Let $x \in \mathbb{R} \cap N[h]$ code $\pi((H_{\theta^+})^V)$ in a natural way. The existence of a model which certifies $\langle \langle \rangle, \langle \rangle, \langle \rangle \rangle$ with respect to $\pi(\mathcal{M})$ is then easily seen to be a $\Sigma^1_1(x)$ statement which holds true in $V[g,h]$, as being witnessed by $V[g]$. By absoluteness, this statement is then also true in $N[h]$. That is, inside $N^{\mathrm{col}(\omega, \pi(2^\theta))}$ there is a model which certifies $\langle \langle \rangle, \langle \rangle, \langle \rangle \rangle$ with respect to $\pi(\mathcal{M})$. By elementarity, in $V^{\mathrm{col}(\omega, 2^\theta)}$ there is therefore a model which certifies $\langle \langle \rangle, \langle \rangle, \langle \rangle \rangle$ with respect to $\mathcal{M}$. $\dashv$

We will now prove some lemmata which will make sure that the generic filter indeed produces a generic iteration leading to $\langle H_\theta ; \in, I \rangle$. If $p \in \mathbb{P}$, then from now on we shall often just say that $\mathfrak{A}$ *certifies* $p$ to express that $\mathfrak{A}$ is a model which certifies $p$ with respect to $\mathcal{M}$.

**Lemma 2.4.** *Let $p \in \mathbb{P}$, let $u$ be finite such that $\mathrm{dom}(p) \subseteq u \subseteq \omega_1$. There is $p' \leqslant p$ such that $u \subseteq \mathrm{dom}(p')$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{col}(\omega, 2^\theta)}$ certify $p$. We may define $p'$ such that $\mathrm{dom}(p') = u$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa_i^{p'} = \kappa_i^{\mathfrak{A}}$ for $i \in u$, $\pi_i^{p'} = \pi_i^p$ for $i \in \mathrm{dom}(p)$, $\pi_i^{p'} = \varnothing$ for $i \in \mathrm{dom}(p') \setminus \mathrm{dom}(p)$, and $\tau_i^{p'} = \tau_i^p$ for $i \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies $p'$, and of course $p' \leqslant p$. $\dashv$

**Lemma 2.5.** *Let $p \in \mathbb{P}$, $i \in \mathrm{dom}(p)$ and $\xi < \theta$. There is a $p' \leqslant p$ and an $\alpha \in \mathrm{dom}(\pi_i^{p'})$ such that $\xi < \pi_i^{p'}(\alpha)$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{col}(\omega,2^\theta)}$ certify $p$. Let $\alpha$ be such that $\pi^{\mathfrak{A}}_{i,\omega_1}(\alpha) > \xi$. (Such an $\alpha$ exists, as the iteration map $\pi^{\mathfrak{A}}_{i,\omega_1}$ is cofinal.) We may define $p'$ such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa^{p'}_j = \kappa^p_j$ for $j \in \mathrm{dom}(p)$, $\pi^{p'}_j = \pi^p_j$ for $j \in \mathrm{dom}(p) \setminus \{i\}$, $\pi^{p'}_i = \pi^p_i \cup \{\langle \alpha, \pi^{\mathfrak{A}}_{i,\omega_1}(\alpha)\rangle\}$, and $\tau^{p'}_j = \tau^p_j$ for $j \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies $p'$, and of course $p' \leqslant p$. ⊣

**Lemma 2.6.** *Let $p \in \mathbb{P}$, $i \in \mathrm{dom}(p)$, $\xi < \zeta$ and $\zeta \in \mathrm{dom}(\pi^p_i)$. There is a $p' \leqslant p$ such that $\xi \in \mathrm{dom}(\pi^{p'}_i)$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{col}(\omega,2^\theta)}$ certify $p$. We may define $p'$ such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa^{p'}_j = \kappa^p_j$ for $j \in \mathrm{dom}(p)$, $\pi^{p'}_j = \pi^p_j$ for $j \in \mathrm{dom}(p) \setminus \{i\}$, $\pi^{p'}_i = \pi^p_i \cup \{\langle \xi, \pi^{\mathfrak{A}}_{i,\omega_1}(\xi)\rangle\}$, and $\tau^{p'}_j = \tau^p_j$ for $j \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies $p'$, and of course $p' \leqslant p$. ⊣

**Lemma 2.7.** *Let $p \in \mathbb{P}$ and $\xi \in H_\theta$. There is a $p' \leqslant p$ such that $\xi \in \mathrm{ran}(\pi^{p'}_i)$ for some $i \in \mathrm{dom}(p')$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{col}(\omega,2^\theta)}$ certify $p$. Let $i < \omega_1$, $i \notin \mathrm{dom}(p)$, and $\bar{\xi}$ be such that $\pi^{\mathfrak{A}}_{i,\omega_1}(\bar{\xi}) = \xi$. We may define $p'$ such that $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{i\}$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa^{p'}_j = \kappa^{\mathfrak{A}}_j$ for $j \in \mathrm{dom}(p')$, $\kappa^{p'}_i = \kappa^{\mathfrak{A}}_i$, $\pi^{p'}_j = \pi^p_j$ for $j \in \mathrm{dom}(p) \setminus \{i\}$, $\pi^{p'}_i = \{\langle \bar{\xi}, \xi\rangle\}$, and $\tau^{p'}_j = \tau^p_j$ for $j \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies $p'$, and of course $p' \leqslant p$. ⊣

**Lemma 2.8.** *Let $p \in \mathbb{P}$, $i \in \mathrm{dom}(p)$, $j \in \mathrm{dom}(p)$, $i < j$, $\xi \in \mathrm{ran}(\pi^p_i)$. There is a $p' \leqslant p$ such that $\xi \in \mathrm{ran}(\pi^{p'}_j)$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{col}(\omega,2^\theta)}$ certify $p$. Let $\bar{\xi}$ be such that $\pi^{\mathfrak{A}}_{j,\omega_1}(\bar{\xi}) = \xi$. We may define $p'$ such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa^{p'}_k = \kappa^{\mathfrak{A}}_k$ for $k \in \mathrm{dom}(p)$, $\pi^{p'}_k = \pi^p_k$ for $k \in \mathrm{dom}(p) \setminus \{j\}$, $\pi^{p'}_j = \pi^p_j \cup \{\langle \bar{\xi}, \xi\rangle\}$, and $\tau^{p'}_k = \tau^p_k$ for $k \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies $p'$, and of course $p' \leqslant p$. ⊣

**Lemma 2.9.** *Let $p \in \mathbb{P}$, $i, i+1 \in \mathrm{dom}(p)$. Let $\xi \in \mathrm{ran}(\pi^p_{i+1})$. There is some $p' \leqslant p$ such that $\xi$ is definable over $\mathcal{M}$ from parameters in $\mathrm{ran}(\pi^{p'}_i) \cup \{\kappa^p_i\}$.*

Proof. Let $\mathfrak{A} \in V^{\mathrm{col}(\omega,2^\theta)}$ certify $p$. Since $M^{\mathfrak{A}}_{i+1} = \mathrm{Ult}(M^{\mathfrak{A}}_i, G^{\mathfrak{A}}_i)$ there is an $f\colon \kappa^p_i = \omega_1^{M^{\mathfrak{A}}_i} \to M^{\mathfrak{A}}_i$, $f \in M^{\mathfrak{A}}_i$ such that $(\pi^p_{i+1})^{-1}(\xi) = \pi^{\mathfrak{A}}_{i,i+1}(f)(\kappa^p_i)$, i.e., $\xi = \pi^{\mathfrak{A}}_{i,\omega_1}(f)(\kappa^p_i)$. Due to the presence of $<$ in $\mathcal{M}$, the function $\pi^{\mathfrak{A}}_{i,\omega_1}(f)$ is definable over $\mathcal{M}$ in some ordinal parameter $\lambda < \theta$. Let $\bar{\lambda}$ be such that $\lambda = \pi^{\mathfrak{A}}_{i,\omega_1}(\bar{\lambda})$. We may define $p'$ such that $\mathrm{dom}(p') = \mathrm{dom}(p)$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa^{p'}_j = \kappa^{\mathfrak{A}}_j$ for $j \in \mathrm{dom}(p')$, $\pi^{p'}_j = \pi^p_j$ for $j \in \mathrm{dom}(p) \setminus \{i\}$,

$$\pi^{p'}_i = \pi^p_i \cup \left\{\langle \bar{\lambda}, \lambda\rangle\right\},$$

and $\tau^{p'}_i = \tau^p_i$ for $i \in \mathrm{dom}_-(p')$. Then $\mathfrak{A}$ also certifies $p'$, and of course $p' \leqslant p$. ⊣

**Lemma 2.10.** *Let $p \in \mathbb{P}$, and let $\lambda \in \operatorname{dom}(p)$ be a limit ordinal. If $\xi \in \operatorname{ran}(\pi_\lambda^p)$, then there is some $p' \leq p$ and some $i < \lambda$ with $i \in \operatorname{dom}(p')$ such that $\xi \in \operatorname{ran}(\pi_i^{p'})$.*

Proof. Let $\mathfrak{A} \in V^{\operatorname{col}(\omega,2^\theta)}$ certify $p$. Because $\operatorname{ran}(\pi^{\mathfrak{A}}_{\lambda,\omega_1}) = \bigcup_{i<\lambda} \operatorname{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$, there is some $i < \lambda$ such that $\xi \in \operatorname{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$. Let us without loss of generality assume that $i \in \operatorname{dom}(p)$. Let $\bar{\xi}$ be such that $\pi^{\mathfrak{A}}_{i,\omega_1}(\bar{\xi}) = \xi$. We may then define $p'$ such that $\operatorname{dom}(p') = \operatorname{dom}(p)$, $\operatorname{dom}_-(p') = \operatorname{dom}_-(p)$, $\kappa_j^{p'} = \kappa_j^p$ for $j \in \operatorname{dom}(p)$, $\pi_j^{p'} = \pi_j^p$ for $j \in \operatorname{dom}(p) \setminus \{i\}$, $\pi_i^{p'} = \pi_i^p \cup \{\langle(\bar{\xi}, \xi\rangle\}$, and $\tau_i^{p'} = \tau_i^p$ for $i \in \operatorname{dom}_-(p')$. Then $\mathfrak{A}$ also certifies $p'$, and of course $p' \leqslant p$. ⊣

**Lemma 2.11.** *Let $p \in \mathbb{P}$, $i \in \operatorname{dom}(p)$ and let $\xi$ be definable over $\mathcal{M}$ from parameters in $\operatorname{ran}(\pi_i^p)$. There is a $p' \leqslant p$ such that $\xi \in \operatorname{ran}(\pi_i^{p'})$.*

Proof. Let $\mathfrak{A} \in V^{\operatorname{col}(\omega,2^\theta)}$ certify $p$. We must have that $\xi \in \operatorname{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$, as $\mathfrak{A}$ certifies $p$ (cf. condition (b)). Let $\pi^{\mathfrak{A}}_{i,\omega_1}(\bar{\xi}) = \xi$. We may define $p'$ such that $\operatorname{dom}(p') = \operatorname{dom}(p)$, $\operatorname{dom}_-(p') = \operatorname{dom}_-(p)$, $\kappa_j^{p'} = \kappa_j^p$ for $j \in \operatorname{dom}(p)$, $\pi_j^{p'} = \pi_j^p$ for $j \in \operatorname{dom}(p) \setminus \{i\}$, $\pi_i^{p'} = \pi_i^p \cup \{\langle\bar{\xi}, \xi\rangle\}$, and $\tau_j^{p'} = \tau_j^p$ for $j \in \operatorname{dom}_-(p')$. Then $\mathfrak{A}$ also certifies $p'$, and of course $p' \leqslant p$. ⊣

**Lemma 2.12.** *Let $p \in \mathbb{P}$, let $i \in \operatorname{dom}(p)$, and suppose that $D \in H_\theta$ is definable over $\mathcal{M}$ from parameters in $\operatorname{ran}(\pi_i^p)$. Suppose also that*

$$\mathcal{M} \models \text{“}D \text{ is dense in the partial order } \langle I^+, \leq_I\rangle.\text{”}$$

*Then there is some $p' \leq p$ and some $X \in D$ which is definable over $\mathcal{M}$ from parameters in $\operatorname{ran}(\pi_i^{p'})$ such that $\kappa_i^p \in X$.*

Proof. Let $\mathfrak{A} \in V^{\operatorname{col}(\omega,2^\theta)}$ certify $p$. Let $\bar{D} \in M_i^{\mathfrak{A}}$ be such that $\pi^{\mathfrak{A}}_{i,\omega_1}(\bar{D}) = D$. As $G_i^{\mathfrak{A}}$ is $\langle (I_i^{\mathfrak{A}})^+, \leq_{I_i^{\mathfrak{A}}}\rangle$-generic over $M_i^{\mathfrak{A}}$, $\bar{D} \cap G_i^{\mathfrak{A}} \neq \varnothing$. There is thus some $\bar{X} \in \bar{D}$ such that $\kappa_i^p = \kappa_i^{\mathfrak{A}} \in \pi^{\mathfrak{A}}_{i,i+1}(\bar{X}) \subset \pi^{\mathfrak{A}}_{i,\omega_1}(\bar{X})$. Let $X = \pi^{\mathfrak{A}}_{i,\omega_1}(\bar{X})$. Then $X \in D$ and $\kappa_i^p \in X$. Due to the presence of $<$ in $\mathcal{M}$, there is some $\lambda < \theta$ such that $X$ is definable over $\mathcal{M}$ from the parameter $\lambda$. Let $\bar{\lambda}$ be such that $\lambda = \pi^{\mathfrak{A}}_{i,\omega_1}(\bar{\lambda})$. We may define $p'$ such that $\operatorname{dom}(p') = \operatorname{dom}(p)$, $\operatorname{dom}_-(p') = \operatorname{dom}_-(p)$, $\kappa_j^{p'} = \kappa_j^{\mathfrak{A}}$ for $j \in \operatorname{dom}(p')$, $\pi_j^{p'} = \pi_j^p$ for $j \in \operatorname{dom}(p) \setminus \{i\}$,

$$\pi_i^{p'} = \pi_i^p \cup \left\{\langle\bar{\lambda}, \lambda\rangle\right\},$$

and $\tau_i^{p'} = \tau_i^p$ for $i \in \operatorname{dom}_-(p')$. Then $\mathfrak{A}$ also certifies $p'$, and of course $p' \leqslant p$. ⊣

Now let $G$ be $\mathbb{P}$-generic over $V$. Set

$$\kappa_i = \kappa_i^p \text{ for some (all) } p \in G \text{ with } i \in \operatorname{dom}(p),$$

$$\pi_i = \bigcup\{\pi_i^p ; p \in G \wedge i \in \operatorname{dom}(p)\}, \text{ and}$$

$$\beta_i = \operatorname{dom}(\pi_i).$$

By Lemmas 2.4, 2.5, and 2.6, $\pi_i\colon\beta_i\to\theta$ is a well-defined cofinal order preserving map, and by Lemma 2.7, $\theta=\bigcup\{\mathrm{ran}(\pi_i); i<\omega_1\}$. For $i<\omega_1$, let $X_i$ be the smallest $X\prec\mathcal{M}$ such that $\mathrm{ran}(\pi_i)\subseteq X$. By Lemma 2.11, $\mathrm{ran}(\pi_i)=X_i\cap\theta$. Let $\tilde\pi_i\colon M_i\cong X_i\prec\mathcal{M}$ be the uncollapsing map, so that $\tilde\pi_i\supset\pi_i$. For $i\le j\le\omega_1$, let $\tilde\pi_{i,j}=\tilde\pi_j^{-1}\circ\tilde\pi_i$. We then have that $\tilde\pi_{i,j}\colon M_i\to M_j$ is then well-defined by Lemma 2.8. For $i\le\omega_1$, let $I_i=\tilde\pi_i^{-1}(I)$ and $\kappa_i=\tilde\pi_i^{-1}(\omega_1)$, and for $i<\omega_1$, let

$$G_i=\{X\in\mathcal{P}(\kappa_i)\cap M_i\ ;\ \kappa_i\in\tilde\pi_{i,i+1}(X)\}.$$

Using Lemmas 2.9, 2.10, and 2.12, we then have the following.

**Lemma 2.13.** *$\langle\langle M_i,\tilde\pi_{i,j},I_i,\kappa_i\ ;\ i\leqslant j\leqslant\omega_1^V\rangle,\langle G_i\ ;\ i<\omega_1\rangle\rangle$ is a generic iteration of $M_0$ such that if $i<\omega_1$, then $M_i$ is countable, and $M_{\omega_1}=\langle H_\theta;\in,I\rangle$.*

Let us now discuss the third component of a condition $p\in\mathbb{P}$.

**Lemma 2.14.** *Suppose that $\mathfrak{A}$ is a model. Let $p\in\mathbb{P}$ and $i\in\mathrm{dom}(p)$. Let $x\in H_\rho$ be such that $\tau_i^p$ is the complete $\mathcal{H}$-type of $x$ over $H_\theta$, i.e., having $\varphi$ range over $\mathcal{H}$-formulae with free variables $u,\vec v$,*

$$\tau_i^p=\{\langle\ulcorner\varphi\urcorner,\vec z\rangle\ ;\ \vec z\in H_\theta\wedge\mathcal{H}\models\varphi[x,\vec z]\}\,.$$

*Then the following are equivalent.*

*i. $\mathfrak{A}$ certifies $p$ with respect to $\mathcal{M}$.*

*ii. $\theta+1\subset\mathrm{wfp}(\mathfrak{A})$, $H_{\theta^+}\in\mathfrak{A}$, $\mathfrak{A}\models\mathsf{ZFC}^-$, for all $S\in I^+$, $\mathfrak{A}\models$ "$S$ is stationary," and inside $\mathfrak{A}$, there is a generic iteration*

$$\langle\langle M_i^{\mathfrak{A}},\pi_{i,j}^{\mathfrak{A}},I_i^{\mathfrak{A}},\kappa_i^{\mathfrak{A}};i\leqslant j\leqslant\omega_1\rangle,\langle G_i^{\mathfrak{A}};i<\omega_1\rangle\rangle$$

*such that if $i<\omega_1$, then $M_i^{\mathfrak{A}}$ is countable, $M_{\omega_1}^{\mathfrak{A}}=\langle H_\theta;\in,I\rangle$, if $i\in\mathrm{dom}(p)$, then $\kappa_i^p=\kappa_i^{\mathfrak{A}}$ and $\pi_i^p\subseteq\pi_{i,\omega_1}^{\mathfrak{A}}$, and if $i\in\mathrm{dom}_-(p)$, then one of the following equivalent conditions holds.*

*a)*

$$\mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})\cup\{x\})\cap H_\theta=\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}}).$$

*b) The map $\pi_{i,\omega_1}^{\mathfrak{A}}\colon M_i\to\mathcal{M}$*

*extends to some elementary map $\tilde\pi\colon H\to\mathcal{H}$ with $\tilde\pi(M_i)=\langle H_\theta;\in,I\rangle$, $\tilde\pi\upharpoonright M_i=\pi_{i,\omega_1}^{\mathfrak{A}}$, and $x\in\mathrm{ran}(\tilde\pi)$.*

*c) $\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})\prec\langle H_\theta;\in,I,<,\tau_i^p\rangle$.*

Proof. i. $\Rightarrow$ ii.(a): Let $y\in\mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})\cup\{x\})\cap H_\theta$. Then $y$ is definable over $\mathcal{H}$ from parameters $\vec z$, $x$ in $\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})\cup\{x\}$. For some $n<\omega$, we then have that $y$ is unique with $(n,y^\frown\vec z)\in\tau_i^p$. As $\mathfrak{A}$ certifies $p$ (cf. condition vi.(e) in Definition 2.2), we then get that in fact $y\in\mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$.

ii.(a) $\Rightarrow$ ii.(b): Let $\tilde{\pi}: H \cong \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1}) \cup \{x\}) \prec \mathcal{H}$, where $H$ is transitive. It is obvious that this map works.

ii.(b) $\Rightarrow$ ii.(a): As $x \in \mathrm{ran}(\tilde{\pi})$ and $\tilde{\pi} \supset \pi^{\mathfrak{A}}_{i,\omega_1}$, $\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1}) \subset \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1}) \cup \{x\}) \cap H_\theta \subset \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\tilde{\pi})) \cap H_\theta = \mathrm{ran}(\tilde{\pi}) \cap H_\theta = \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$.

ii.(a) $\Rightarrow$ ii.(c): We need to show that if $\vec{z} \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$ and $\varphi$ is a formula (of the language associated with $\langle H_\theta; \in, I, <, \tau^p_i\rangle$) such that

$$\langle H_\theta; \in, I, <, \tau^p_i\rangle \vDash \exists v \varphi(v, \vec{z}), \tag{2.1}$$

then there is some $u \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$ with

$$\langle H_\theta; \in, I, <, \tau^p_i\rangle \vDash \varphi(u, \vec{z}).$$

There is some recursive $\ulcorner\psi\urcorner \mapsto \ulcorner\psi^*\urcorner$ (assigning to each formula of the language associated with $\langle H_\theta; \in, I, <, \tau^p_i\rangle$ a formula of the language associated with $\langle H_\rho; \in, H_\theta, I, <, x\rangle$) such that for all $\vec{w} \in H_\theta$,

$$\langle H_\theta; \in, I, <, \tau^p_i\rangle \vDash \psi(\vec{w})$$

iff

$$\langle H_\rho; \in, H_\theta, I, <, x\rangle \vDash \psi^*(\vec{w}).$$

Hence if (2.1) holds, then there is some $u \in H_\theta$ such that

$$\langle H_\rho; \in, H_\theta, I, <, x\rangle \vDash \varphi^*(u, \vec{z}).$$

There is then also some such $u \in H_\theta$ which is in $\mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1}) \cup \{x\})$, so that $u \in \mathrm{ran}(\pi_{i,\omega_1})^{\mathfrak{A}}$ by ii.(a). But then

$$\langle H_\theta; \in, I, <, \tau^p_i\rangle \vDash \varphi(u, \vec{z}),$$

where $u \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$.

ii.(c) $\Rightarrow$ i.: Let $n < \omega$ and $\vec{z} \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$. Suppose there to be some $y \in H_\theta$ such that $(n, y^\frown\vec{z}) \in \tau^p_i$. Then

$$\langle H_\theta; \in, I, <, \tau^p_i\rangle \vDash \exists y (n, y^\frown\vec{z}) \in \tau^p_i,$$

so that there is some $y \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$ with

$$\langle H_\theta; \in, I, <, \tau^p_i\rangle \vDash (n, y^\frown\vec{z}) \in \tau^p_i,$$

as needed for condition vi.(e) in Definition 2.2. $\dashv$

It is easy to see that if $X \in I$ and $X \in \mathrm{ran}(\tilde{\pi}_{i,\omega_1})$, where $i < \omega_1$, then $\{\kappa_j; i \leq j < \omega_1\} \subset \omega_1 \setminus X$. This means that no set in $I$ will be stationary in $V^{\mathbb{P}}$.

### 2.1.3 $\mathbb{P}(I,\theta)$ is stationary set preserving

**Lemma 2.15.** *If $S \in I^+$, then $S$ is stationary in $V^{\mathbb{P}}$.*

Proof. Let $S \in I^+$, and let $p \in \mathbb{P}$ and $\dot{C}$ be such that $p \Vdash \dot{C}$ is club in $\check{\omega_1}$. We need to see that there is some $p' \leqslant p$ and some $\alpha < \omega_1$ such that $p' \Vdash \check{\alpha} \in \dot{C} \cap \check{S}$.

Let

$$R = \{(r,\delta); r \in \mathbb{P}, \delta < \omega_1, \text{ and } r \Vdash_{\mathbb{P}} \check{\delta} \in \dot{C}\}.$$

Notice that $p, R, \leq_{\mathbb{P}} \in H_\rho$. Let $\tau$ the the complete $\mathcal{H}$-type of $\langle p, R, \leq_{\mathbb{P}}\rangle$ over $H_\theta$. Let $\mathfrak{A} \in V^{\mathrm{col}(\omega,2^\theta)}$ certify $p$ with respect to $\mathcal{M}$. Recall that $H_\theta \in \mathfrak{A}$ and $\omega_1^{\mathfrak{A}} = \omega_1^V$. We have that $\langle \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1}); i < \omega_1\rangle$ is a continuous tower of countable substructures of $H_\theta$ with $\bigcup\{\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1}); i < \omega_1\} = H_\theta$. Since $S$ is stationary in $\mathfrak{A}$, $H_{\theta^+} \in \mathfrak{A}$ and thus $\tau \in \mathfrak{A}$, we may therefore pick an $\alpha < \omega_1$ such that

i. $\kappa^{\mathfrak{A}}_\alpha = \alpha$ and $\mathrm{dom}(p) \subseteq \alpha$,

ii. $\mathrm{ran}(\pi^{\mathfrak{A}}_{\alpha,\omega_1}) \prec \langle H_\theta; \in, I, <, \tau\rangle$, and

iii. $\alpha \in S$.

We may define $p'$ such that $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{\alpha\}$, $\mathrm{dom}_-(p') = \mathrm{dom}(p)_- \cup \{\alpha\}$, $\kappa^{p'}_i = \kappa^p_i$ for all $i \in \mathrm{dom}(p)$, $\kappa^{p'}_\alpha = \alpha$, $\pi^{p'}_i = \pi^p_i$ for all $i \in \mathrm{dom}(p)$, $\pi^{p'}_\alpha = \varnothing$, $\tau^{p'}_i = \tau^p_i$ for all $i \in \mathrm{dom}_-(p)$, and $\tau^{p'}_\alpha = \tau$. Using Lemma 2.14, we see that $\mathfrak{A}$ still certifies $p'$ by the above choice of $\alpha$. Also, notice that if $i \in \mathrm{dom}_-(p)$, then $\tau^p_i$ is (trivially) definable over $\mathcal{H}$ from the parameter $p$, so that because $\tau$ is the complete $\mathcal{H}$-type of $\langle p, R, \leq_{\mathbb{P}}\rangle$ over $H_\theta$, we get that there is an $n < \omega$ such that

$$\tau^p_i = \{(m,\vec{z})\ ;\ (n, m^\frown\vec{z}) \in \tau\}.$$

We thus have $p' \in \mathbb{P}$, and of course $p' \leqslant p$.

We claim that $p' \Vdash \check{\alpha} \in \dot{C} \cap \check{S}$. Suppose not. Then $p'$ does not force $\dot{C} \cap \check{\alpha}$ to be unbounded in $\check{\alpha}$. Pick $q \leqslant p'$ and $\xi < \alpha$ such that

$$q \Vdash \sup(\dot{C} \cap \check{\alpha}) = \check{\xi}. \tag{2.2}$$

Let the model $\mathfrak{B}$ certify $q$ with respect to $\mathcal{M}$. By Lemma 2.14,

$$\mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\}) \cap H_\theta = \mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}). \tag{2.3}$$

Let us now set

$$q' = \langle\langle \kappa^q_i; i \in \mathrm{dom}(q) \restriction \alpha\rangle, \langle \pi^q_i; i \in \mathrm{dom}(q) \restriction \alpha\rangle, \langle \tau^q_i; i \in \mathrm{dom}_-(q) \restriction \alpha\rangle\rangle.$$

Of course, $q \leqslant q' \leqslant p$. If $i \in \mathrm{dom}_-(q') = \mathrm{dom}_-(q) \restriction \alpha$, then there is some $n < \omega$ and some $\vec{u} \in \mathrm{ran}(\pi^q_\alpha)$ such that

$$\tau^{q'}_i = \{(m,\vec{z})\ ;\ (n, \vec{u}^\frown m^\frown \vec{z}) \in \tau^q_\alpha = \tau\}.$$

By the choice of $\tau$, we must then have that

$$\tau_i^{q'} \in \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\})$$

for every $i \in \mathrm{dom}_{-}(q')$, because if

$$\tau = \{\langle \ulcorner\varphi\urcorner, \vec{z}\rangle; \vec{z} \in H_\theta \wedge \mathcal{H} \models \varphi[\langle p, R, \leq_{\mathbb{P}}\rangle, \vec{z}]\},$$

then

$$\tau_i^{q'} = \tau_i^q = \{\langle m, \vec{z}\rangle; \vec{z} \in H_\theta \wedge \mathcal{H} \models \varphi[\langle p, R, \leq_{\mathbb{P}}\rangle, \vec{u}^\frown m^\frown \vec{z}]\}.$$

This implies that in fact

$$q' \in \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\}). \tag{2.4}$$

Because $q' \Vdash_{\mathbb{P}}$ "$\dot{C}$ is club in $\check{\omega}_1$," there is some $\gamma > \xi$ and some $q'' \leq_{\mathbb{P}} q'$ such that $q'' \Vdash_{\mathbb{P}} \check{\gamma} \in \dot{C}$, i.e., $(q'', \gamma) \in R$, and therefore by (2.4)

$$\mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\}) \models \exists\gamma > \xi\ \exists q'' \leq_{\mathbb{P}} q'\ (q'', \gamma) \in R,$$

which means that there is some $q'' \leqslant q'$ with

$$q'' \in \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\}) \tag{2.5}$$

such that

$$q'' \Vdash_{\mathbb{P}} \sup(\dot{C} \cap \check{\alpha}) > \check{\xi}.$$

In particular, $\mathrm{dom}(q'') \subseteq \alpha$. We must now have that

$$q'' \text{ and } q \text{ are incompatible.}$$

We derive a contradiction by constructing some $q^* \leqslant q'', q$.

Let

$$\tilde{\pi} \colon H \cong \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\omega_1}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\}) \prec \mathcal{H},$$

where $H$ is transitive. By (2.3), $M^{\mathfrak{B}}_\alpha = \tilde{\pi}^{-1}(\langle H_\theta; \in, I\rangle) \in H$ and $\tilde{\pi} \upharpoonright M^{\mathfrak{B}}_\alpha = \pi^{\mathfrak{B}}_{\alpha,\omega_1}$. In $V^{\mathrm{col}(\omega,2^\theta)}$, there is a model $\mathfrak{C}$ which certifies $q''$. In $\mathcal{H}^{\mathrm{col}(\omega,2^\theta)}$, there is hence some generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that $M_{\omega_1} = \langle H_\theta; \in, I\rangle$ and for all $i \in \mathrm{dom}(q'')$, $\kappa_i^{q''} = \kappa_i$ and $\pi_i^{q''} \subseteq \pi_{i,\omega_1}$. By the elementarity of $\tilde{\pi}$, there is hence in $H^{\mathrm{col}(\omega,\tilde{\pi}^{-1}(2^\theta))} \subseteq V^{\mathrm{col}(\omega,2^\theta)}$ some generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \alpha\rangle, \langle G_i; i < \alpha\rangle\rangle$$

such that $M_\alpha = \tilde{\pi}^{-1}(\langle H_\theta; \in, I\rangle) = M^{\mathfrak{B}}_\alpha$ and for all $i \in \mathrm{dom}(q'')$, $\kappa_i^{q''} = \kappa_i$ and $\tilde{\pi}^{-1}(\pi_i^{q''}) \subseteq \pi_{i,\alpha}$, i.e., $\pi_i^{q''} \subseteq \tilde{\pi} \circ \pi_{i,\alpha} = \pi^{\mathfrak{B}}_{\alpha,\omega} \circ \pi_{i,\alpha}$. Because $M^{\mathfrak{B}}_\alpha$ is countable in $\mathfrak{B}$, $\theta + 1 \subset \mathrm{wfp}(\mathfrak{B})$, and $\mathfrak{B} \in V^{\mathrm{col}(\omega,2^\theta)}$, there is therefore by absoluteness some generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \alpha\rangle, \langle G_i; i < \alpha\rangle\rangle \in \mathfrak{B}$$

such that $M_\alpha = M_\alpha^{\mathfrak{B}}$ and for all $i \in \mathrm{dom}(q'')$, $\kappa_i^{q''} = \kappa_i$ and $\pi_i^{q''} \subseteq \pi_{\alpha,\omega_1}^{\mathfrak{B}} \circ \pi_{i,\alpha}$. Let

$$\langle\langle M_i^*, \pi_{i,j}^*, I_i^*, \kappa_i^*; i \leqslant j \leqslant \omega_1\rangle, \langle G_i^*; i < \omega_1\rangle\rangle \in \mathfrak{B} \tag{2.6}$$

be defined as follows. If $i \le j \le \alpha$, then we set $M_i^* = M_i$, $\pi_{i,j}^* = \pi_{i,j}$, $I_i^* = I_i$, $\kappa_i^* = \kappa_i$, and if $i < \alpha$, then we set $G_i^* = G_i$. If $\alpha \le i \le j \le \omega_1$, then we set $M_i^* = M_i^{\mathfrak{B}}$ (there is no conflict for $i = \alpha$, as $M_\alpha^{\mathfrak{B}} = M_\alpha$), $\pi_{i,j}^* = \pi_{i,j}^{\mathfrak{B}}$, $I_i^* = I_i^{\mathfrak{B}}$, $\kappa_i^* = \kappa_i$, and if $\alpha \le i < \omega_1$, then we set $G_i^* = G_i^{\mathfrak{B}}$. Finally, if $i \le \alpha \le j$, then we set $\pi_{i,j}^* = \pi_{\alpha,j}^{\mathfrak{B}} \circ \pi_{i,\alpha}$. The existence of the generic iteration (2.6) inside $\mathfrak{B}$ clearly shows that $\mathfrak{B}$ in fact certifies $q''$. However, as $\mathrm{dom}(q'') \supseteq \mathrm{dom}(q) \upharpoonright \alpha$, the very same generic iteration (2.6) shows that $\mathfrak{B}$ certifies $q$.

Let us now define $q^* \in \mathbb{P}$ as follows. Let $\mathrm{dom}(q^*) = \mathrm{dom}(q) \cup \mathrm{dom}(q'')$ and $\mathrm{dom}_-(q^*) = \mathrm{dom}(q)_- \cup \mathrm{dom}_-(q'')$. (Neither $\mathrm{dom}(q)$ and $\mathrm{dom}(q'')$ nor $\mathrm{dom}(q)_-$ and $\mathrm{dom}_-(q'')$ need to be disjoint, but $\mathrm{dom}(q) \cap \alpha \subseteq \mathrm{dom}(q'')$ and $\mathrm{dom}(q)_- \cap \alpha \subseteq \mathrm{dom}_-(q'')$.) For $i \in \mathrm{dom}(q^*)$ set $\kappa_i^{q^*} = \kappa_i^*$. For $i \in \mathrm{dom}_-(q'')$ set $\tau_i^{q^*} = \tau_i^{q''}$, and for $i \in \mathrm{dom}_-(q)$, set $\tau_i^{q^*} = \tau_i^q$. Also, for $i \in \mathrm{dom}(q'')$ set $\pi_i^{q^*} = \pi_i^{q''}$. Finally, for $i \in \mathrm{dom}(q) \setminus \alpha$, we need some adjustment in order to actually get a condition. By (2.5), there is some finite $\vec{u} \subseteq \mathrm{ran}(\pi_{\alpha,\omega_1}^{\mathfrak{B}})$ such that

$$q'' \in \mathrm{Hull}^{\mathcal{H}}(\{\vec{u}, \langle p, R, \leq_{\mathbb{P}}\rangle\}).$$

We then also have some $n < \omega$ such that for every $i \in \mathrm{dom}_-(q'')$,

$$\tau_i^{q''} = \tau_i^{q^*} = \{(m, \vec{z}) \;;\; (n, \vec{u}^\frown m^\frown \vec{z}) \in \tau_\alpha^{q^*} = \tau_\alpha^{p'} = \tau\}.$$

We may assume without loss of generality that $\pi_{i,\omega_1}^*{}'' \mathrm{dom}(\pi_i^{q^*}) \subseteq \vec{u}$ for $i \in \mathrm{dom}(q'') \subseteq \alpha$. For $j \in \mathrm{dom}(q^*)$, $j \geqslant \alpha$, we then set

$$\pi_j^{q^*} = \pi_{j,\omega_1}^* \upharpoonright ((\pi_{j,\omega_1}^*)^{-1}(\vec{u}) \cup \mathrm{dom}(\pi_j^{q''})).$$

It is now straightforward to see that $q^* \in \mathbb{P}$. Notice that if $i \in \mathrm{dom}_-(q^*) \cap \alpha = \mathrm{dom}_-(q'')$ and $j \in \mathrm{dom}_-(q^*) \setminus \alpha = \mathrm{dom}_-(q) \setminus \alpha$, and if

$$\tau_\alpha^{q^*} = \tau_\alpha^q = \{(m, \vec{z}); (k, \vec{v}^\frown m^\frown \vec{z}) \in \tau_j^{q^*} = \tau_j^q\},$$

where $\vec{v} \in \mathrm{ran}(\pi_j^{q^*}) = \mathrm{ran}(\pi_j^q)$, then

$$\tau_i^{q^*} = \tau_i^{q''} = \{(m, \vec{z}); (n, \vec{u}^\frown m^\frown \vec{z} \in \tau_\alpha^{q^*}\} = \{(m, \vec{z}); (k, \vec{v}^\frown n^\frown \vec{u}^\frown m^\frown \vec{z} \in \tau_j^{q^*}\}$$

and $\vec{v}$, $\vec{u} \subseteq \mathrm{ran}(\pi_j^{q^*})$. Of course, $q^* \leqslant q, q''$. We have reached a contradiction. $\dashv$

This finishes the proof of Theorem 2.1.

## 2.2 Consequences

A straightforward adaptation yields the following result.

**Theorem 2.16.** *Let $I$ be a precipitous ideal on $\omega_1$, and let $\theta > \omega_1$ be a regular cardinal. Suppose that $H_\theta^\#$ exists. There is a poset $\mathbb{P}$, preserving the stationarity of all sets in $I^+$, such that if $G$ is $\mathbb{P}$-generic over $V$, then in $V[G]$ there is a generic iteration*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

*such that if $i < \omega_1$, then $M_i$ is countable and $M_{\omega_1} = \langle H_\theta^\#; \in, I\rangle$. In particular, $M_0$ is generically $\omega_1 + 1$ iterable. If $I = \mathsf{NS}_{\omega_1}$, then $\mathbb{P}$ is stationary set preserving.*

Proof. Let $\rho > 2^{2^\theta}$, and let $\mathbb{P} = (\mathrm{col}(\rho, \rho) \times \mathrm{col}(\theta^+, \theta^+)) * \mathbb{P}(I, \theta^+)$, where $\mathbb{P}(I, \theta^+)$ is as in Theorem 2.1. Let

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

be a generic iteration which is added by forcing with $\mathbb{P}$. Setting $N_i = \pi^{-1}_{i,\omega_1}(H_\theta)$, we will have that $\pi^{-1}_{i,\omega_1}(H_\theta^\#) = N_i^\#$. The iterability of $M_0$ follows from Lemma 1.14. Notice that $\langle N_0^\#; \in, I_0\rangle$ is generically $\omega_1 + 1$ iterable iff $\langle L[N_0]; \in, I_0\rangle$ is generically $\omega_1 + 1$ iterable. $\dashv$

**Lemma 2.17 (Woodin).** *Let $M$ be a countable transitive model of $\mathsf{ZFC}^*$ + "$\omega_1$ exists," and let $I \subseteq \mathcal{P}(\omega_1^M)$ be such that $\langle M; \in, I\rangle \models$ "$I$ is a uniform and normal ideal on $\omega_1^M$." Let $\alpha < \omega_1$, and suppose $\langle M; \in, I\rangle$ to be generically $\alpha + 1$ iterable. Let $z_0$ be a real which codes $\langle M; \in, I\rangle$, let $z_1$ be a real which codes $\alpha$, and let $z = z_0 \oplus z_1$. Let*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \alpha\rangle, \langle G_i; i < \alpha\rangle\rangle$$

*be a generic iteration of $\langle M; \in, I\rangle$ of length $\alpha + 1$. Then $M_\alpha \cap \mathrm{OR} < \omega_1^{L[z]}$.*

Proof. The proof is taken from [Woo99, p. 56f.]. Let $A \subset \mathbb{R}$ be defined by $x \in A$ iff $x$ codes a countable ordinal $\xi$ (which we write as $\xi = \|x\|$) such that for some generic iteration

$$\langle\langle M_i', \pi_{i,j}', I_i, \kappa_i'; i \leqslant j \leqslant \alpha\rangle, \langle G_i'; i < \alpha\rangle\rangle$$

of $\langle M; \in, I\rangle$ of length $\alpha + 1$, $\xi \subseteq M_\alpha'$. The set $A$ is $\Sigma^1_1(z)$, so that by the Boundedness Lemma (cf. [Jec03, Corollary 25.14]),

$$\sup\{\xi; \exists x \in A \xi = \|x\|\} < \omega_1^{L[z]}.$$

In particular, $M_\alpha \cap \mathrm{OR} < \omega_1^{L[z]}$. $\dashv$

**Lemma 2.18.** *Suppose $I$ to be a precipitous ideal on $\omega_1$. Let $\theta \geq \omega_2$ be regular, and suppose that $H_\theta^\#$ exists. Let $\mathbb{P} = \mathbb{P}'(I, \theta)$ be as in Theorem 2.16, and let $G$ be $\mathbb{P}$-generic over $V$. In $V[G]$, let*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle \in V[G]$$

*be a generic iteration such that if $i < \omega_1$, then $M_i$ is countable and $M_{\omega_1} = \langle H_\theta^\#; \in, I\rangle$. Let $z \in \mathbb{R} \cap V[G]$ code $\langle \pi^{-1}_{0,\omega_1}(H_\theta); \in, I_0\rangle$. Then $\theta < \omega_1^{+L[z]}$. In particular, $V[G] \models \theta < \underset{\sim}{\delta}^1_2$.*

Proof. For a canonical choice of $z$, $z^{\#}$ exists in $V[G]$ and $z^{\#}$ codes $\langle M_0; \in, I_0\rangle$. It therefore suffices to prove $\theta < \omega_1^{+L[z]}$. Suppose that $\omega_1^{+L[z]} \leq \theta$. Let us work in $V[G]$ to derive a contradiction. Let $X \prec H_\Omega$ be countable (where $\Omega$ is regular and large enough) such that $z^{\#}$ and

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

are both elements of $X$, and let $\sigma: N \cong X \prec H_\Omega$, where $N$ is tranitive. Let $\alpha = X \cap \omega_1 = \omega_1^N$. Since $z^{\#} \in X$, we have that

$$\mathcal{P}(\alpha) \cap L[z] \subseteq \mathcal{P}(\alpha) \cap N,$$

so that $\sigma^{-1}(\omega_1^{L[z]}) = \alpha^{+L[z]}$. Also,

$$\sigma^{-1}(\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle) =$$

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leqslant j \leqslant \alpha\rangle, \langle G_i; i < \alpha\rangle\rangle,$$

so that $\sigma^{-1}(\theta) = M_\alpha \cap \mathrm{OR}$. Let $g \in V[G]$ be $\mathrm{col}(\omega, \alpha)$-generic over $N$. Then $M_\alpha \cap \mathrm{OR} \geq \alpha^{+L[z]} = \omega_1^{L[z \oplus g]}$. This contradicts Lemma 2.17. $\dashv$

Recall that Bounded Martin's Maximum, BMM, may be formulated as follows. If $\mathbb{Q} \in V$ is a stationary set preserving forcing, then

$$H_{\omega_2}^V \prec_{\Sigma_1} H_{\omega_2}^{V^{\mathbb{Q}}}.$$

It was shown in [Sch04] that BMM implies that $V$ is closed under sharps. Of course, having a precipitous ideal on $\omega_1$ also yields that the reals are closed under sharps.

**Corollary 2.19.** *Suppose that* BMM *holds and* $\mathsf{NS}_{\omega_1}$ *is precipitous. Then* $u_2 = \omega_2$.

Proof. Let $\alpha < \omega_2$. Let $\varphi \equiv \exists z \in \mathbb{R}(\alpha < \omega_1^{+L[z]})$. The statement $\varphi$ is $\Sigma_1$ over $H_{\omega_2}$ in the parameters $\omega_1$, $\alpha$, and $\varphi$ holds in $V^{\mathbb{P}}$, where $\mathbb{P} = \mathbb{P}'(NS_{\omega_1}, \omega_2)$. Therefore, $\varphi$ must hold in $V$. As $\alpha$ was arbitrary, we have shown that $u_2^V = \omega_2$. $\dashv$

Recall that the Bounded Semiproper Forcing Axiom, BSPFA, may be formulated as follows. If $\mathbb{Q} \in V$ is a semiproper forcing, then

$$H_{\omega_2}^V \prec_{\Sigma_1} H_{\omega_2}^{V^{\mathbb{Q}}}.$$

For a formulation of the Reflection Principle RP cf. [Jec03, p.688].

**Corollary 2.20.** *Suppose* BSPFA *and* RP *both hold. Then* $u_2 = \omega_2$.

Proof. The Reflection Principle RP implies that all stationary set preserving forcings are semiproper, and it implies that $\mathsf{NS}_{\omega_1}$ is precipitous (cf. [Jec03, p.688]). The rest of the proof is the same as that of the previous corollary. $\dashv$

# 3 The strength of BPFA and a precipitous ideal on $\omega_1$

The main goal of this chapter is to analyse how consistency strength arise from the interplay between the existence of one precipitous ideal on $\omega_1$ and some forcing axioms, namely BPFA and $\mathrm{BPFA}^{\mathrm{uB}}$. In the last chapter we have seen that BMM together with the precipitousness of $\mathsf{NS}_{\omega_1}$ implies to $u_2 = \omega_2$. Schindler in [Sch04] showed that BMM implied that the universe is closed under the sharp operator, more recently he showed in [Sch] that the existence of a precipitous ideal and $\underset{\sim}{\delta}^1_2 = \omega_2$ implies the existence of an inner model with a Woodin cardinal. Hence with the result of last section, BMM and $\mathsf{NS}_{\omega_1}$ is precipitous implies the existence of an inner model with a Woodin cardinal. We present similar inner model techniques and apply them to BPFA + there is a precipitous ideal on $\omega_1$.

The first goal of this chapter is to prove the following theorem:

**Theorem 3.1.** *Suppose* BPFA *holds and that there is a precipitous ideal on $\omega_1$. Then there is an inner model with a Woodin cardinal.*

We will first prove the following theorem as an intermediary result.

**Theorem 3.2.** BPFA + "$\exists$ *a precipitous ideal on $\omega_1$" implies $0^\P$.*

The two proofs are by contradiction, we will show that BPFA implies that the cardinal successor of $\omega_1$ in $K$ is strictly less then $\omega_2$ on one side and that the existence of a precipitous ideals on $\omega_1$ on the other side implies that the cardinal successor of $\omega_1$ is computed correctly in $K$ on the other side. We will expose the analysis of BPFA in section 3.2 and give the full result here. In section 3.3 we will turn to the the precipitous ideal on $\omega_1$ below $0^\P$. Recent result by Schindler in [Sch], will show that those result hold true assuming that there is no inner model with a Woodin cardinal.

We will finish this chapter by looking at a strengthening of BPFA allowing predicates to be universally Baires sets and showing that this axiom leads to even stronger consistency strength:

**Theorem 3.3.** *Suppose* $\mathrm{BPFA}^{uB}$ *holds and there is a precipitous ideal on $\omega_1$. Then Projective Determinacy holds.*

Throughout this chapter, we assume that we work in some $\mathsf{ZFC}$ model $V$ and unless the contrary is explicitly noted, we will drop the index $V$ and write $\omega_1$ for $\omega_1^V$ and $\omega_2$ for $\omega_2^V$.

The main forcing tool we will be using in the next section is to "seal" the height of certain premice. This will be done by making sure one can not extend a certain fine structural component of these mice, namely their square sequence. The square sequence was invented by Jensen in [Jen72], where he showed that $L$ has a square sequence for all $\kappa$. It has been a constant in inner model theory so far that every inner model has a square sequence. Work on square sequences in recent fine structural extender models have been mostly due to Schimmerling and Zeman (see [SZ01] and [SZ04]).

## 3.1 Definitions

**Definition 3.4.** We call a sequence $\langle C_\alpha, \alpha \in S\rangle$ a *$C$-sequence in $S$* if $S$ is unbounded in some regular cardinal $\kappa$ and each $C_\alpha$ is club in $\alpha$. We say that $\langle C_\alpha, \alpha \in S\rangle$ is a *$C$-sequence* if it is a $C$-sequence in $S$ and $S$ is some stationary subset of the limit ordinals of $\kappa$.

A coherent sequence $\langle C_\alpha, \alpha \in S\rangle$ is a $C$-sequence with the property that if $\alpha$ is a limit point of $C_\beta$ then $\alpha \in S$ and $C_\alpha = C_\beta \cap \alpha$.

**Definition 3.5.** Let $\kappa$ be a cardinal. $\square_\kappa$ is the combinatorial principle stating:

> "There is a coherent sequence $\langle C_\alpha; \alpha < \kappa^+,\ \alpha \text{ limit ordinal}\rangle$ such that for all $\alpha$, $\operatorname{otp}(C_\alpha) \leq \kappa$."

Such a sequence $\langle C_\alpha; \alpha < \kappa^+,\ \alpha \text{ limit ordinal}\rangle$ is called a *square sequence* or in symbols a *$\square_\kappa$-sequence.*

**Definition 3.6.** Let $\langle C_\alpha, \alpha \in S\rangle$ be a coherent sequence in some regular $\kappa$. We say that $C \subseteq \kappa$ is a *thread* if $C$ is a club in $\kappa$ such that for all limit points $\alpha$ of $C$, $C_\alpha = C \cap \alpha$. If such $C$ exists, we say that the square sequence is *threadable*

**Remark 3.7.** If $\langle C_\alpha, \alpha \in S\rangle$ is a square sequence, then it is not threadable.

Proof. Let $C$ be a thread, $C$ is a club in some $\kappa^+ = \sup(S)$, hence there is a $\kappa + \omega$th limit point of $C$, say $\beta$. Since $C$ is a thread we have on one hand that $C_\beta = C \cap \beta$, thus $\operatorname{otp}(C_\beta) > \kappa$. On the other hand, as $\langle C_\alpha, \alpha \in S\rangle$ is a square sequence, $\operatorname{otp}(C_\beta) \leq \kappa$, a contradiction! $\dashv$

Let $\langle C_\alpha, \alpha \in S\rangle$ be a $\square_\kappa$-sequence. Let us now define $T$ the canonical *tree derived from the square sequence*. The domain of $T$ are the limits ordinal between $\kappa$ and $\kappa^+$ and the order is:

$$\alpha <_T \beta \iff \alpha \in \operatorname{Lim}(C_\beta)$$

Remark that $T$ is a $\kappa + 1$ tree and that all the maximal branches are of the form $b = \{\delta\} \cup \{\alpha < \delta; \alpha \in \operatorname{Lim}(C_\delta)\}$ for some $\delta < \kappa^+$ with $\operatorname{cf}(\delta) = \kappa$. Obviously these branches can not be extended anymore.

As announced, the main result we will be using is the following, taken from [SZ01, Theorem 2]

**Theorem 3.8 (Schimmerling and Zeman).** *If $K$ is a Mitchell-Steel core model, then $K$ satisfy $\square_\kappa$ for all $\kappa$.*

Throughout the rest of this chapter we write $\omega_1$ for $\omega_1^V$.

## 3.2 Using BPFA

**Lemma 3.9.** *Suppose that there is no inner model with a Woodin cardinal.* BPFA *implies that* $\omega_1^{+K} < \omega_2$.

Let us first prove a "warm up" case, where the inner model theory is slightly easier, and then go on to the full result.

**Lemma 3.10.** *Suppose* $\neg 0^{\P}$. *Let* $\omega_1 \subseteq \mathcal{M} \in H_{\omega_2}$ *be a premouse. There is a tree* $T$ *of height* $\omega$ *uniformly definable from* $\mathcal{M}$ *such that* $\mathcal{M}$ *is iterable if and only if there is no branch through* $T$. *Moreover* $T$ *is in every model* $H$ *of* $\mathsf{ZFC}^-$ *containing* $\mathcal{M}$ *and* $\omega_1$ *and* $H \models$ "$\mathcal{M}$ *is iterable* $\Longleftrightarrow [T] = \varnothing$".

Proof. The tree $T$ will be the so-called tree searching for a countable model witnessing the non-iterability of $\mathcal{M}$, as we will be using many searching trees in the following, we will define it in great detail here and refer to that construction later.

Let $L_{\mathcal{M}}$ be a language with three predicates $\in$, $\vec{E}$ and $E$ and infinitely many constants $\langle c_n; n < \omega\rangle$. Let further $\langle \psi_n; n < \omega\rangle$ be an enumeration of all $L_{\mathcal{M}}$-formulae without free variables such that if $c_i$ occurs in $\psi_n$ then $i < n$. Let $\langle \varphi_n^1; n < \omega\rangle$ be an enumeration of all $\mathsf{ZFC}^-$ formulae and $L$ be a language with one predicate $\in$ and with constants $\mathcal{N}, \mathcal{I}, \mathcal{N}_\infty, \langle a_n; n < \omega\rangle, \langle c_n; n < \omega\rangle$ and let $\langle \varphi_n^2; n < \omega\rangle$ be an enumeration of all $L$ formulae without free variables such that

i. $\varphi_0^2 =$ "$\mathcal{N}$ is a premouse and $\mathcal{I}$ is a putative iteration of $\mathcal{N}$ with a last ill-founded model $\mathcal{N}_\infty$",

ii. if $a_i$ or $c_i$ occurs in $\varphi_n^2$ then $i < n$.

The idea is to search for a countable model $M$ of $\mathsf{ZFC}^-$, an elementary map $\sigma : \mathcal{N} \to \mathcal{M}$ and an $\in$-preserving map $\pi : M \cap \mathrm{OR} \to \omega_1$ such that $M = \{a_i; i < n\}$, moreover in $M$ there is a countable submodel of $\mathcal{M}$, $\mathcal{N} = \{c_n; n < \omega\}$ and an iteration of $\mathcal{N}$ that is ill-founded.

Nodes in our tree will be of the form

$$p = \langle \sigma^p, \Theta^{\mathcal{M},p}, h^{\mathcal{M},p}, \Theta^p, h^p, \pi^p\rangle \in {}^{<\omega}(\mathcal{M} \times 2 \times \omega \times 2 \times \omega \times \omega_1)$$

with $n = \mathrm{dom}(p)$. Let us give a short explanation of all components: $\sigma^p$ will be a finite approximation to the fully elementary map $\sigma : \mathcal{N} \to \mathcal{M}$, $\Theta^{\mathcal{M},p}$ will be a finite approximation to the truth function of $\mathcal{N}$, $h^{\mathcal{M},p}$ will be a function mapping existential sentences $\exists x \varphi(x)$ to some witness in $\mathcal{N}$, $\Theta^p$ will be a finite approximation to the truth function of $M$, $h^p$ will be mapping existential sentences true in $M$ as seen by $\Theta^p$ to their witness in $M$ and finally $\pi^p$ will be an $\in$-preserving map mapping every ordinal of $M$ to some ordinal less than $\omega_1$.

We say that $p$ is correct if for all $i < \mathrm{dom}(p)$, letting $\bar{\psi}_k$ denote the formula $\psi_k$ where all $c_i$ have been replaced by $\sigma^p(c_i)$, we have that if $\Theta^{\mathcal{M},p}(i) = 0$, $\mathcal{M} \models \bar{\psi}_k$ and if $\Theta^{\mathcal{M},p}(i) = 1$, $\mathcal{M} \models \neg\bar{\psi}_k$. Moreover if $\psi_k$ is an existential formulae, that is there is a $\bar{\psi}$ such that $\psi_k \equiv \exists x \bar{\psi}(x)$, then $j = h^{\mathcal{M},p}(k) < k$ and letting $\theta_k$ be the formula where we replaced every $c_i$ by $\sigma^p(c_i)$ in $\bar{\psi}$, we have that $\mathcal{M} \models \theta_k(\sigma^p(c_{h^{\mathcal{M},p}(k)}))$. Else $h^{\mathcal{M},p}(k) = 0$. Similarly if $\varphi_k^2 \equiv \exists x \bar{\varphi}_k^2$, we require that $h^p(k) < k$. let $T_p$ be the set containing all the following sentences:

i. $\{$“$\mathcal{N} \models \psi_i$”; $i < n \wedge \Theta^{\mathcal{M},p}(i) = 0\}$

ii. $\{$“$\mathcal{N} \models \neg\psi_i$”; $i < n \wedge \Theta^{\mathcal{M},p}(i) = 1\}$

iii. $\{\varphi_i^1;\ i < n\}$

iv. $\{\varphi_i^2;\ i < n \wedge \Theta^p(i) = 0\}$

v. $\{\neg\varphi_i^2;\ i < n \wedge \Theta^p(i) = 1\}$

vi. $\{$“$\mathcal{N} \models \bar{\psi}_i(c_{h^{\mathcal{M},p}(i)})$”; $i < n \wedge \Theta^{\mathcal{M},p}(i) = 0 \wedge \psi_i \equiv \exists x \bar{\psi}_i(x)\}$

vii. $\{\bar{\varphi}_i(c_{h^p(i)});\ i < n \wedge \Theta^p = 0 \wedge \varphi_i^2 \equiv \exists x \bar{\varphi}_i(x)\}$

We call $p$ certified if for all $i, j \in \mathrm{dom}(p)$ with $T_p \models$ “$c_i, c_j$ are ordinals”, we have that if $T_p \models$ “$c_i \in c_j$” then $\pi^p(i) \in \pi^p(j)$. This will make sure that the ordinals of $M$ are well-founded. We further require $\Theta^p(0) = 0$, that is the model we produce thinks that $\mathcal{N}$ is not iterable and that whenever $\varphi_i^2 \equiv$ “$a_k \in \mathcal{N}$”, $\Theta^p(k) = 0$ if and only if there is a $l \in \mathrm{dom}(p)$ such that $T^p \models$ “$a_k = c_l$”. This last condition will make sure that $\mathcal{N} = \{c_i;\ i < \omega\}$.

Finally let $T$ be the tree of all conditions $p$ that are correct and certified such that $T_p$ is consistent and $\Theta^p(0) = 0$. The ordering is inclusion.

Since we are below $0^{¶}$, by Definition 1.27, non-iterability means have a putative iteration whose last model is ill-founded. If $\mathcal{M}$ is not iterable then there is such an iteration $\mathcal{I}$ and taking a countable submodel $M \cong X \prec H_{\omega_2}$ with $\mathcal{M}, \mathcal{I} \in X$, $M$ describes a branch through $T$. On the other hand, if $\mathcal{M}$ is iterable, suppose there was a branch $b$ through $T$, Let $M$ be the model in that branch. Since $M$ is transitive, as witnessed by $\bigcup_{p \in b} \pi^p$, the putative iteration $\mathcal{I}$ of $\mathcal{N}$ with an ill-founded last model as witnessed by $b$ is also one in $V$. This is a contradiction to $\mathcal{M}$ being a mouse! $\dashv$

**Lemma 3.11.** *Suppose* $\neg 0^{¶}$ *Let* $\mathcal{M} \in H_{\omega_2}$ *be a premouse. The following are equivalent*

*i.* $\mathcal{M}$ *is a mouse,*

*ii. there is an* $M \in H_{\omega_2}$ *with* $\omega_1 \cup \{\mathcal{M}\} \subseteq M$ *and* $M \models \mathsf{ZFC}^-$ *such that* $M \models$*“*$\mathcal{M}$ *is a mouse”,*

*iii. for all* $M \in H_{\omega_2}$ *with* $\omega_1 \cup \{\mathcal{M}\} \subseteq M$ *and* $M \models \mathsf{ZFC}^-$ *we have that* $M \models$*“*$\mathcal{M}$ *is a mouse”.*

Proof. It is clear that if $\mathcal{M}$ is a mouse, the two other condition holds as well. Or put in the other way, every ill-founded iteration of $\mathcal{M}$ in such a model $M$ is an ill-founded iteration in $V$. Hence we have that i. $\implies$ iii. $\implies$ ii.

Let us now prove “ii. $\implies$ i.”. Suppose we have a $M \in H_{\omega_2}$ with $M \models \mathsf{ZFC}^-$ and $\omega_1 \cup \{\mathcal{M}\} \subseteq M$, $M \models$“$\mathcal{M}$ is a mouse” but $\mathcal{M}$ is actually not iterable in $V$. Since we are below 0¶ this means that there is a putative iteration $\mathcal{I}$ on $\mathcal{M}$ such that the last model is ill-founded model. Let $X \prec H_{\omega_2}$ be countable with $\mathcal{I}, \mathcal{M} \in X$ and let $\sigma : N \to X$ be the uncollapsing map. Therefore there is a countable $\bar{\mathcal{M}} = \sigma^{-1}(\mathcal{M})$, a fully elementary map $\sigma \upharpoonright \bar{\mathcal{M}} : \bar{\mathcal{M}} \to \mathcal{M}$ and a countable putative iteration $\bar{\mathcal{I}} = \sigma^{-1}(\mathcal{I})$ on $\bar{\mathcal{M}}$ such that the last model is ill-founded. $\langle N, \sigma \upharpoonright \bar{\mathcal{M}}, \sigma \upharpoonright \mathrm{OR}, \bar{\mathcal{N}}, \bar{\mathcal{I}} \rangle$ describe a branch through our tree $T$. Hence $T$ is ill-founded in $V$.

Now look at the tree $T$ defined in the previous lemma. Since $T$ is in every model of $\mathsf{ZFC}^-$ that contains $\omega_1$ and $\mathcal{M}$, by absoluteness of foundedness, if it has a branch in $V$, it is ill-founded in every model containing $\omega_1$ and $\mathcal{M}$.

Hence $T$ is ill-founded in $M$, taking a branch through the tree in $M$, we have a countable submodel of $\mathcal{M}$ with an ill-founded iteration, a contradiction to the iterability of $\mathcal{M}$ in $M$! $\dashv$

This shows that for a premouse $\mathcal{M} \in H_{\omega_2}$ being iterable is a $\Delta_1^{H_{\omega_2}}(\mathcal{M}, \omega_1)$ property below 0¶. With that key observation, we are ready to prove the "warm up" case:

**Lemma 3.12.** *Suppose $\neg 0^{\P}$.* BPFA *implies that $\omega_1^{+K} < \omega_2$.*

Proof. We work towards a contradiction. As we assumed $\neg 0^{\P}$, the Jensen core model $K$ as described in Definition 1.34 exist. So let us suppose that $\omega_1^K = \omega_2$. Notice that by the results of Schimmerling and Zeman, $K$ has a square sequence at every $\kappa$.

Claim 1. *There is a proper forcing $\mathbb{P}^*$ such that, if $G$ is $\mathbb{P}^*$-generic over $V$, in $H_{\omega_2}^{V[G]}$ there is a mouse $\mathcal{M}$ such that $\mathcal{M} \trianglerighteq K \| \omega_1^V$, $\mathcal{M}$ has the strong condensation property, $\mathrm{cf}(\omega_1^{+\mathcal{M}}) = \omega_1$ and there is a function $f : \omega_1^{+\mathcal{M}} \to \omega$ specializing the tree arising from the restriction of the $\square_{\omega_1}^{\mathcal{M}}$-sequence to some club in the height of $\mathcal{M}$.*

Proof. The forcing is constructed in two steps. The model $\mathcal{M}$ will be $K \| \omega_2$, we first force in order to get make the model $\mathcal{M}$ of size $\omega_1$, then specialize the tree arising from its square sequence. Let the first forcing $\mathbb{Q}$ be the $\omega$-closed forcing that adds a club of order type $\omega_1$ in $\omega_2$. Set $\mathcal{M} = K \| \omega_2^V$, we will show that $\mathcal{M}$ is the model with the desired properties. Notice that after forcing with $\mathbb{Q}$, we already have that $\mathrm{cf}(\omega_1^{+\mathcal{M}}) = \mathrm{cf}(\omega_2^V) = \omega_1$. Let $T$ be the tree arising from the $\square_{\omega_1}$-sequence of $\mathcal{M}$, that is $T$ is the tree on $]\omega_1, \omega_1^{+\mathcal{M}}[ \cap \mathrm{Lim}$ with order $\alpha <_T \beta \iff \alpha \in \mathrm{Lim}(C_\beta)$, where $\langle C_\alpha ; \alpha \in ]\omega_1, \omega_1^{+\mathcal{M}}[ \cap \mathrm{Lim} \rangle$ is the $\square_{\omega_1}^{\mathcal{M}}$-sequence. Let us prove that in $V^{\mathbb{Q}}$, there is no $\omega_1$ branch in $T \upharpoonright \dot{G}$, where $\dot{G}$ is the canonical name for the $\mathbb{Q}$-generic filter.

Suppose that $p \Vdash$ "$\tau$ is a cofinal branch through $T \upharpoonright \dot{G}$" and let $(p,p) \in G \times H$ be $\mathbb{Q} \times \mathbb{Q}$-generic over $V$. We first show that $b = \tau^G$ is unique in $V[G]$. Suppose not, and let $b'$ be a second branch through $T \upharpoonright \dot{G}$. Remark that since $\bigcup b$ and $\bigcup b'$ are both club in $\sup G = \omega_2$, they are of ordertype $\omega_1$. Hence $\bigcup b \cap \bigcup b'$ is a club in $\omega_2$ and by coherency $b = b'$. Go back to $V[G \times H]$, we claim that $\tau^G = \tau^H$. $\bigcup \tau^G$ and $\bigcup \tau^H$ are both clubs in $\omega_2$ and again by coherency of the $\square_{\omega_1}$-sequence, they must be equal. Thus $b = \tau^G = \tau^H \in V[G] \cap V[H] = V$. But $\bigcup b$ is a club of ordertype $\omega_1$ and cofinal in $\omega_2$, a contradiction!

Let $\mathbb{P}$ be the forcing in $V^{\mathbb{Q}}$ with conditions of the form $p : \omega_1^{+\mathcal{M}} \rightarrowtail \omega$ such that $p$ is a partial function, $\mathrm{dom}(p) \subseteq T \upharpoonright \dot{G}$ is finite and for all $\xi, \xi' < \omega_1^{+\mathcal{M}}$ if $\xi'$ is a limit point of

$C_\xi$, then $p(\xi) \neq p(\xi')$. The conditions are ordered by inclusion. As the forcing has no $\omega_1$ branch, by [Sch95, page 198 ff], it is a c.c.c. forcing. Hence

$$V^{\mathbb{Q}*\mathbb{P}} \models \text{“}\operatorname{cf}(\omega_1^{+\mathcal{M}}) = \operatorname{cf}(\omega_2^V) = \omega_1\text{”}.$$

Thus setting $\mathbb{P}^* = \mathbb{Q} * \mathbb{P}$, $\mathbb{P}^*$ produces a model $\mathcal{M}$ with the desired properties. As it is the $*$-product of an $\omega$-closed forcing with a c.c.c. forcing, it is proper. $\dashv$

Now look at the following formulae:

i. $\varphi_0(\mathcal{M}, f, C, \omega_1) \equiv f$ specialize the tree derived from the restriction of the $\square_{\omega_1}^{\mathcal{M}}$-sequence to $C$, where $C$ is a club in $\mathcal{M} \cap \mathrm{OR}$ of order type $\omega_1$. $\varphi_0$ is a $\Sigma_0$-formula over $H_{\omega_2}$ with the parameters $\mathcal{M}$, $f$, $C$ and $\omega_1$.

ii. $\varphi_1(\mathcal{M}, K\|\omega_1, \omega_1) \equiv \mathcal{M} \triangleright K\|\omega_1$. $\varphi_1$ is a $\Sigma_0$-formula over $H_{\omega_2}$ with parameters $\mathcal{M}$, $K\|\omega_1$ and $\omega_1$.

iii. $\varphi_2(\mathcal{M}) \equiv \mathcal{M} \models \mathsf{ZFC}^-$. $\varphi_2$ is a $\Delta_1$-formula over $H_{\omega_2}$ with parameter $\mathcal{M}$.

iv. $\varphi_3(\mathcal{M}, \omega_1) \equiv$ “$\mathcal{M}$ is an iterable premouse”. $\varphi_3$ is a $\Delta_1$ statement over $H_{\omega_2}$ with parameters $\mathcal{M}$ and $\omega_1$ as shown in Lemma 3.11

v. $\varphi_4 \equiv \mathcal{M} \models$ “$\omega_1$ is the largest cardinal”. $\varphi_4$ is a $\Delta_1$-formula over $H_{\omega_2}$ with parameters $\mathcal{M}$ and $\omega_1$.

vi. $\varphi_5(\mathcal{M}) \equiv$ “ for every $\alpha < \omega_1$ if $\alpha$ is the critical point of the uncollapsing map $\mathcal{N} \to \mathrm{Hull}(\alpha \cup \{p(\mathcal{M})\})$, $\mathcal{N} \triangleleft \mathcal{M}$

We thus have that the formula:

$$\Psi \equiv \exists \mathcal{M} \exists f \exists C\ \varphi_1(\mathcal{M}, K\|\omega_1, \omega_1) \wedge \varphi_2(\mathcal{M}) \wedge \varphi_3(\mathcal{M}) \wedge \varphi_4(\mathcal{M}, \omega_1) \wedge \varphi_0(\mathcal{M}, f, C, \omega_1)$$

is clearly $\Sigma_1$ over $H_{\omega_2}$.

Since BPFA holds in $V$, we have that $H_{\omega_2}^V \prec_{\Sigma_1} H_{\omega_2}^{V^{\mathbb{P}^*}}$. $\Psi$ holds in $H_{\omega_2}^{V^{\mathbb{P}^*}}$, hence there is a $\mathcal{M} \triangleright K\|\omega_1$ and a $f$ such that $\varphi_0$ to $\varphi_5$ hold in $V$.

We have seen that $\Psi(K\|\omega_1, \omega_1)$ is a $\Sigma_1$-formula over $H_{\omega_2}$, moreover $\Psi(K\|\omega_1, \omega_1)$ holds in $H_{\omega_2^V}$, as witnessed by $K\|\omega_2$, $C$ and $f$, where $C$ is the club added by the first forcing and $f$ is the function added by the second forcing. By BPFA we have that there is some $\mathcal{M}, C, f \in V$ witnessing the truth of $\Psi(K\|\omega_1, \omega_1)$.

We claim that $\mathcal{M}$ and $K\|\omega_2$ are lined up. Suppose not and let $X$ be a countable submodel of $H_{\omega_2}$ with $\mathcal{M} \in X$, let $\sigma : M \to X$ be the transitive collapse. $K^N$ is a countable submodel of $K$ as witnessed by $\sigma \restriction K^N$ hence $K^N \triangleleft K$. Let $\alpha = X \cap \mathrm{OR}$ we have that $\sigma \restriction \mathcal{N} : \mathcal{N} \to \mathrm{Hull}^{\mathcal{M}}(\alpha \cup \{p(\mathcal{M})\})$ is an elementary embedding with critical point $\alpha$, hence by $\varphi_5$, $\mathcal{N} \triangleleft \mathcal{M}$ but $\mathcal{N}$ is countable and $K\|\omega_1 \triangleleft \mathcal{M}$ hence $\mathcal{N} \triangleleft K$. This shows that $\mathcal{N}$ and $K^N$ are lined up, a contradiction since $M \models$ “$\mathcal{N}$ and $K^N$ are not lined up ”!

Now this implies again that $\mathcal{M} \triangleleft K\|\omega_2$, but then the $\square_{\omega_1}$-sequence of $K$ extends the $\square_{\omega_1}$-sequence of $\mathcal{M}$, a contradiction since it was specialized! $\dashv$

If one looks closely at the previous proof everything but one argument would hold as well if we assumed that there are no inner model with a Woodin cardinal. The missing argument being the complexity of iterability . The next lemma will take care of iterability, we will then show how to fix the argument.

**Lemma 3.13.** *Suppose there is no inner model with a Woodin cardinal. Let $\mathcal{M} \in H_{\omega_2}$ be a premouse such that $\mathcal{M} \models$ "there are no definably Woodin cardinals". The following are equivalent*

*i. $\mathcal{M}$ is a mouse,*

*ii. there is an $M \in H_{\omega_2}$ with $\omega_1 \cup \{\mathcal{M}\} \subseteq M$ and $M \models \mathsf{ZFC}^-$ such that $M \models$ "$\mathcal{M}$ is a mouse",*

*iii. for all $M \in H_{\omega_2}$ with $\omega_1 \cup \{\mathcal{M}\} \subseteq M$ and $M \models \mathsf{ZFC}^-$ we have that $M \models$ "$\mathcal{M}$ is a mouse".*

Proof. We look at the very same tree $T$ as in Lemma 3.10. The only difference is that non-iterability doesn't necessarily mean that the last model is ill-founded. Let $\varphi$ be the formula

> "$\mathcal{N}$ is a premouse and $\mathcal{I}$ is a putative iteration of $\mathcal{N}$ with a last ill-founded model $\mathcal{N}_\infty$"

and $\varphi'$ the formula

> "$\mathcal{N}$ is a premouse, and either
>
> i. $\mathcal{I}$ is a putative iteration of $\mathcal{N}$ with a last ill-founded model $\mathcal{N}_\infty$ or
>
> ii. $\mathcal{I}$ is an iteration of $\mathcal{N}$ with no cofinal branch $b$ such that $\mathcal{N}_\infty \triangleleft \mathcal{M}_b^{\mathcal{I}}$, where $\mathcal{N}_\infty$ is the $Q$-structure of $\mathcal{I}$, $\mathcal{N}_\infty \models$ '$\delta(\mathcal{I})$ is not definably Woodin' and $\mathcal{N}_\infty = \mathcal{J}_\alpha(\mathcal{M}(\mathcal{I}))$".

We modify the tree by switching the formula $\varphi$ with the formula $\varphi'$ in the enumeration.

Claim 1. *Suppose $M$ is a model of $\mathsf{ZFC}^-$ and $\omega_1 \cup \{\mathcal{M}\} \subseteq M$, then*

$$M \models \text{"}\mathcal{M} \textit{ is iterable if and only if } T \textit{ is well-founded."}$$

Proof. "$\Longleftarrow$" by contraposition. Suppose $\mathcal{M}$ is not iterable and let $\mathcal{I}$ be a putative iteration tree witnessing it. Take a countable substructure $X$ with $\mathcal{M}, \mathcal{I} \in X$. Let $\sigma : N \to X$ be the uncollapsing map. We can construct a branch through $T$ by searching for $\langle N, \sigma \restriction \mathrm{OR}, \sigma \restriction \sigma^{-1}(\mathcal{M}), \sigma^{-1}(\mathcal{M}), \sigma^{-1}(\mathcal{I})\rangle$.

"$\Longrightarrow$" Now suppose $\mathcal{M}$ is iterable and $T$ has a branch $b_T$. Let $\bar{M}$ be the countable $\mathsf{ZFC}^-$ model described by $b_T$ and $\mathcal{N} \in \bar{M}$ be the elementary substructure of $\mathcal{M}$ with a bad iteration $\mathcal{I}$. If $\mathcal{I}$ has a last ill-founded model, we already have argued in the proof of Lemma 3.11 that this would give a contradiction. So suppose that the second possibility occurs, that is $\bar{M} \models$"$\mathcal{I}$ is an iteration of $\mathcal{N}$ with no cofinal branch $b$ such that there is a $Q \triangleleft \mathcal{M}_b^{\mathcal{I}}$ with $Q \models$ "$\delta(\mathcal{I})$ is not Woodin" and $Q = \mathcal{J}_\alpha(\mathcal{M}(\mathcal{I}))$". By the absoluteness of the strategy $\Sigma$ according to which we play the iteration game, $\mathcal{I}$ is an iteration tree played according to $\Sigma$ in $M$ as well. For any reals, $x$ and $y$, let $\varphi(x,y)$ be the statement:

if $x$ is a code for a cofinal branch through $\mathcal{I}$ and $y$ is a code for $\mathcal{J}_\alpha(\mathcal{M}(\mathcal{I}))$ then $y \models$"$\delta(\mathcal{I})$ is Woodin".

The statement $\exists x \exists y\ \varphi(x,y)$ is $\Sigma^1_1(\mathcal{I},\alpha)$ and hence is absolute between $\mathsf{ZFC}^-$-models containing $\mathcal{I}$ and $\alpha$. Thus $\mathcal{I}$ has no branch in $M$ as well. This shows that $\mathcal{N}$ is not iterable in $M$. But $\mathcal{M}$ is iterable in $M$ and there is an elementary embedding $\sigma : \mathcal{N} \to \mathcal{M}$, a contradiction! ⊣

$T$ is a countable tree with finites nodes and is definable from $\mathcal{M}$ and $\omega_1$. Hence $T$ is in every $\mathsf{ZFC}^-$ model $M$ containing $\mathcal{M}$ and $\omega_1$. By absoluteness of well-foundedness for finite trees, $T$ is ill-founded in $V$ if and only if it is in $M$. Hence by the previous claim, $\mathcal{M}$ is iterable in $V$ if and only if it is in $M$. ⊣

This lemma shows that iterability is $\Delta_1$ over $H_{\omega_2}$.

**Lemma 3.14.** *Suppose there is no inner model with a Woodin cardinal.* BPFA *implies that* $\omega_1^{+K} < \omega_2$.

Proof. We can go through the proof of Lemma 3.12. Let us make a few minor modification. Let $K$ be the Mitchell-Steel core model below one Woodin cardinal. Let $\mathbb{P}$ be the forcing first shooting an $\omega_1$ club into $\omega_2$ and the specializing the square sequence of $K\|\omega_2$ restricted to points in the club we just added. Now look at $\Psi$, the conjunction of the formulae:

i. $\varphi_0(\mathcal{M}, f, C, \omega_1) \equiv f$ specialize the restriction of the tree derived from the $\square^{\mathcal{M}}_{\omega_1}$-sequence restricted to $C$, where $C$ is a club in $\mathcal{M} \cap \mathrm{OR}$ of order type $\omega_1$. $\varphi_0$ is a $\Sigma_1$-formula over $H_{\omega_2}$ with the parameters $\mathcal{M}$, $f$, $C$ and $\omega_1$.

ii. $\varphi_1(\mathcal{M}, K\|\omega_1, \omega_1) \equiv \mathcal{M} \trianglerighteq K\|\omega_1$. $\varphi_1$ is a $\Sigma_0$-formula over $H_{\omega_2}$ with parameters $\mathcal{M}$, $K\|\omega_1$ and $\omega_1$.

iii. $\varphi_2(\mathcal{M}) \equiv \mathcal{M} \models \mathsf{ZFC}^-$. $\varphi_2$ is a $\Delta_1$-formula over $H_{\omega_2}$ with parameter $\mathcal{M}$.

iv. $\varphi_3(\mathcal{M}, \omega_1) \equiv$ "$\mathcal{M}$ is an iterable premouse". $\varphi_3$ is a $\Delta_1$ statement over $H_{\omega_2}$ with parameters $\mathcal{M}$ and $\omega_1$ as shown in Lemma 3.11

v. $\varphi_4 \equiv \mathcal{M} \models$ "$\omega_1$ is the largest cardinal". $\varphi_4$ is a $\Delta_1$-formula over $H_{\omega_2}$ with parameters $\mathcal{M}$ and $\omega_1$.

vi. $\varphi_5(\mathcal{M}) \equiv$ " for every $\alpha < \omega_1$ if $\alpha$ is the critical point of the uncollapsing map $\mathcal{N} \to \mathrm{Hull}(\alpha \cup \{p(\mathcal{M})\})$, $\mathcal{N} \triangleleft \mathcal{M}$

We have seen that $\exists f \exists C \exists \mathcal{M} \Psi(\mathcal{M}, f, C, K\|\omega_1, \omega_1)$ is a $\Sigma_1$-formula over $H_{\omega_2}$. Moreover $\Psi(K\|\omega_2, f, C, K\|\omega_1, \omega_1)$ holds in $H_{\omega_2^V}$, where $C$ is the club in $\omega_2$ added by the first part of the forcing and $f$ is the function added by the second part of the forcing. By BPFA we have that there is some $\mathcal{M} \in V$ witnessing the truth of $\exists f \exists C \exists \mathcal{M} \Psi(\mathcal{M}, f, C, K\|\omega_1, \omega_1)$.

As in the $\neg 0^\P$ case, $\mathcal{M}$ and $K\|\omega_2$ are lined up. As $\mathcal{M}$ is only of cardinality $\omega_1$, this implies again that $\mathcal{M} \triangleleft K\|\omega_2$, but then the $\square_{\omega_1}$-sequence of $K$ extends the $\square_{\omega_1}$-sequence of $\mathcal{M}$, a contradiction since it was specialized! ⊣

## 3.3 Consequences from precipitousness

**Lemma 3.15.** *Suppose* $\neg 0^{\P}$. *The existence of a precipitous ideal on* $\omega_1$ *implies that* $\omega_1^{+K} = \omega_2^V$

Proof. We go for a contradiction, as $\mathsf{GCH}$ holds in $K$ it implies that $\mathcal{P}(\omega_1) \cap K$ has size $\omega_1$ in $V$. Let $f : \omega_1 \to \bigcup_{n<\omega} \mathcal{P}([\omega_1]^n) \cap K$ be a bijection. Let $I$ be precipitous on $\omega_1$ and let $G$ be a $\mathcal{P}(\omega_1) \smallsetminus I$-generic over $V$. Let $j : V \to M = \mathrm{Ult}(V, G)$ be the associated ultrapower map.

Since $0^{\P}$ does not exist, the restriction of the embedding

$$j \upharpoonright K : K \to K^M$$

is a normal iteration by Theorem 1.35, $\mathcal{T}$ say. Let $E = E_0^{\mathcal{T}}$ be the first extender used in the iteration and $\kappa$ its critical point. Since the iteration is normal, and $\mathrm{cp}(j) = \omega_1$, $\kappa = \omega_1$. Let us first show by the ancient Kunen argument that $E$ is in $M$. We have that $j(f) \in M$, therefore

$$f = \left\{ \langle \alpha, j(f)(\alpha) \cap \bigcup_{n<\omega} (\omega_1)^n \rangle ; \alpha \in \omega_1 \right\} \in M.$$

Now we can define $E$ by:

$$\langle a, x \rangle \in E \iff \exists \xi < \omega_1 \ a \in j(f)(\xi) \text{ and } x = f(\xi).$$

$E$ can not be on the $K^M$ sequence, since it has been used in the first step of an iteration to $K^M$. If we show that $\mathrm{Ult}(K^M, E)$ is iterable, we will be finished, since $E$ would have appeared on the $K^M$ sequence during its construction. Let $\rho_0 : K \to \mathrm{Ult}(K, E)$ be the ultrapower map and $\mathcal{U} = \rho_0(\mathcal{T})$. We copy the iteration $\mathcal{T}$ via $\rho_0$. Hence we get the following diagram:

$$\begin{tikzcd}
K \arrow[rr, "i_{E_0^{\mathcal{T}}}"'] \arrow[dd, "\rho_0"] & & \mathcal{M}_1^{\mathcal{T}} \arrow[dd, "\rho_1"] \arrow[dr] & \\
& & & \mathrm{Ult}(\mathcal{M}_1^{\mathcal{T}}, E) \arrow[dl, dotted, "h_1"] \\
\mathrm{Ult}(K, E) = M_1^{\mathcal{T}} \arrow[rr, "i_{E_0^{\mathcal{U}}}"'] & & \mathcal{M}_1^{\mathcal{U}} &
\end{tikzcd}$$

Let us explain how $\rho_1$ and $h_1$ are defined:

$$\begin{aligned} \rho_1 : & \mathcal{M}_1^{\mathcal{T}} \longrightarrow \mathcal{M}_1^{\mathcal{U}} \\ & i_{E_0^{\mathcal{T}}}(f)(a) \longmapsto i_{E_0^{\mathcal{U}}}(\rho_0(f))(\rho_0(a)) \end{aligned}$$

Let us show that this map is an embedding. Let $\varphi$ be a formula.

$$\begin{aligned}\mathcal{M}_1^{\mathcal{T}} \vDash \varphi(i_{E_0^{\mathcal{T}}}(f)(a)) &\iff \{u; K \vDash \varphi(f(u))\} \in (E_0^{\mathcal{T}})_a \\ &\iff \{u; \mathrm{Ult}(K,E) \vDash \varphi(\rho_0(f)(u))\} \in \rho_0(E_0^{\mathcal{T}})_{\rho_0(a)} = (E_0^{\mathcal{U}})_{\rho_0(a)} \\ &\iff \mathcal{M}_1^{\mathcal{U}} \vDash \varphi(i_{E_0^{\mathcal{U}}}(\rho_0(f))(\rho_0(a)))\end{aligned}$$

We claim that $E$ is the $(\kappa, \mathrm{lh}(E))$-extender derived from $\rho_1$. Let $\xi \in \rho_1(X)$ with $\xi < \mathrm{lh}(E)$ and $X \subseteq \mathrm{cp}(E)$. If we prove that:

$$\xi \in \rho_1(X) \iff \xi \in \rho_0(X),$$

we are finished. For $X \subseteq \mathrm{OR}$, let us call $f_X$ the function that maps a $\gamma < \kappa$ to $X \cap \gamma$. Since $\mathcal{P}(\kappa) \cap \mathcal{M}_1^{\mathcal{T}} = \mathcal{P}(\kappa) \cap K$ and $X = i_{E_0^{\mathcal{T}}}(f_X)(\kappa)$.

$$\begin{aligned}\rho_1(X) = \rho_1(i_{E_0^{\mathcal{T}}}(f_X)(\kappa)) &= i_{E_0^{\mathcal{U}}}(f_{\rho_0(X)})(\rho_0(\kappa)) \\ &= f_{i_{E_0^{\mathcal{U}}} \circ \rho_0(X)}(\rho_0(\kappa)) \\ &= i_{E_0^{\mathcal{U}}} \circ \rho_0(X) \cap \rho_0(\kappa) \\ &= \rho_0(X) \cap \rho_0(\kappa)\end{aligned}$$

The last equality holds, since $\rho_0(\kappa) = \mathrm{cp}(i_0^{\mathcal{U}})$. Now since $\xi < \mathrm{lh}(E) < \rho_0(\kappa)$ we have that, for $\xi < \mathrm{lh}(E)$:

$$\xi \in \rho_1(X) \iff \xi \in \rho_0(X) \cap \rho_0(\kappa) \iff \xi \in \rho_0(X)$$

As $E$ is the extender derived from $\rho_1$, there is a unique map $h_1$ such that the diagram commutes.

Inductively we can extend the diagram to:

$$\begin{tikzcd}
K \arrow[r, "i_{E_0^{\mathcal{T}}}"'] \arrow[dd, "\rho_0"] & \mathcal{M}_1^{\mathcal{T}} \arrow[rr] \arrow[dd, "\rho_1"] \arrow[dr] & & \mathcal{M}_\infty^{\mathcal{T}} \arrow[dd, "\rho_\infty"] \arrow[dr] & \\
& & \mathrm{Ult}(\mathcal{M}_1^{\mathcal{T}}, E) \arrow[dl, dotted, "h_1"] & & \mathrm{Ult}(\mathcal{M}_\infty^{\mathcal{T}}, E) \arrow[dl, dotted] \\
\mathrm{Ult}(K,E) \arrow[r, "i_{E_0^{\mathcal{U}}}"'] & \mathcal{M}_1^{\mathcal{U}} \arrow[rr] & & \mathcal{M}_\infty^{\mathcal{U}} &
\end{tikzcd}$$

$E$ is the extender derived from each $\rho_\alpha$. But $\mathcal{M}_\infty^{\mathcal{T}} = K^M$, hence we can embed $\mathrm{Ult}(K^M, E)$ into $\mathcal{M}_\infty^{\mathcal{U}}$. As $\mathcal{M}_\infty^{\mathcal{U}}$ is an iterate of $K$, it is iterable, and thus $\mathrm{Ult}(K^M, E)$ is iterable itself. $\dashv$

A strengthening of this result has been found by Ralf Schindler in [Sch, Theorem 5.2]:

**Theorem 3.16.** *Suppose that there is no inner model with a Woodin cardinal, and let $K$ denote the Stell-Mitchell core model. Assume $\kappa$ to be such that there is a precipitous ideal on $\kappa$. Then $\kappa^{+K} = \kappa^{+V}$.*

## 3.4 Core model induction

In this section we want to show that, assuming a stronger hypothesis $\mathrm{BPFA}^{\mathrm{uB}}$, we can do a core model induction and prove the closure of the universe under the $\mathcal{M}_n^\#$ operator for every $n$.

**Theorem 3.17.** *Suppose* $\mathrm{BPFA}^{uB}$ *holds and that there there is a precipitous ideal on* $\omega_1$*. Then Projective determinacy holds.*

The proof will go through the next subsections. We first prove the closure under the sharp operator.

### 3.4.1 Below a Woodin

**Lemma 3.18.** *Suppose there is a precipitous ideal on* $\omega_1$*. Then* $\mathbb{R}$ *is closed under #'s.*

Proof. Let $I$ be a precipitous ideal on $\omega_1$ and let $G$ be a $\mathcal{P}(\omega_1)\smallsetminus I$-generic over $V$. Let $j : V \to \mathrm{Ult}(V, G)$ be the associated ultrapower map. For any $x \in \mathbb{R}$ the restriction of $j$ to $L[x]$ gives an elementary embedding from $L[x]$ into itself. Hence $x^\#$ exists in $V[G]$. But sharps can't be added by forcing, hence $x^\# \in V$. ⊣

**Lemma 3.19.** *Suppose there is a precipitous ideal on* $\omega_1$*. Then* $H_{\omega_2}$ *is closed under #'s.*

Proof. Let $I$ be a precipitous ideal on $\omega_1$ and let $G$ be a $\mathcal{P}(\omega_1)\smallsetminus I$-generic over $V$. Let $j : V \to \mathrm{Ult}(V, G) = M$ be the associated ultrapower map. Let $A \subseteq \omega_1$. By elementarity $\mathbb{R}$ is closed under sharps in $M$, on the other side $A = j(A) \cap \omega_1^V \in \mathbb{R}^M$. Hence $A^\#$ exists in $V[G]$ and again in $V$ since forcing can't add sharps. ⊣

**Lemma 3.20.** *Suppose* BPFA *holds and that there is a precipitous ideal on* $\omega_1$*, then* $V$ *is closed under #'s.*

Proof. This follows from Lemma 3.19 and [Sch, Theorem 0.1]. ⊣

### 3.4.2 Some toolboxes

In order for the proof to go smoothly in the higher case, let us make some observation on the absoluteness of iterability and some mouse operators between class sized models. Especially between $M$ and $V[G]$ where $G$ is $\mathcal{P}(\omega_1)\smallsetminus I$-generic over $V$ and $M = \mathrm{Ult}(V, G)$, where $I$ is a precipitous ideal on $\omega_1$.

Let us fix the following statements:

i. $A_n \equiv$ "$V$ is closed under $\mathcal{M}_n^\#$"

ii. $(i)_n \equiv$ "$\mathbb{R}$ is closed under $\mathcal{M}_n^\#$"

iii. $(ii)_n \equiv$ "$H_{\omega_2}$ is closed under $\mathcal{M}_n^\#$"

The good thing about $\omega_1$-iterability is that it is absolute in a strong sense:

**Lemma 3.21.** *Suppose $A_n$ holds. Let $x \in \mathbb{R}$ and suppose that $\mathcal{M}^{\#}_{n+1}(x)$ does not exist. Let $\mathcal{P}$ be a $(n+1)$-small countable $x$-premouse with no definably Woodin cardinal. $\mathcal{P}$ is $\omega_1$-iterable if and only if one of the following condition holds:*

*i. $n$ is even and $\mathcal{M}^{\#}_n(\mathcal{P}) \models$ "$\mathcal{P}$ is $\omega_1$-iterable",*

*ii. $n$ is odd and $\mathcal{M}^{\#}_n(\mathcal{P})[G] \models$ "$\mathcal{P}$ is $\omega_1$-iterable", where $G$ is $\mathrm{col}(\omega, \delta_0)$-generic over $\mathcal{M}^{\#}_n(\mathcal{P})$, $\delta_0$ being the smallest Woodin cardinal of $\mathcal{M}^{\#}_n(\mathcal{P})$.*

If one of the two above condition holds, we say that $\mathcal{M}^{\#}_n(x)$ *witness the iterability of $x$* let us give a sketch of the proof:

Proof. We consider first the case $n$ is even.

Suppose $A_n$ and that $\mathcal{M}^{\#}_{n+1}(x)$ does not exist. By [Ste95, Corollary 4.9] and [KMS83]:

$$\mathcal{M}^{\#}_n \prec_{\utilde{\Sigma}^1_{n+2}} V.$$

Moreover as $\mathcal{M}^{\#}_{n+1}(\mathcal{P})$ does not exist, "$\mathcal{P}$ is not $\omega_1$-iterable" is a $\utilde{\Sigma}^1_{n+2}$ statement by [Ste95, Lemma 1.5]. Hence we have the equivalence we wanted to show.

Now let us look at the odd case. Assume $A_{n+1}$, where $n$ is even. Suppose $V \models$ "$\mathcal{P}$ is not $\omega_1$-iterable". We have already discussed that "$\mathcal{P}$ is not $\omega_1$-iterable" is a $\utilde{\Sigma}^1_{n+3}$ statement, say $\exists z \varphi(z)$. Let $y$ be a real such that $V \models \varphi(y)$. By Woodin's genericity iteration (cf. [Ste, Theorem 7.14]), $y$ is $\mathrm{col}(\omega, \delta_0)$-generic over $\mathcal{M}^{\#}_{n+1}(\mathcal{P})$, where $\delta_0$ is the last Woodin of $\mathcal{M}^{\#}_{n+1}(\mathcal{P})$. But we can rewrite $\mathcal{M}^{\#}_{n+1}(\mathcal{P})[y]$ as some $\mathcal{M}^{\#}_n(X)$ which, by hypothesis, is $\utilde{\Sigma}^1_{n+2}$ absolute, hence $\mathcal{M}^{\#}_{n+1}(\mathcal{P})[y] \models \varphi(y)$. In turn, this implies that $\mathcal{M}^{\#}_{n+1}(\mathcal{P})[y] \models$ "$\mathcal{P}$ is not $\omega_1$-iterable".

Now suppose that $V \models$ "$\mathcal{P}$ is $\omega_1$-iterable". For every iteration $\mathcal{T}$ of limit length, doing a $L[E, \mathcal{M}(\mathcal{T})]$ construction inside $\mathcal{M}^{\#}_{n+1}(\mathcal{P})$ give rise to a $Q$-structure for $\mathcal{T}$ of $\mathcal{P}$ inside $\mathcal{M}^{\#}_{n+1}(\mathcal{P})$. But if the $Q$-structure exists, we already have argued that the existence of a branch is only a $\utilde{\Sigma}^1_1$-fact, hence it is also true in $\mathcal{M}^{\#}_{n+1}(\mathcal{P})$. This shows that $\mathcal{P}$ is $\omega_1$-iterable in $\mathcal{M}^{\#}_{n+1}(\mathcal{P})$ as well. ⊣

**Lemma 3.22.** *Suppose $A_n$ holds and that $I$ is a precipitous ideal on $\omega_1$. Let $G$ be $\mathcal{P}(\omega_1) \setminus I$-generic over $V$ and $M = \mathrm{Ult}(V, G)$ the generic ultrapower. Let $X \subseteq \omega_1$ and $M \models \mathcal{P} = \mathcal{M}^{\#}_n(X)$, then $V[G] \models \mathcal{P} = \mathcal{M}^{\#}_n(X)$ and $\mathcal{P}$ is iterable in $V[G]$.*

Proof. We proceed by induction. Let $\Phi_n$ be the statement:

> "There is tree $T \in M$ such that
>
> $$V[G] \models p[T] = \{(x, y);\ y \text{ is a code for } \mathcal{M}^{\#}_n(x)\}\text{"}$$

and let $\Psi_n$ be the statement:

> "for $X \subseteq \omega_1$, if $M \models \mathcal{P} = \mathcal{M}^{\#}_n(X)$ then $V[G] \models \mathcal{P} = \mathcal{M}^{\#}_n(X)$ and $\mathcal{P}$ is iterable in $V[G]$".

We will show that:

$$A_n + \Psi_n \Rightarrow \Phi_n \text{ and } A_{n+1} + \Phi_n \Rightarrow \Psi_{n+1}.$$

$\Psi_0$ is trivial, let us prove $\Phi_0$. Assume that the universe is closed under sharps. Let $T_0$ be the tree searching for

i. a fully elementary map $\pi : \bar{H} \to j((H_\theta^V)^\#)$

ii. some $x$, small generic over $\bar{H}$ and

iii. some $y \in \mathbb{R}$ that codes an $x$-sound $x$-premouse derivable from $\bar{H}$.

Suppose $V[G] \models$ "$y$ codes $x^\#$". Let $\tau$ be a name for $x$ and $\theta$ large enough such that $\mathcal{P}(\omega_1) \smallsetminus I \in H_\theta^V$. Notice that by $\Psi_0$,

$$j((H_\theta^V)^\#) = j(H_\theta^V)^\#.$$

Since $x$ is countable in $V[G]$, we can find a countable substructure $X$ of $j(H_\theta^V)^\#[G]$ with $x, G, \mathcal{P}(\omega_1) \smallsetminus I \in X$. Notice that $x$ is small generic over $j(H_\theta^V)^\#$ by construction. Let $N$ be the transitive collaps of $X$ and $\pi$ the uncollapsing map. $N$ is of the form $\bar{H}[g]$, where $g = \pi^{-1}(G)$. As $x$ is countable, $\pi^{-1}(x) = x$ and we have $x \in \bar{H}[g]$. By elementarity, $x$ is small generic over $\bar{H}$. On the other side, since $\pi \restriction \bar{H} : \bar{H} \to j(H_\theta^V)^\#$ is an elementary embedding, we can embed $\bar{H}$ into a sharp. Hence $\bar{H}$ is a sharp. But then we can derive $y$ from $\bar{H}$ and thus $(x, y) \in p[T_0]$.

Now let $(x, y) \in p[T_0]$, since $\bar{H}$ is embeddable in $j((H_\theta^V)^\#)$ via $\pi$, it is truly a sharp, and thus $V[G] \models$ "$y$ is a code for $x^\#$".

Let us prove "$A_{n+1} + \Phi_n \Rightarrow \Psi_{n+1}$" now. Suppose $T_n$ is the tree given by $\Phi_n$. Let $M \models \mathcal{M} = \mathcal{M}_{n+1}^\#(X)$, for some $X \subseteq \mathrm{OR}$. We already know that $\mathcal{M}$ is $X$-sound and projects to $X$. Moreover, every initial segment of $\mathcal{M}$ is $(n+1)$-small, hence if we can prove that $\mathcal{M}$ is iterable in $V[G]$, we will automatically get that $V[G] \models \mathcal{P} = \mathcal{M}_{n+1}^\#(X)$. Let us define the tree $U_{n+1}^{\mathcal{M}}$ searching for a countable submodel of $\mathcal{M}$ witnessing its non-iterability. For the rest of the argument we will drop the index $\mathcal{M}$, that is $U_{n+1} = U_{n+1}^{\mathcal{M}}$. $U_{n+1}$ is the tree searching for:

i. a countable model $\bar{M}$ of $\mathsf{ZFC}^-$

ii. a countable mouse $\mathcal{P} \in \bar{M}$,

iii. an elementary map $\sigma : \mathcal{P} \to \mathcal{M}$,

iv. an iteration tree $\mathcal{T} \in \bar{M}$ on $\mathcal{P}$ played according to the iteration strategy $\Sigma_{\mathcal{P}}^t$ such that either:

   a) the last model is ill-founded,

   b) in $\bar{M}$ there is no cofinal branch $b$ such that $\mathcal{Q}(b, \mathcal{T})$ exists and

$$\mathcal{Q}(b, \mathcal{T}) \triangleleft \mathcal{M}_n^\#(\mathcal{M}(\mathcal{T})).$$

Remark first that by absoluteness of the strategy $\Sigma^t$ and by the uniqueness of the branches, if $\mathcal{T}$ is an iteration tree played according to $\Sigma^t$ in some $\bar{M}$ as above, it is an iteration tree played according to $\Sigma^t$ in $V$. Hence by $\underset{\sim}{\Pi}^1_1$ absoluteness, the existence of a branch for $\mathcal{T}$ in $\bar{M}$ is equivalent to the existence of a branch in $V$, as long as the $Q$-structure for the tree exists in $\bar{M}$ (and in $V$). As the $Q$-structure is given by an initial segment of $\mathcal{M}_n^\#(\mathcal{M}(\mathcal{T}))$, searching for $\mathcal{M}_n^\#(\mathcal{M}(\mathcal{T}))$ will solve that matter. Notice that $\mathcal{M}(\mathcal{T})$ does not depend on $b$, hence we can simultaneously search for $\mathcal{M}(\mathcal{T})$ and some $y$ such that if $(\mathcal{M}(\mathcal{T}), y) \in T_n$, that is every part of the finite attempt to describe $(\mathcal{M}(\mathcal{T}), y)$ is a point in $T_n$ extending the previous one in $T_n$. The rest of the construction of $U_{n+1}$ is just as usual.

Let us now prove that $U_{n+1}$ gives the desired result. Suppose $X \subseteq \mathrm{OR}$ and that $M \vDash \mathcal{P} = \mathcal{M}^\#_{n+1}(X)$. If $\mathcal{P}$ is not iterable in $V[G]$, let $\mathcal{T}$ be a tree on $\mathcal{P}$ such that either, its last model is ill-founded, or there are no branch $b$ through the tree with $\mathcal{Q}(b,\mathcal{T}) \lhd \mathcal{M}_n^\#(\mathcal{M}(\mathcal{T}))$. Let $\theta$ be large enough such that $\mathcal{T} \in H_\theta$ and let $Y \prec H_\theta$ be a submodel with $\mathcal{P}, \mathcal{T} \in Y$. In the transitive collapse of $Y$ there is a countable substructure $\bar{\mathcal{P}} \prec \mathcal{P}$ and a tree $\mathcal{T}$ such that, either the last model of $\mathcal{T}$ is ill-founded or $\mathcal{T}$ is of limit length and has no cofinal branch $b$ with a $Q$-structure that is iterable above $\delta(\mathcal{T})$ . Suppose the later holds, we claim that having a $Q$-structure is equivalent to condition iv. b)). If $\mathcal{Q}(b,\mathcal{T}) \lhd \mathcal{M}_n^\#(\mathcal{M}(\mathcal{T}))$ then $\mathcal{Q}(b,\mathcal{T})$ is iterable above $\delta(\mathcal{T})$ and hence a $Q$-structure. Suppose $\mathcal{Q}(b,\mathcal{T})$ is a $Q$-structure, hence it is iterable and $\delta(\mathcal{T})$-sound. We can compare $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{M}_n^\#(\mathcal{M}(\mathcal{T}))$. As $\delta(\mathcal{T})$ is a cut point for both models and they agree below $\delta(\mathcal{T})$, the coiteration is above $\delta(\mathcal{T})$ and thus they are lined up. Since $\mathcal{P}$ is a substructure of $\mathcal{M}^\#_{n+1}$ every initial segment of an iterate of $\mathcal{P}$ is $(n+1)$-small, hence $\mathcal{Q}(b,\mathcal{T})$ is $(n+1)$-small and $\mathcal{Q}(b,\mathcal{T}) \lhd \mathcal{M}_n^\#(\mathcal{M}(\mathcal{T}))$. This shows that the tree $U_{n+1}$ is ill-founded in $V[G]$ and thus ill-founded in $M$ as well. Let $\mathcal{P}$ be the countable substructure of $\mathcal{M}$ described by a branch of $U_{n+1}$ in $M$. On one hand $\mathcal{P}$ should be iterable, on the other $U_{n+1}$ describes an iteration tree $\mathcal{T}$ witnessing the non-iterability of $\mathcal{P}$ a contradiction!

Now we can prove that "$A_n + \Psi_n \Rightarrow \Phi_n$": Let $U_n$ be the tree constructed above witnessing the iterability of some mouse $\mathcal{M}$. We construct $T_n$ as the tree searching for:

i. a fully elementary map $\pi : H \to j(\mathcal{M}_n^\#(H^V_\theta))$

ii. some $x$, small generic over $H$ and

iii. some $y \in \mathbb{R}$ that codes an $x$-sound $x$-premouse derivable from $H[x]$.

$T_n$ is clearly in $M$ as well as in $V[G]$. Suppose that $(x,y) \in [T_n]$. Then by construction $y$ is a $x$-sound $x$-premouse, moreover every initial segment of $y$ is $n$-small. $j(\mathcal{M}_n^\#(H^V_\theta))$ is iterable in $M$, by $\Psi_n$ it is iterable in $V[G]$ as well. Since $H$ is an elementary substructure of $\mathcal{M}_n^\#(H^V_\theta)$, there is a $\bar{H}$ such that $H = \mathcal{M}_n^\#(\bar{H})$. As $y$ is derived from $\mathcal{M}_n^\#(\bar{H})$, it is iterable in $V[G]$. This shows that $y$ is indeed $\mathcal{M}_n^\#(x)$.

Now suppose that $x \in \mathcal{P}(\omega_1) \cap V[G]$ and $y$ codes $\mathcal{M}_n^\#(x)$. We have to show that $(x,y) \in p[T_n]$. $x$ is generic over some $H^M_\eta$, let $\theta$ be large enough such that $x$ is generic over $j(H^V_\theta)$ as well and $\mathcal{P}(\omega_1) \setminus I \in H^V_\theta$. Let $X$ be a countable substructure of $j(\mathcal{M}_n^\#(H^V_\theta))[G]$ such that $x, G, \mathcal{P}(\omega_1) \smallsetminus I \in X$. Let $N$ be the transitive collapse of $X$ and $\pi$ the uncollapsing map. By construction, there is some $g$ such that $N = H[g]$, moreover $x$ is small generic over $H$. We have that $H \prec j(\mathcal{M}_n^\#(H_\theta^V))$, $H \in V[G]$. But then there is again a $\bar{H}$ such that $\mathcal{M}_n^\#(\bar{H}) = H$. As $x$ is small generic over $\mathcal{M}_n^\#(\bar{H})$ and as $y$ codes $\mathcal{M}_n^\#(x)$, $y$ codes an $x$-sound $x$-premouse derivable from $\mathcal{M}_n^\#(\bar{H})[x]$. Hence $\pi \restriction H, x, y$ give a branch of $T$ and $(x, y) \in p[T_n]$. $\dashv$

The previous lemma actually showed that:

**Lemma 3.23.** *Suppose $A_n$ holds and that $I$ is a precipitous ideal on $\omega_1$. Let $G$ be $\mathcal{P}(\omega_1) \smallsetminus I$-generic over $V$ and $M = \mathrm{Ult}(V, G)$ the generic ultrapower. Let $X \subseteq \omega_1$ and $\mathcal{M}$ an $X$-premouse such that either $\mathcal{M}$ is $(n+1)$-small and $M \vDash$ "$\mathcal{M}$ is iterable" or $M \vDash$ "$\mathcal{M} = \mathcal{M}_{n+1}^\#(X)$". Then $V[G] \vDash$ "$\mathcal{M}$ is iterable".*

If we look carefully at the $\Phi_n \Rightarrow \Psi_{n+1}$ argument in the proof of Lemma 3.22, we didn't need $A_{n+1}$ for the construction of the tree. Hence in case $\mathcal{P} = \mathcal{M}_{n+1}^\#(X)$ it is sufficient to have $A_n$. Hence if $\mathcal{M}$ is $(n+1)$-small or equal to $\mathcal{M}_{n+1}^\#$ we can go through the rest of the proof with the weakened hypothesis.

The proof of Lemma 3.22 also shows that:

**Lemma 3.24.** *Suppose $A_n$ holds. The set $U_n = \{(x, y);\ y \text{ is a code for } \mathcal{M}_n^\#(x)\} \subseteq \mathbb{R}^2$ is a universally Baire set.*

### 3.4.3 The induction

We prove the main theorem by induction. The proof of our main theorem will follow the following "strategy": Assuming $\mathrm{BPFA}^{\mathrm{uB}}$ and the existence of a precipitous ideal, we will show that:

$$A_n \Rightarrow A_n + (i)_{n+1} \Rightarrow A_n + (ii)_{n+1} \Rightarrow A_{n+1}$$

**Lemma 3.25.** *Suppose* $\mathrm{BPFA}^{uB}$ *and $A_n$ holds and that there is a precipitous ideal on $\omega_1$, then $(i)_{n+1}$.*

Proof. Will will prove the lemma by contradiction, let us suppose that there is an $x \in \mathbb{R}$ such that $K(x)$ exist, is $(n+1)$-small and has no Woodin cardinals. We first want to use [Sch, Theorem 5.2], saying that if we have a precipitous ideal on $\kappa$, then $\kappa^{+K(x)} = \kappa^{+V}$. There is one key fact we have to check, in order for that proof to go through in our case: the iterability of $K^M(x)$. If it wasn't, there would be an initial segment, say $K^M(x)\|\theta$ that is not iterable. This would be a contradiction to Lemma 3.23. Hence we can follow the same argument as in [Sch, Theorem 5.2] and we can suppose that:

$$\omega_1^{+K(x)} = \omega_1^{+V}.$$

Now we follow the same proof as in Lemma 3.12. By the previous argument, we know that $\omega_1^{+K(x)} = \omega_2$. Let $\mathcal{M} = K(x)\|\omega_2$, let $\mathbb{P}$ be the forcing adding a club $C$ of order-type $\omega_1$ and let $\mathbb{Q}$ be the forcing in $V^{\mathbb{P}}$ specializing the tree arising from the square-sequence of $\mathcal{M}$ restricted to ordinals in $C$. $\mathbb{P} * \mathbb{Q}$ is a proper forcing. Let $G$ be $\mathbb{P} * \mathbb{Q}$-generic over $V$. In $V[G]$ there is a mouse $\mathcal{M}$ with a height of cofinality $\omega_1$, which is an end extension of $K(x)\|\omega_1$ such that the restriction of the square-sequence of $\mathcal{M}$ to some club $C$ is specialized. Let us take a closer look at the complexity of this statement.

Let $\Psi$ be the following formula:

> there is a $\mathcal{M}$ a club $C \subseteq \mathcal{M} \cap \mathrm{OR}$ and a $f : \omega_1 \to \omega$ with the following properties:
>
> i. $\mathcal{M}$ is a $(n+1)$-small premouse,
>
> ii. $K(x)\|\omega_1 \lhd \mathcal{M}$,
>
> iii. $\mathcal{M}$ has a largest cardinal $\omega_1$,
>
> iv. the height of $\mathcal{M}$ has cofinality $\omega_1$,
>
> v. $f$ specialize the square-sequence of $\mathcal{M}$ restrcted to $C$,
>
> vi. $\mathcal{M}$ has the strong condensation property,
>
> vii. for every $y \prec \mathcal{M}$ countable, $\mathcal{M}_n^{\#}(y)$ witness the $\omega_1$-iterability of $y$.

We claim that $\Psi$ is a $\Sigma_1$ formula over $H_{\omega_2}^{V[G]}$ with the parameter $U_n$. Obviously we have to check item vi and vii. Let $\langle \lambda_i, i < \omega_1 \rangle$ be a sequence of ordinals cofinal in $\mathcal{M} \cap \mathrm{OR}$ such that each $\mathcal{M}_i = \mathcal{M}\|\lambda_i$ projects to $\omega_1$. Every countable submodel of $\mathcal{M}_i$ has to be the transitive collaps of $\mathrm{Hull}^{\mathcal{M}_i}(\alpha \cup \{p(\mathcal{M}_i)\})$, for some $\alpha$. This shows that all the collapses of countable substructures of $\mathcal{M}$ are initial segments of some $\mathrm{Hull}^{\mathcal{M}_i}(\alpha \cup \{p(\mathcal{M}_i)\})$, for some $\alpha, i < \omega_1$. We can rewrite item vi. and vii. by the formula $\Psi'$:

> there is a sequence $\langle \lambda_i;\ i < \omega_1 \rangle$ such that
>
> i. $\langle \lambda_i;\ i < \omega_1 \rangle$ is cofinal in $\mathcal{M} \cap \mathrm{OR}$,
>
> ii. each $\mathcal{M}\|\lambda_i$ projects to $\omega_1$,
>
> iii. for every $\alpha < \omega_1$ if $\mathcal{N}$ is the transitive collapse of $\mathrm{Hull}^{\mathcal{M}\|\lambda_i}(\alpha \cup \{p(\mathcal{M}\|\lambda_i)\})$ then $\mathcal{N} \lhd \mathcal{M}$,
>
> iv. for every $\alpha < \omega_1$ if $z \in \mathbb{R}$ is a code for $\mathcal{N}$, where $\mathcal{N}$ is the transitive collapse of $\mathrm{Hull}^{\mathcal{M}\|\lambda_i}(\alpha \cup \{p(\mathcal{M}\|\lambda_i)\})$ then for all $y$ such that $(z, y) \in U_n$, $y$ witness the $\omega_1$-iterability of $z$.

$\Psi'$ is a $\Sigma_1^{H_{\omega_2}}$ formula with parameters $\mathcal{M}$, $\omega_1$, $K\|\omega_1$ and $U_n$. Let us prove that $\Psi'$ equivalent to item vi. and vii. provided the other item of $\Psi$ holds true.

It is clear that item vi. and vii. implies $\Psi'$. Let us now suppose that $\Psi'$ holds. Let $\mathcal{P}$ be a countable premice and $\sigma : \mathcal{P} \to \mathcal{M}$ be an embedding. Since the cofinality of $\mathcal{M}$ is uncountable, Setting $\lambda = \sup \sigma''\mathcal{P} \cap \mathrm{OR}$, $\sigma : \mathcal{P} \to \mathcal{M}\|\lambda$ is also an embedding. Remark that since $x \in \mathbb{R}$, $\mathcal{P}$ is an $x$-mouse. Let $\lambda_i > \lambda$ and let $\alpha = \omega_1^{\mathcal{P}}$ be the critical point of $\sigma$. We have that $\mathcal{P} \lhd \mathcal{N}$, where $\mathcal{N} \cong \mathrm{Hull}^{\mathcal{M}|\ \lambda_i}(\alpha \cup \{p(\mathcal{M}|\ \lambda_i)\})$. By $\Psi'$ $\mathcal{N}$ is $\omega_1$-iterable, as witnessed by $\mathcal{M}_n^{\#}(\mathcal{N})$ thus $\mathcal{P}$ is $\omega_1$-iterable as well, hence it is also witnessed by $\mathcal{M}_n^{\#}(\mathcal{P})$. By $\Psi'$ we also have that $\mathcal{N} \lhd \mathcal{M}$ hence, $\mathcal{P} \lhd \mathcal{M}$.

This shows that $\Psi$ is a $\Sigma_1$ formula in parameters $\omega_1$, $U_n$ in $H_{\omega_2}$. $\Psi$ is true in $V[G]$ as witnessed by $K(x)\|\omega_2$, $C$ and $f$, where $C$ is the club added by $\mathbb{P}$ and $f$ the function added by $\mathbb{Q}$. As $\mathbb{P} * \mathbb{Q}$ is proper, by BPFA$^{\mathrm{uB}}$, $\Psi$ holds in $V$ as well. Let $\mathcal{M}$ and $f$ be such that $\Psi(\mathcal{M}, f)$ holds in $V$.

We claim that $\mathcal{M}$ and $K(x)$ are lined up. Suppose not and let $M$ be a countable substructure of $H_{\omega_3}$ containing $\mathcal{M}$ and $K(x)\|\omega_2$. Let $\overline{\mathcal{M}}, \overline{K} \in \overline{M}$ be such that

$$\sigma(\overline{\mathcal{M}}, \overline{K}) = (\mathcal{M}, K\|\omega_2),$$

where $\sigma : \overline{M} \to M$ is the uncollapsing map. By $\Psi$, $\overline{\mathcal{M}}$ is an initial segment of $\mathcal{M}$ but since $\overline{\mathcal{M}}$ is countable and $K(x)\|\omega_1 \lhd \mathcal{M}$, we have that $\overline{\mathcal{M}} \lhd K(x)$, on the other side by condensation $\overline{K} \lhd K$. Hence $\overline{\mathcal{M}}$ and $\overline{K}$ are lined up, but $\overline{M}$ believes that there are not, a contradiction.

Since $\mathcal{M}$ has cardinality $\omega_1$, we must have that $\mathcal{M} \lhd K(x)\|\omega_2$. This implies that the $\square_{\omega_1}$-sequence of $K(x)\|\omega_2$ is extending the $\square_{\omega_1}$-sequence of $\mathcal{M}$, a contradiction as the $\square_{\omega_1}$-sequence of $\mathcal{M}$ has a specializing function. Hence $K(x)$ does not exist and thus by $K$-existence dichotomy $\mathcal{M}^{\#}_{n+1}(x)$ exists. $\dashv$

**Lemma 3.26.** *Suppose* BPFA$^{uB}$*,* $A_n$ *and* $(i)_{n+1}$ *holds and that there is a precipitous ideal on* $\omega_1$*. Then* $(ii)_{n+1}$ *hold.*

Proof. Let $X \in H_{\omega_2}$ be the set such that $K(X)$ exists, is $n$-small and has no Woodin cardinals. Let $I$ be a precipitous ideal on $\omega_1$ and $G$ $\mathcal{P}(\omega_1) \smallsetminus I$-generic over $V$. Let $j : V \to \mathrm{Ult}(V, G)$ be the associated ultrapower map. By elementarity $(i)_{n+1}$ holds in $M$. As $X$ is a real in $M$, $\mathcal{M}^{\#}_{n+1}(X)$ exists in $M$ and is iterable in $M$. As $A_n$ holds we have by Lemma 3.23 that $\mathcal{M}^{\#}_{n+1}(X)$ is iterable in $V[G]$. Since $K^V(X) = K^{V[G]}(X)$ by generic absoluteness, it is also iterable in $V[G]$. Let $\langle \mathcal{T}, \mathcal{U} \rangle$ be the coiteration of $\mathcal{M}^{\#}_{n+1}(X)$ with $K(X)$. As $K(X)$ is universal, the $\mathcal{M}^{\#}_{n+1}(X)$ side of the coiteration has no drop. Since $K(X)$ is $(n+1)$-small, at least the critical point $\kappa$ of the top extender of $\mathcal{M}^{\#}_{n+1}(X)$ has to be iterated out of the universe, a contradiction to the universality of $K(X)$. $\dashv$

**Lemma 3.27.** *Suppose* BPFA$^{uB}$*,* $A_n$ *and* $(ii)_{n+1}$ *hold and that there is a precipitous ideal on* $\omega_1$*. Then* $A_{n+1}$ *hold.*

Proof. Let $X \subseteq \mathrm{OR}$ be such that $\mathcal{M}^{\#}_{n+1}(X)$ does not exist. Without loss of generality we suppose that $\kappa = \sup(X)$ is a cardinal. Let $\mathbb{P}^*$ be the $\omega_1$-closed forcing adding a surjection $g : \omega_2 \to \kappa$. Let us first work in $V[g]$. Remark that $g$ is generic over each $X$-mouse $\mathcal{M}$, hence we can work with $\mathcal{M}[g]$ which we can reorganize as an $X'$-mouse, where $X' \subseteq \omega_2$. Since we can also form $K(X)[g]$, we have that $K(X')$ exist and thus, by $K$-existence dichotomy, $\mathcal{M}^{\#}_{n+1}(X')$ does not exist. Let $S(X')$ be the stack of all $X'$-mice which are $\omega_2$-sound and projects to or below $\omega_2$ (for more on stacks see [JSSS09]). That is $\mathcal{P} \lhd S(X')$ if there is a $\mathcal{Q} \rhd \mathcal{P}$ such that $\mathcal{Q}$ is an $X'$-mouse which is sound above $\omega_2$ and projects to $\omega_2$. $S(X')$ is an $X'$-mouse, $S(X') \vDash \mathsf{ZFC}^-$ and its largest cardinal is $\omega_2$. Fix $\lambda = \mathrm{OR} \cap S(X')$.

Claim 1. $\mathrm{cf}(\lambda) \geqslant \omega_2$

Proof. Suppose not and let $Y$ be a substructure of $S(X')$ of size $\omega_1$ such that the height of $Y$ is cofinal in $\lambda$. Let $\pi : \mathcal{N} \to Y$ be the uncollapsing map. Then $\mathcal{N}$ is some $\bar{X}$-mouse where $\bar{X} = \pi^{-1}(X')$. $\bar{X}$ can be coded as a subset of $\omega_1$, since the forcing was $\omega_1$-closed $\bar{X} \in V$ and thus by $(ii)_{n+1}$, $\mathcal{M}^{\#}_{n+1}(\bar{X})$ exists in $V$. But we have already seen that $\mathcal{M}^{\#}_{n+1}(\bar{X})$ is absolute between forcing extensions, hence $\mathcal{M}^{\#}_{n+1}(\bar{X})$ is iterable in $V[g]$ as well. This implies that $\mathcal{N} \lhd \mathcal{M}^{\#}_{n+1}(\bar{X})$ as $\mathcal{N}$ is $(n+1)$-small. Let $\mathcal{Q}$ be the minimal $\bar{X}$-mouse such that $\mathcal{N} \lhd \mathcal{Q} \lhd \mathcal{M}^{\#}_{n+1}(\bar{X})$, $\mathcal{Q}$ is $\bar{X}$-sound and projects to $\bar{X}$. Let $n < \omega$ be such that $\rho_{\mathcal{Q}}(n+1) \leq \sup(\bar{X}) < \rho_{\mathcal{Q}}(n)$. Let $E$ be the extender derived by $\pi$. By [MS95], if we let

$$\mathcal{Q}^* = \mathrm{Ult}_n(\mathcal{Q}, E)$$

be the ultrapower by the long extender $E$, $\mathcal{Q}^*$ is an $X'$-mouse and is iterable. Since we chose $Y$ such that $\sup(Y \cap \mathrm{OR}) = \lambda$, we have that $\mathrm{ran}(\pi) \cap \mathrm{OR}$ is cofinal in $\lambda = S(X') \cap \mathrm{OR}$. This implies that $\mathcal{Q}^* \rhd S(X')$. But $\mathcal{Q}^*$ is an $X'$-sound $X'$-mouse that projects to $X'$, a contradiction to the definition of the stack $S(X')$. ⊣

Again we can look at the forcing $\mathbb{P}$ adding a club of order-type $\omega_1$ in $\lambda$ and $\mathbb{Q}$ the forcing specializing the tree arising from the square-sequence of $S(X')$ restricted to elements in the club added by $\mathbb{P}$. Let $G$ be $\mathbb{P} * \mathbb{Q}$-generic over $V[g]$.

Let $\Psi$ be the formula:

> there is a $X \subseteq \omega_1$, a $\mathcal{M}$, a club $C$ in $\mathcal{M} \cap \mathrm{OR}$ and a $f$ such that $X \subseteq \omega_1$, $\mathcal{M}$ is an iterable $X$-premouse, $\mathrm{cf}(\mathrm{OR} \cap \mathcal{M}) = \omega_1$, $f$ specializes the restriction of the tree arising from the square-sequence of $\mathcal{M}$ to $C$.

The core of our argument will be to show that this sentence can be formulated in a $\Sigma_1$ way with parameters from $H_{\omega_2}$ and one universally Baire set.

Claim 2. *There is a $\Sigma_1$-formula with parameters in $H_{\omega_2}$ and two free variables $\Phi$, such that:*

$$V[g,G] \models \text{“}S(X') \textit{ is iterable”} \textit{ if and only if } V[g,G] \models \Phi(S(X'), U_n)$$

Proof. Let $\bar{\lambda} < \lambda$ be such that $S(X') \| \bar{\lambda}$ projects to $\omega_1$. Let $\sigma : \mathcal{P} \to S(X') \| \bar{\lambda}$ be an elementary embedding where $\mathcal{P}$ is countable. Let $\alpha$ be the critical point of $\sigma$. $\mathcal{P}$ is a $X' \cap \alpha$-mouse. Since $S(X')$ is $(n+1)$-small and iterable, we have that $\mathcal{P}$ is $(n+1)$-small and $\omega_1$-iterable, witnessed by $\mathcal{M}^{\#}_n(\mathcal{P})$. Since $S(X') \| \bar{\lambda}$ projects to $X'$, we have that $\mathcal{P}$ projects to $X' \cap \alpha$. Since both mouse are sound, this implies that $\mathcal{P} \cong \mathrm{Hull}^{S(X') \| \bar{\lambda}}(\alpha \cup \{p(S(X') \| \bar{\lambda})\})$.

Let $\Phi(S, U_n)$ be the following formula:

> For all $\bar{\lambda} < \lambda$, where $\lambda = \mathrm{OR} \cap S$ such that $\rho_\omega(S \| \bar{\lambda}) = \omega_1$, for all $\alpha < \omega_1$ such that $\alpha = \omega_1 \cap \mathrm{Hull}^{S \| \bar{\lambda}}(\alpha \cup \{p(S \| \bar{\lambda})\})$ and for all transitive $\mathcal{N}$ such that $\mathcal{N} = \mathrm{Hull}^{S \| \bar{\lambda}}(\alpha \cup \{p(S \| \bar{\lambda})\})$, $\mathcal{N}$ is $\omega_1$-iterable as witnessed by y, where $y$ is such that $(\mathcal{N}, y) \in U_n$.

$\Phi$ is the formula we were looking for, as proved by the previous argumentation. ⊣

Hence rephrase $\Psi$ by:

> There is a $Y \subseteq \omega_1$, a $Y$-premouse $\mathcal{M}$, a club $C$ in $\mathcal{M} \cap \mathrm{OR}$ and a $f : \omega_1 \to \omega$ with the following properties:
>
> i. $\mathcal{M}$ is a $(n+1)$-small $Y$-premouse,
>
> ii. the largest cardinal of $\mathcal{M}$ is $\omega_1$,
>
> iii. the height of $\mathcal{M}$ has cofinality $\omega_1$,
>
> iv. $f$ specialize the square-sequence of $\mathcal{M}$ restricted to $C$,
>
> v. cofinally many initial segments of $\mathcal{M}$ projects to $Y$,
>
> vi. $\Phi(\mathcal{M}, U_n)$.

Since $\mathbb{P}^* * \mathbb{P}$ is $\omega$-closed and $\mathbb{Q}$ is c.c.c. the $*$-product is proper. Thus by $\mathrm{BPFA}^{\mathrm{uB}}$, $\Psi$ holds in $V$.

Let $Y$, $\mathcal{M}$ and $f$ be witnesses of $\Psi$ in $V$. Since $(ii)_n$ holds, $\mathcal{M}^{\#}_{n+1}(Y)$ exists. There are cofinally many initial segment of $\mathcal{M}$ that are sound and which projects to $Y$. Let $\mathcal{N} \lhd \mathcal{M}$ be such an initial segment, $\mathcal{M}^{\#}_{n+1}(Y)$ wins the coiteration with $\mathcal{N}$, hence $\mathcal{N}$ is not moved in the coiteration. Moreover since there are both sound and projects to $Y$, there are actually lined up. This implies that $\mathcal{M} \lhd \mathcal{M}^{\#}_{n+1}(Y)$. Hence there is a thread to the square-sequence of $\mathcal{M}$ in $\mathcal{M}^{\#}_{n+1}(Y)$, as given by the square-sequence of $\mathcal{M}^{\#}_{n+1}(Y)$. But there is a specializing function for the square-sequence of $\mathcal{M}$, a contradiction! $\dashv$

# 4 Ideal Extenders

This chapter is devoted to the analysis of the consistency strength of various generic embeddings and their construction. In the first section, we recapitulate some analysis of the relationship between precipitous ideals and $< \kappa$-complete ultrafilters and present a forcing construction that allows to change the power-set of a measurable cardinal without killing the measurability. In the second section, we will define ideal extender, which we think, are the natural counterpart to precipitous ideals in the strong context. That is, ideal extenders are to strong cardinals, what precipitous ideals are to measurable cardinals. That the techniques to produce such ideal extenders, by a levy-collapse, are just the same as in the measurable case strengthen that hypothesis. We will finish that section by showing that the existence of these ideal extender have at least the consistency strength of a strong cardinal. In the third section, we will discuss how to produce finitely many of such ideals. Sadly it seems that it becomes more difficult, consistency wise, to get a combinatorial witness to the "generical strongness" of a cardinal. By switching to a more general concept of generically strong, we show that they are equiconsistent to the "same amount" of strong cardinals. Finally in the last section, we use the techniques developed in the previous sections to show that given $\omega$ supercompact, we can construct a model in which every $\aleph_n$ is generically strong.

## 4.1 In the case of a measurable

Let $\kappa$ be a cardinal. The levy collapse of $\kappa$ to $\omega_1$, $\mathrm{col}(\omega, < \kappa)$, is the set of all finite function $p$ such that $\mathrm{dom}(p) \subseteq \kappa \times \omega$ and for all $\langle \alpha, n \rangle \in \mathrm{dom}(p)$ $p(\langle \alpha, n \rangle) < \alpha$. We say that p is stronger than q, $p \leqslant_{\mathrm{col}(\omega,<\kappa)} q$, if $q \subseteq p$.

**Fact 4.1.** *If $G$ is $\mathrm{col}(\omega, < \kappa)$-generic over $V$, $V[G] \vDash \kappa = \omega_1$.*

Let $\mathbb{P} = \mathrm{col}(\omega, < \kappa)$ and for $\nu < \kappa$, we set

$$\mathbb{P}^\nu = \{p \in \mathbb{P}; \forall \langle \alpha, n \rangle \in \mathrm{dom}(p)\ \alpha \geqslant \nu\},$$

similarly

$$\mathbb{P}_\nu = \mathrm{col}(\omega, < \nu) = \{p \in \mathbb{P}; \forall \langle \alpha, n \rangle \in \mathrm{dom}(p)\ \alpha < \nu\}.$$

It is easy to see that $\mathbb{P}$ is isomorphic to the product $\mathbb{P}_\nu \times \mathbb{P}^\nu$, for all $\nu$.

Let $\kappa$ be a measurable cardinal and $U$ a normal $< \kappa$-complete ultrafilter on $\kappa$. Let

$$\pi : V \to M = \mathrm{Ult}(V, U)$$

be the ultrapower generated by $U$. We can split $\pi(\mathbb{P})$ in $\mathbb{P}$ and $\pi(\mathbb{P})^\kappa$. Let $G$ be $\mathbb{P}$-generic over $V$ and $H$ be $\pi(\mathbb{P})^\kappa$-generic over $V[G]$. Every condition in $q \in \pi(\mathbb{P})^\kappa$ can be represented by a family $\langle q_\alpha; \alpha < \kappa\rangle$ , that is $[\langle q_\alpha; \alpha < \kappa\rangle]_U = q$, where all the $q_\alpha$ are in $\mathbb{P}$. Moreover for $U$-almost all $\alpha$ $q_\alpha \in \mathbb{P}^\alpha$, since $[q] \in \pi(\mathbb{P})^\kappa \iff \{\alpha; q_\alpha \in \mathbb{P}^\alpha\} \in U$.

In $V[G \times H]$ we can define a new $V[G]$-ultrafilter $W$ by:

$$\tau^G \in W \iff \kappa \in (\pi(\tau))^{G\times H}.$$

Let $\dot{W}$ be the canonical name for $W$. For every $p \in \mathbb{P}$ and $q \in \pi(\mathbb{P})^\kappa$,

$$\langle p, q\rangle \Vdash \dot{X} \in \dot{W} \iff \text{for } U\text{-measure one many } \alpha, p \cup q_\alpha \Vdash \check{\alpha} \in \dot{X}.$$

We will use this last remark to show that $W$ is generic over $V[G]$ for the following forcing: $\mathbb{Q} = \{X \in V[G]; \forall Y \in U\ Y \cap X \neq \varnothing\}$, where $X \leqslant_{\mathbb{Q}} Y$ if and only if $X \subseteq Y$. We already know that $W$ is a $V[G]$ ultrafilter, so we only have to prove that it is generic. Suppose $X = \{X_i; i < \theta\}$ is a maximal antichain in $V[G]$ and for all $i < \theta$, $X_i \notin W$. Let $\dot{X}, \dot{X}_i$ be names for $X$ and $X_i$. Let $p \in G$ and $q \in H$ be such that:

$$\langle p, q\rangle \Vdash \forall i < \theta\ \dot{X}_i \notin \dot{W}$$

By the last remark $q = [\langle q_\alpha; \alpha < \kappa\rangle]_U$ and for each $i$ there is a set $A_i \in U$ such that for all $\alpha \in A_i$ $p \cup q_\alpha \Vdash \alpha \notin \dot{X}_i$ . Now let $T = \{\alpha, q_\alpha \in G\}$. We first prove that $T \cap X_i \notin \mathbb{Q}$. For each $i < \theta$, if $\alpha \in T \cap A_i$ we have that $q_\alpha \in G$ and $\alpha \notin X_i$. Therefore $T \cap X_i \cap A_i = \varnothing$ but $A_i \in U$ hence $T \cap X_i \notin \mathbb{Q}$. Thus $T$ is incompatible with all $X_i$. If we can prove that $T \in \mathbb{Q}$ we would have that $\{X_i; i < \theta\}$ wasn't a maximal antichain, a contradiction. Let $Z \in U$, we have to prove that $T \cap Z \neq \varnothing$. We want to show that $q_\alpha \in G$ for some $\alpha \in Z$. Let

$$E = \{r \in \mathbb{P}; r \leqslant q_\alpha \text{ for some } \alpha \in Z\}$$

Let us show that $E$ is dense. Take some $p \in \mathbb{P}$ and let $\beta$ be the minimal such that $p \in \mathbb{P}_\beta$. Now since $Z$ is unbounded in $\kappa$ there is a $\alpha \in Z \smallsetminus \beta$. But then, $p$ and $q_\alpha$ have disjoint domains in a way that $p \cup q_\alpha \in \mathbb{P}$ and thus $r = p \cup q_\alpha$ is the strengthening of $p$ that we were looking for. Thus $E \cap G \neq \varnothing$ and $T \cap Z \neq \varnothing$.

What we basically did is, starting with some embedding:

$$\pi : V \to M = \mathrm{Ult}(V, G)$$

to lift up $\pi$ to some

$$\tilde{\pi} : V[G] \to M[G, H],$$

moreover if $W$ is the ultrafilter derived from $\tilde{\pi}$, $W$ is generic over $V[G]$ for the forcing $\mathbb{Q}$. This case was easy, because the forcing adding $G$ was basically below $\kappa$, the critical point of $\pi$. But there are ways to lift up embeddings even when forcing above of a large cardinal. Let us first show a way to deal with it in the case of a measurable.

**Lemma 4.2.** *Assume* GCH. *Let $\kappa$ be measurable, $X_\kappa$ the set of all cardinals less or equal to $\kappa$ and $\mathbb{P}$ the easton support iteration of* $\mathrm{col}(\xi, \xi)$*, the forcing adding a cohen subset of $\xi$, for all $\xi \in X_\kappa$. Let $G$ be $\mathbb{P}$-generic over $V$, then in $V[G]$, $\kappa$ is still measurable.*

Proof. Let $U$ be an normal ultrafilter witnessing the measurability of $\kappa$. Let

$$j : V \to M = \mathrm{Ult}(V, U)$$

be the associated ultrapower map. $j(\mathbb{P})$ is the easton support iteration of $\mathrm{col}(\xi, \xi)$ for all $\xi \in j(X_\kappa) = X_{j(\kappa)}$, let $j(\mathbb{P})^\kappa$ for the part of the forcing starting after $\kappa$, that is $j(\mathbb{P}) = \mathbb{P} * \mathbb{P}^\kappa$. Let $G$ be $\mathbb{P}$-generic over $V$, since $(H_\kappa)^V = (H_\kappa)^M$ and $\mathcal{P}(\kappa) \cap V = \mathcal{P}(\kappa) \cap M$ we have that $G$ is $\mathbb{P}$-generic over $M$. If we can show that there is an $\tilde{G} \in V[G]$ such that $G * \tilde{G}$ is $j(\mathbb{P})$-generic over $M$ and $j"G = \tilde{G} \cap \mathrm{ran}(j)$, we will be able to lift the embedding $j$ to an embedding

$$\tilde{j} : V[G] \to M[G \times \tilde{G}]$$

By the Factor Lemma [Jec03, Lemma 21.8 pp. 396] it suffices to define $\tilde{G}$ such that it is $j(\mathbb{P})^\kappa$-generic over $M[G]$.

By [Kan03, Proposition 5.7 (b)], $2^\kappa \leq (2^\kappa)^M < j(\kappa)$. Notice that for the same reasons, we also have $j(\kappa^+) < (2^\kappa)^+ = \kappa^{++}$. Hence

$$\mathrm{card}^V(\{D \in M, D \text{ is dense in } j(\mathbb{P})^\kappa\}) = \kappa^+ = 2^\kappa$$

Every dense set of $j(\mathbb{P})^\kappa$ in $M[G]$ is of the form $j(f)(\kappa)^G$, where $f$ is a function from $\kappa$ to $V^{\mathbb{P}}$. Let $\langle f_i; i < \kappa^+\rangle$ be an enumeration in $V$ of functions representing all open dense sets of $j(\mathbb{P})^\kappa$ in $M[G]$. There is an enumeration with size $\kappa^+$ since in $V[G]$ there are at most $\kappa^+$ many such dense sets and $\mathbb{P}$ has the $\kappa^+$-c.c. Since $M$ is $\kappa$-closed each initial segment $\langle j(f_i)(\kappa)^G; i < \alpha\rangle$ is in $M$, for $\alpha < \kappa^+$. But $j(\mathbb{P})^\kappa$ is also $\kappa$-closed, hence $\bigcap_{i<\alpha} j(f_i)(\kappa)^G$ is a dense set of $j(\mathbb{P})^\kappa$ in $M$. Now one can construct in $V[G]$ a sequence $\langle p_\alpha; \alpha < \kappa^+\rangle$ with the following properties:

i. $p_0 = \bigcup j"(G \cap \mathrm{col}(\kappa, \kappa))$

ii. $p_\alpha < p_\beta$ for $\alpha < \beta$

iii. $p_\alpha \in \bigcap_{i<\alpha} j(f_i)(\kappa)^G$

Each element of the sequence is in $M$, and the sequence itself is in $V[G]$. Moreover $j"G \subseteq G * \tilde{G}$. Now we can lift $j$ by using the classical definition:

$$j(\tau^G) = j(\tau)^{G * \tilde{G}} \qquad \dashv$$

for $\tau$ a $\mathbb{P}$-name in $V$.

## 4.2 One ideally strong cardinal

### 4.2.1 The definition of ideal extenders

**Definition 4.3.** Let $\kappa$ be a cardinal, $\lambda > \kappa$ an ordinal and let $X$ be a set. For every finite subset $a$ of $X$ let us fix one bijection between $a$ and its cardinality. We identify finite sets of ordinals with their increasing enumeration, finite subsets of $X$ with their previously fixed bijection.

i. A $\langle\kappa, X\rangle$-*system of filters* is a set

$$F \subseteq \left\{\langle a, x\rangle \in [X]^{<\omega} \times \mathcal{P}([\kappa]^{<\omega}); x \subseteq [\kappa]^{\overline{\overline{a}}}\right\},$$

such that for all $a \in [X]^{<\omega}$, $F_a = \{x; \langle a, x\rangle \in F\}$ is a non trivial filter that is $F_a \neq \mathcal{P}([\kappa]^{\overline{\overline{a}}})$. We set $\operatorname{supp}(F) = \{a \in [X]^{<\omega}; F_a \neq \{X\}\}$.

ii. Let $F$ be a $\langle\kappa, X\rangle$-system of filters. Let $a, b \in \operatorname{supp}(F)$, such that $a \subseteq b$. Let $s : \overline{\overline{a}} \to \overline{\overline{b}}$ be such that $a(n) = b(s(n))$. For a set $x \in \mathcal{P}([\kappa]^{\overline{\overline{a}}})$, we define

$$x_{a,b} = \left\{\langle u_i; i < \overline{\overline{b}}\rangle \in [\kappa]^{\overline{\overline{b}}}; \langle u_{s(j)}; j < \overline{\overline{a}}\rangle \in x\right\}.$$

For a function $f : [\kappa]^{\overline{\overline{a}}} \to V$, we define $f_{a,b} : [\kappa]^{\overline{\overline{b}}} \to V$ by

$$f_{a,b}(\langle u_i; i < \overline{\overline{b}}\rangle) = f(\langle u_{s(j)}; j < \overline{\overline{a}}\rangle).$$

iii. A $\langle\kappa, X\rangle$-system of filters $F$ is called *compatible* if for all $a \subseteq b \in \operatorname{supp}(F)$

$$x \in F_a \iff x_{a,b} \in F_b.$$

iv. Let $a \in [X]^{<\omega}$ and $x \in [\kappa]^{\overline{\overline{a}}}$, we say that $F' = \operatorname{span}\{F, \langle a, X\rangle\}$ is the *span of $F$ and $\langle a, x\rangle$* if it is the smallest compatible system of filters such that $F \subseteq F'$ and $\langle a, x\rangle \in F'$.

v. Let $F$ be a $\langle\kappa, \lambda\rangle$-system of filters. The *associated forcing* $\mathbb{P}_F$ consists of all conditions $p = F^p$, where $F^p$ is a compatible $\langle\kappa, \lambda\rangle$-system of filters, $\operatorname{supp}(p) = \operatorname{supp}(F^p) \subseteq \operatorname{supp}(F)$ is finite and $F^p$ is generated by one point $x \in (F_a)^+$ for some $a \in \operatorname{supp}(p)$, i.e. $F^p$ is the span of $F$ and $\langle a, x\rangle$. $p \leqslant_{\mathbb{P}} q$ if and only if $\operatorname{supp}(q) \subseteq \operatorname{supp}(p)$ and for all $a \in \operatorname{supp}(q)$, $F^q_a \subseteq F^p_a$, that is if $F^q \subseteq F^p$.

Let $F$ be a compatible $(\kappa, \lambda)$-systems of filters and $G$ be $\mathbb{P}_F$-generic over $V$. Set $\dot{E}_F = \bigcup \dot{G}$, where $\dot{G}$ is the canonical name for the generic filter. Clearly $\dot{E}^G_F$ is a system of filters again. For any $a \in [\lambda]^{<\omega}$ and $X \in F^+_a$ we have that

$$A = \left\{\operatorname{span}\{F, \langle a, X\rangle\}, \operatorname{span}\left\{F, \langle a, [\kappa]^{\overline{\overline{a}}} \setminus X\rangle\right\}\right\}$$

is an antichain in $\mathbb{P}_F$. This shows that each $\dot{E}^G_{F,a} = (\bigcup \dot{G})_a$ is an ultrafilter. Moreover $\dot{E}_F$ has the compatibility property. Let us now look how we can translate the normal and $\omega$-closed concept to this situation.

**Definition 4.4.** Let $\kappa$, $\lambda$ be as in the previous definition.

i. We call a $\langle\kappa,\lambda\rangle$-system of filters *potentially normal* if for every $p \in \mathbb{P}_F$, for every $a \in \operatorname{supp}(p)$ and for every $f : [\kappa]^{\overline{\overline{a}}} \to V$ if there is a $j < \overline{\overline{a}}$ such that

$$\{u, f(u) \in u_j\} \in F_a^p,$$

it follows that there is a dense set $D$ below $p$ such that for every $p' \in D$ there is a $\xi \in \operatorname{supp}(p')$ with

$$\left\{v, f_{a,a\cup\{\xi\}}(v) = v_i\right\} \in F^{p'}_{a\cup\{\xi\}},$$

where $i$ is such that $s(i) = \xi$, $s$ being the enumeration of $a \cup \{\xi\}$.

ii. We call a $\langle\kappa,\lambda\rangle$-system of filter *precipitous* if for all $p \in \mathbb{P}_F$ and for all systems $\langle\langle p_s, X_s, a_s\rangle; s \in {}^{<\omega}\theta\rangle$ such that:

a) $p_\varnothing = p$,

b) $a_s \subseteq a_{s^\frown i}$ for all $i < \theta$,

c) $p_{s^\frown i}$ contains the span of $p_s$ and $\langle a_{s^\frown i}, X_{s^\frown i}\rangle$ for all $i$,

d) $\{p_{s^\frown i}; i < \theta\}$ is a maximal antichain below $p_s$,

there is an $x \in {}^\omega\theta$ and a $\tau : \bigcup_{s\subseteq x} a_s \to \kappa$ such that $\tau\text{"}a_s \in X_s$ for all $s \subseteq x$.

**Definition 4.5.** Let $\kappa < \lambda$ be ordinals. $F$ is a $\langle\kappa,\lambda\rangle$*-ideal extender* if it is a compatible and potentially normal $\langle\kappa,\lambda\rangle$-system of filters such that for each $a \in \operatorname{supp}(F)$, $F_a$ is $< \kappa$-closed.

Let $F$ be a compatible $\langle\kappa,\lambda\rangle$-systems of filters and $G$ be $\mathbb{P}_F$-generic over $V$. By compatibility and potential normality, we can see that $\dot{E}_F^G$ is a $\langle\kappa,\lambda\rangle$-extender over $V$. Hence we can construct the formal ultrapower, regardless of it being well-founded or not.

**Lemma 4.6.** *Let $F$ be a $\langle\kappa,\lambda\rangle$-ideal extender and $G$ be $\mathbb{P}_F$-generic over $V$. Let $\varphi(u)$ be a formula in the language of set theory in one free variable $u$. Łoś's theorem holds for generic ultrapowers, that is* $\operatorname{Ult}(V, \dot{E}_F^G) \models \varphi([a,f])$ *if and only if*

$$\left\{\vec{\alpha} \in [\kappa]^{\overline{\overline{a}}}; V \models \varphi(f(\vec{\alpha}))\right\} \in \dot{E}^G_{F,a}.$$

Proof. We proceed by induction on the rank of the formula. For atomic formulae this holds by definition. We only prove the lemma for the negation and the existencial quantifier as the other cases are easy.

Let $\varphi \equiv \neg\psi$. It follows from the fact that each $\dot{E}_F^G$ is a system of ultrafilters, that is if $\langle a, x\rangle \notin \dot{E}_F^G$ then $\langle a, [\kappa]^{\overline{\overline{a}}} \setminus x\rangle \in \dot{E}_F^G$.

Let $\varphi([c,g]) \equiv \exists v\psi(v,[c,g])$. We first show that:

$$\operatorname{Ult}(V, \dot{E}_F^G) \models \varphi([c,g]) \implies \left\{\vec{\alpha} \in [\kappa]^{\overline{\overline{c}}}; V \models \varphi(g(\vec{\alpha}))\right\} \in \dot{E}^G_{F,c}$$

Let $[b,f]$ be such that $\psi([b,f],[c,g])$ holds. By induction hypothesis, there is a $\langle b\cup c, x\rangle \in G$ witnessing that $\psi([b,f],[c,g])$ is true, that is

$$x = \left\{\vec{\alpha} \in [\kappa]^{\overline{\overline{b\cup c}}}; V \models \psi(f_{b,b\cup c}(\vec{\alpha}), g_{c,b\cup c}(\vec{\alpha}))\right\} \in \dot{E}^G_{F,b\cup c}.$$

Since $\dot{E}^G_{F,b\cup c}$ is a filter, by a compatibility argument we can show that:

$$x_c^{b\cup c} \subseteq \left\{\vec{\beta} \in [\kappa]^{\overline{\overline{c}}}; V \models \exists x\psi(x, g(\vec{\beta}))\right\} \in \dot{E}^G_{F,c}.$$

Let us now prove the other direction, that is:

$$\left\{\vec{\alpha} \in [\kappa]^{\overline{\overline{c}}}; V \models \varphi(g(\vec{\alpha}))\right\} \in \dot{E}^G_{F,c} \implies \mathrm{Ult}(V, \dot{E}^G_F) \models \varphi([c,g]).$$

Let

$$y = \left\{\vec{\beta} \in [\kappa]^{\overline{\overline{c}}}; V \models \exists x\psi(x, g(\vec{\beta}))\right\} \in \dot{E}^G_{F,c},$$

and $f$ the function that assigns to some $\vec{\beta}$ some set $x$ such that

$$V \models \psi(x, g(\vec{\beta})),$$

if one exists and the empty set else. $f$ gives a witness for the fact that $\exists v\psi(v, g(\vec{\beta}))$ on a $\dot{E}^G_{F,c}$ measure one set. Thus by induction hypothesis

$$\mathrm{Ult}(V, \dot{E}^G_F) \models \psi([c,f],[c,g]). \qquad \dashv$$

**Lemma 4.7.** *A $\langle\kappa,\lambda\rangle$-ideal extender is precipitous if and only if the generic ultrapower given by any generic over the associated forcing is well-founded.*

Proof. Suppose first that $F$ is precipitous and that there is a condition $p \in \mathbb{P}_F$ such that $p \Vdash$ "the ultrapower by $\dot{E}_F$ is ill-founded". That is there is a system $\langle[\dot{a}^n, \dot{f}^n], n < \omega\rangle$ such that

$$p \Vdash [\dot{a}^n, \dot{f}^n] > [\dot{a}^{n+1}, \dot{f}^{n+1}].$$

Without loss of generality we can fix a system $\langle\langle p_s, X_s, a_s\rangle, s \in {}^{<\omega}\theta\rangle$ with $p_\varnothing = p$ such that $\{p_{s^\frown i}; i < \theta\}$ is a maximal antichain below $p_s$ and

$$p_s \Vdash \mathrm{dom}(\dot{f}_n) = \check{X}_s \in \dot{E}_{\check{a}_s} \wedge \dot{f}_n = \check{f}_s \wedge \check{a}_s = \dot{a}_n.$$

By precipitousness we then have a $x \in {}^\omega\theta$ and a $\tau : \bigcup_{s\subseteq x} a_s \to \kappa$ such that $\tau" a_s \in X_s$ for all $s \subseteq x$. Since all conditions are below $p$,

$$p_{x\restriction n+1} \Vdash \text{“}[\dot{a}^n, \dot{f}^n] > [\dot{a}^{n+1}, \dot{f}^{n+1}]\text{”}.$$

Moreover

$$p_{x\restriction n+1} \Vdash \text{“}\mathrm{dom}(\dot{f}_n) = \check{X}_{x\restriction n} \in \dot{E}_{\check{a}_{x\restriction n}} \wedge \check{a}_{x\restriction n} = \dot{a}_n\text{”}$$

and

$$p_{x\restriction n+1} \Vdash \text{“}\mathrm{dom}(\dot{f}_{n+1}) = \check{X}_{x\restriction n+1} \in \dot{E}_{\check{a}_{x\restriction n+1}} \wedge \check{a}_{x\restriction n+1} = \dot{a}_{n+1}\text{”}.$$

Thus $\tau\text{”}a_{x\restriction n} \in \mathrm{dom}(\check{f}_{x\restriction n})$ and $f_{x\restriction n}(\tau\text{”}a_{x\restriction n}) > f_{x\restriction n+1}(\tau\text{”}a_{x\restriction n+1})$, but this is a descending sequence of ordinals in $V$, contradiction!

Suppose now that for every generic, the ultrapower is well-founded. Consider the system $\mathcal{T} = \langle\langle p_s, X_s, a_s\rangle; s \in {}^{<\omega}\theta\rangle$ such that:

i. $p_\varnothing = p$,

ii. $a_s \subseteq a_{s^\frown i}$ for all $i < \theta$,

iii. $p_{s^\frown i}$ contains the span of $p_s$ and $\langle a_{s^\frown i}, X_{s^\frown i}\rangle$ for all $i$,

iv. $\{p_{s^\frown i}; i < \theta\}$ is a maximal antichain below $p_s$.

Let us show that $x, \tau$ exists such that $\tau''a_s \in X_s$ for $s \subseteq x$. Let $G$ be a generic filter such that $p_\varnothing \in G$. Since for all $n < \omega$ the set $\{p_s, \mathrm{lh}(s) = n\}$ is a maximal antichain below $p_\varnothing$, there is one $s$ such that $p_s \in G$, let $x$ be the union of all such $s$, notice that $x \in {}^\omega\theta$ is well defined. Let $\pi : V \to \mathrm{Ult}(V, G)$ be the ultrapower map. We write $a_s^{\pi(\mathcal{T})}$ for the second components of the condition at the $s$-node of $\pi(\mathcal{T})$, similarly for $X_s^{\pi(\mathcal{T})}$ and $p_s^{\pi(\mathcal{T})}$. Let

$$\tau : \bigcup_{s\subseteq x} \pi(a_s) \to \pi(\kappa)$$

be defined as follows:

> if $\xi \in \bigcup_{s\subseteq x} \pi(a_s)$ then there is a $s$ such that $\xi \in \pi(a_s)$, since $a_s$ is finite there is a $\overline{\xi} \in a_s$ such that $\xi = \pi(\overline{\xi})$, let $\tau(\xi) = \overline{\xi}$.

Hence we have that:

$$\mathrm{Ult}(V, G) \vDash \text{“}\tau : \bigcup_{s\subseteq x} \pi(a_s) \to \pi(\kappa)\text{”}$$

and

$$\tau\text{”}\pi(a_s) \in \pi(X_s) \text{ for all } s \subseteq x.$$

By elementarity $\pi(a_s) = a_{\pi(s)}^{\pi(\mathcal{T})}$ and $\pi(X_s) = X_{\pi(s)}^{\pi(\mathcal{T})}$.

Let us argue why $x$ and $\tau$ exists in $\mathrm{Ult}(V, G)$: let $T$ be the tree of height $\omega$, with finite conditions searching[1] for a $x'$ and a $\tau'$ such that

> $\tau' : \bigcup_{s\subseteq x'} a_s^{\pi(\mathcal{T})} \to \pi(\kappa)$ and $\tau'\text{”}a_s^{\pi(\mathcal{T})} \in X_s^{\pi(\mathcal{T})}$ for all $s \subseteq x'$.

This tree is in $V[G]$ as well as in $\mathrm{Ult}(V, G)$, setting $x' = \pi''x$ we can see that it is ill-founded in $V[G]$, hence it is ill-founded in $\mathrm{Ult}(V, G)$. A branch through the tree gives some $x$ and $\tau$ with the above properties, hence

> $\mathrm{Ult}(V, G) \vDash$ “$\exists x \exists \tau$ such that $\tau : \bigcup_{s\subseteq x} a_s^{\pi(\mathcal{T})} \to \pi(\kappa)$ and $\tau\text{”}a_s^{\pi(\mathcal{T})} \in X_s^{\pi(\mathcal{T})}$ for all $s \subseteq x$”.

By elementarity

$$V \vDash \text{“}\exists x \exists \tau \text{ such that } \tau : \textstyle\bigcup_{s\subseteq x} a_s \to \kappa \text{ and } \tau\text{”}a_s \in X_s \text{ for all } s \subseteq x\text{”}. \qquad \dashv$$

[1] we already constructed such a type of tree in 3.10

### 4.2.2 Forcing ideal extenders and ideally strong cardinals

**Lemma 4.8.** *Let $\kappa$ be $\alpha$-strong in $V$, $\mu < \kappa$ some cardinal and let $E$ be the $\langle \kappa, \lambda \rangle$-extender derived by the ultrapower map witnessing the $\alpha$-strongness. Let $W = V[G]$ where $G$ is* $\mathrm{col}(\mu, <\kappa)$*-generic over $V$. Set*

$$F = \left\{ \langle a, x \rangle ; x \subseteq [\kappa]^{\overline{\overline{a}}} \textit{ and } \exists y \textit{ such that } \langle a, y \rangle \in E \ y \subseteq x \right\}$$

*then $F$ is precipitous.*

Proof. Let us first start with a simple general consideration that is useful in many cases when considering ultrapowers and the Levy collapse:

Claim 1. *Suppose $V \models$ "$E$ is a $(\kappa, \lambda)$-extender". Let $\pi : V \to M = \mathrm{Ult}(V, E)$ be the associated ultrapower map. Then for each $G$ $\mathrm{col}(\mu, <\kappa)$-generic over $V$ and each condition $q \in \mathrm{col}(\mu, <\pi(\kappa))^M$ such that $q \restriction \mu \times \kappa \in G$, there is a $M$-generic $G^*$ such that $\{q\} \cup G \subseteq G^*$, moreover there is a canonical map $\tilde{\pi} : V[G] \to M[G^*]$ such that $\pi \subseteq \tilde{\pi}$.*

Proof. Since $(H_{\kappa^+})^V = (H_{\kappa^+})^M$, $G$ is also generic over $M$. In $M[G]$ we can look for a $\mathrm{col}(\mu, ]\kappa, \pi(\kappa)[)$-generic filter $\tilde{G}$ such that $q \restriction \mu \times ]\kappa, \pi(\kappa)[ \in \tilde{G}$. Let $G^*$ be the filter generated by $G \cup \tilde{G}$, now we can define an embedding $\tilde{\pi} : V[G] \to M[G^*]$ as follow: for every name $\tau \in V^{\mathrm{col}(\mu, <\kappa)}$, let $\tilde{\pi}(\tau^G) = (\pi(\tau))^{G^*}$. It is easy to check that $\tilde{\pi}$ is well defined and an embedding. ⊣

Let us now turn to $F$, we first want to prove that for each $\mathrm{col}(\mu, <\pi(\kappa))$-generic over $M$ filter $G^*$, we can construct an extender $E^{G^*}$ that extends $F$ such that the following diagram commutes:

$$\begin{tikzcd}
V[G] \arrow[rr, "\tilde{\pi}"] \arrow[dr, "j"'] & & M[G^*] \\
& \mathrm{Ult}(V[G], E^{G^*}) \arrow[ur, "k"'] &
\end{tikzcd}$$

where $j$ is the associated ultrapower map and $k$ still needs to be defined and $G = G^* \cap \mathrm{col}(\mu, <\kappa)$. We define $E^{G^*}$ by:

$$\langle a, x \rangle \in E^{G^*} \iff a \in \tilde{\pi}(x),$$

for $a \in [\lambda]^{<\omega}$ and $x \subseteq \mathcal{P}([\kappa]^{\overline{\overline{a}}})$ and $k$ by:

$$k([f, a]) = \tilde{\pi}(f)(a),$$

where $a$ is as before and $f : \kappa^{\overline{\overline{a}}} \to V[G]$. It is easy to check that $k$ is well defined. Hence $\mathrm{Ult}(V, E^{G^*})$ is transitive.

Let us do a few remark similar to the case of a measurable before turning to the genericity of $E^{G^*}$. Each condition in $\mathrm{col}(\mu, <\pi(\kappa))$ can be split in $p \in \mathrm{col}(\mu, <\kappa)$ and a $q \in \mathrm{col}(\mu, [\kappa, \pi(\kappa)[)$, moreover $q$ can be represented in the ultrapower by $a_q \in [\lambda]^{<\omega}$ and a function $f^q : \kappa^{\overline{\overline{a}}_q} \to \mathrm{col}(\mu, < \kappa)$. Let $s$ be an enumeration of $a_q \cup \{\kappa\}$ and $i$ such that $s(i) = \kappa$, we have:

$$\left\{\vec{\xi}; f^q_{a_q, a_q \cup \{\kappa\}}(\vec{\xi}) \in \mathrm{col}(\mu, [\xi_i, \kappa[)\right\} \in E_{a_q \cup \{\kappa\}}.$$

Let $\dot{E}$ be the canonical name for $E^{G^*}$, $a \in [\kappa]^{<\lambda}$ and $\dot{X} \in V^{\mathrm{col}(\mu, <\kappa)}$ some set such that there are $\langle p, q \rangle \in \mathbb{P} * j(\mathbb{P})^\kappa$ with

$$p \cup q \Vdash^{j(\mathbb{P})} \langle \check{a}, \dot{X} \rangle \in \dot{E}.$$

By definition of $\pi$ we then have:

$$p \cup q \Vdash^{j(\mathbb{P})} \check{a} \in \pi(\dot{X})^2.$$

Setting $\mathrm{id}^a : [\kappa]^{\overline{\overline{a}}} \to [\kappa]^{\overline{\overline{a}}}$, this leeds to:

$$\left\{\vec{\xi}; p \cup f^q_{a_q, a \cup a_q}(\vec{\xi}) \Vdash^{\mathbb{P}} \mathrm{id}^a_{a, a \cup a_q}(\vec{\xi}) \in \dot{X}\right\} \in E_{a \cup a_q}.$$

Let $G_F = \left\{p \in \mathbb{P}_F; F^p \subseteq E^{G^*}\right\}$. We want to prove that $G_F$ is $\mathbb{P}_F$-generic over $V[G]$. Let $p \in G$ and $q \in G^* \restriction \mathrm{col}(\mu[\kappa, \pi(\kappa)[)$ such that

$$p \Vdash \text{“}\dot{A} = \left\{\dot{F}^i; i < \theta\right\} \subseteq \mathbb{P}_F \text{ is an antichain”}$$

moreover for each $i < \theta$, $p \cup q \Vdash$ “$F^i \nsubseteq E^{G^*}$”. Let each $\dot{F}^i$ be generated by $\langle \check{a}_i, \dot{X}_i \rangle$, we have

$$p \cup q \Vdash \text{“}\dot{X}_i \notin \dot{E}_{\check{a}_i}\text{”}.$$

By the previous observation, we have sets $A_i \in E_{a_i \cup a_q}$ such that for all $\vec{\xi} \in A_i$:

$$p \cup f^q_{a_q, a_i \cup a_q}(\vec{\xi}) \Vdash \mathrm{id}^{a_i}_{a_i, a_i \cup a_q}(\vec{\xi}) \notin \dot{X}_i.$$

Let $T = \left\{\vec{\xi};\ f^q(\vec{\xi}) \in G\right\}$ and let $F'$ be the span of $F$ and $\langle a_q, T \rangle$. We first show that $F'$ is a condition: for a $Z \in E_{a_q}$, we have to show that $Z \cap T \neq \varnothing$. Let

$$D = \left\{r; r \leqslant_{\mathrm{col}(\mu, <\kappa)} q_{\vec{\xi}} \text{ for some } \vec{\xi} \in Z\right\}.$$

$D$ is dense, since each condition has size less then $\mu$, $Z$ is unbounded and $\mu$ is regular, therefore we can choose some $q_{\vec{\xi}}$ such that

$$\sup(\mathrm{dom}(r)) < \min(\mathrm{dom}(q_{\vec{\xi}})).$$

Let $r \in D \cap G$, there is a $\vec{\xi} \in Z$ such that $r \leqslant_{\mathrm{col}(\mu, <\kappa)} q_{\vec{\xi}}$, thus $\vec{\xi} \in T$, and we have $T \cap Z \neq \varnothing$. Let us now show that $T \cap X_i \notin F^+$, it suffices to prove that there is a set $X \in E_{a_i \cup a_q}$ such that

$$T_{a_q, a_i \cup a_q} \cap X_{i a_i, a_i \cup a_q} \cap X = \varnothing.$$

Let $\vec{\xi} \in A_i$. If $\xi \in T_{a_q, a_i \cup a_q}$, $q_\xi \in G$. Since

$$p \cup f^q_{a_q, a_i \cup a_q}(\vec{\xi}) \Vdash \mathrm{id}^{a_i}_{a_i, a_i \cup a_q}(\vec{\xi}) \notin \dot{X}_i.$$

We have that $\xi \notin X_{i a_i, a_i \cup a_q}$, hence the $A_i$ where the set we sought, and $\langle X_i; i < \theta \rangle$ isn't a maximal antichain, a contradiction!

[2] notice that $\pi(\dot{X})$ is a $\mathrm{col}(\mu, \pi(\kappa))$ name

Claim 2. *Let $G$ be $\mathrm{col}(\mu, < \kappa)$-generic over $V$. For each condition $p \in \mathbb{P}_F$, there is a $G^*$ $\mathrm{col}(\mu, < \pi(\kappa))$-generic that extends $G$, such that $F^p \subseteq E^{G^*}$.*

Proof. Let $\dot{p} \in V$ be a name for a condition in $\mathbb{P}_F$ . Fix $\tau \in V$ and $q \in \mathrm{col}(\mu, < \kappa)$ such that $q \Vdash$ "$F^{\dot{p}}$ is the span of $\check{F}$ and $\langle \check{a}, \tau \rangle$", for some finite set of ordinals $a \in [\lambda]^{<\omega}$. Without loss of generality we can assume that $\tau = \left\{ \langle p, \check{\vec{\alpha}} \rangle ; p \Vdash \check{\vec{\alpha}} \in \tau \right\}$. We want to show that we can find a $q' < q \in \mathrm{col}(\mu, < \pi(\kappa))$ such that $a \in \tilde{\pi}(\tau)$, for every $\mathrm{col}(\mu, < \pi(\kappa))$-generic $G^*$ with $q' \in G^*$. Let

$$y = \{\vec{\alpha}; \exists r < q \langle r, \vec{\alpha} \rangle \in \tau\} .$$

Clearly, $\langle a, y \rangle$ has to be in $E$, else $\tau$ would be a null set in $V[G]$. Hence

$$a \in \pi(y) = \pi(\{\vec{\alpha}; \exists r < q \langle r, \vec{\alpha} \rangle \in \tau\}) = \{\vec{\alpha}; \exists r \langle r, \vec{\alpha} \rangle \in \pi(\tau)\} .$$

This shows that there is a $q' \in \mathrm{col}(\mu, < \pi(\kappa))$, $q' < q$ such that $\langle q', a \rangle \in \pi(\tau)$. Let $G^*$ be $\mathrm{col}(\mu, < \pi(\kappa))$-generic with $q' \in G^*$, $G^*$ has the desired properties. ⊣

Let us prove now that $F$ is potentially normal and precipitous. Suppose first that $F$ is not precipitous, then there is a generic over $\mathbb{P}_F$ such that the associated ultrapower is ill-founded. This is then forced by a condition $p \in \mathbb{P}_F$. By the previous result we can find $G^*$ $\mathrm{col}(\mu, < \pi(\kappa))$-generic such that $F^p \subseteq E^{G^*}$. Thus $\mathrm{Ult}(V[G], E^{G^*})$ should be ill-founded, a contradiction since you can embed it in $M[G^*]$. Similarly suppose that $F$ is not potentially normal. Let $p \in \mathbb{P}_F$ such that there is $f : [\kappa]^{\overline{\overline{a}}} \to V$ with

$$\{u, f(u) \in u_j\} \in F^p_a$$

for some $a \in \mathrm{supp}(p)$, such that for no $q \leqslant_{\mathbb{P}_F} p$ there is a $\xi$ with $a \cup \{\xi\} \subseteq \mathrm{supp}(q)$ and

$$\left\{v, f_{a, a\cup\{\xi\}}(v) = v_i\right\} \in F^q_{a\cup\{\xi\}}.$$

Let $G^*$ be $\mathrm{col}(\mu, < \pi(\kappa))$-generic such that $F^p \subseteq E^{G^*}$. $E^{G^*}$ is a normal extender, since it is an extender derived from an embedding. Hence there is a $\xi$ such that

$$V[G^*] \vDash A = \left\{v, f_{a, a\cup\{\xi\}}(v) = v_i\right\} \in E^{G^*}_{a\cup\{\xi\}}.$$

Let $F^p$ be generated by $\langle b, x \rangle$, and define

$$y = x_{b, b\cup a\cup\{\xi\}} \cap A_{a\cup\{\xi\}, b\cup a\cup\{\xi\}}.$$

Let $q \in \mathbb{P}_F$ be such that $F^q$ is the filter generated by $\langle b \cup a \cup \{\xi\}, y \rangle$. Then $q \leqslant_{\mathbb{P}_F} p$ and

$$\left\{v, f_{a, a\cup\{\xi\}}(v) = v_i\right\} \in F^q_{a\cup\{\xi\}},$$

a contradiction! ⊣

**Definition 4.9.** Let $\kappa$ be a regular cardinal. We call a regular cardinal $\kappa$ *ideally strong* if and only if for all $A \subseteq OR$, $A \in V$, there is some $\langle \kappa, \nu \rangle$-ideal extender $E$ such that, whenever $G$ is $E$-generic over $V$, $A \in \mathrm{Ult}(V, G)$

**Theorem 4.10.** *Let $\kappa$ be a strong cardinal in $V$ and $\lambda$ be a cardinal. Let $G$ be* $\mathrm{col}(\lambda, <\kappa)$*-generic over $V$. In $V[G]$, $\kappa$ is ideally strong.*

Proof. Let $A \subseteq V[G]$. There is a name $\tau \in V$ for $A$. Let $\tilde{E}$ be the extender witnessing the strongness of $\kappa$ with respect to $\tau$. That is $\tilde{j} : V \to \mathrm{Ult}(V, \tilde{E})$ is such that $\tau \in \mathrm{Ult}(V, \tilde{E})$.

Now let $E$ be the ideal extender derived by $\tilde{E}$ in $V[G]$, as we have seen previously if $H$ is $E$-generic over $V$ and $j : V[G] \to \mathrm{Ult}(V[G], H)$ is the associated ultrapower, then $j \restriction V = \tilde{j}$. Moreover $G \in \mathrm{Ult}(V[G], H)$ thus we have that $A = \tau^G \in \mathrm{Ult}(V[G], H)$, which finishes the proof.

### 4.2.3 Iteration of ideal extenders

Let us now discuss the iteration of generic ultrapower by ideal extender.

**Definition 4.11.** A sequence:

$$\langle \langle M_i, E_i, \pi_{i,j}; i \leqslant j \leqslant \theta \rangle, \langle G_i; i < \theta \rangle \rangle$$

is a putative generic iteration of $M$ (of length $\theta + 1$) if and only if the following holds:

i. $M_0 = M$,

ii. for all $i < \theta$ $M_i \vDash$ "$E_i$ is an ideal extender",

iii. for all $i < \theta$ $G_i$ is $E_i$-generic over $M_i$,

iv. for all $i + 1 \leqslant \theta$ $M_{i+1} = \mathrm{Ult}(M_i, G_i)$ and $\pi_{i,i+q}$ is the associated generic ultrapower,

v. for all $i \leqslant j \leqslant k \leqslant \theta$ $\pi_{j,k} \circ \pi_{i,j} = \pi_{i,k}$,

vi. if $\lambda < \theta$ is a limit ordinal, then $\langle M_\lambda, \pi_{i,\lambda}; i < \lambda \rangle$ is the direct limit of the system $\langle M_i, \pi_{i,j}; i \leqslant j < \lambda \rangle$.

We call

$$\langle \langle M_i, E_i, \pi_{i,j}; i \leqslant j \leqslant \theta \rangle, \langle G_i; i < \theta \rangle \rangle$$

a generic iteration of $M$ (of length $\theta + 1$) if $M_\theta$ is well-founded. We call

$$\langle \langle M_i, E_i, \pi_{i,j}; i \leqslant j \leqslant \theta \rangle, \langle G_i; i < \theta \rangle \rangle$$

a putative generic iteration of $\langle M, E \rangle$ if the following additional clause holds true:

vii. for all $i + 1 < \theta$ $E_{i+1} = \pi_{i,i+1}(E_i)$.

Let $E$ be an ideal extender. We say that $G$ is $E$-generic if $G$ is a $\mathbb{P}_E$-generic filter.

**Lemma 4.12.** *Let $M$ be a countable transitive* ZFC *model and $F$ be a precipitous $\langle \kappa, \lambda \rangle$-ideal extender over $M$. Let $\theta < \sup\{M \cap \mathrm{OR}, \omega_1^V\}$. Then $M$ is $<\theta$-iterable by $F$. That is every putative iteration of $\langle M, F \rangle$ of length less or equal to $\theta$ is an iteration.*

Proof. This proof is an adaptation of Woodin's proof to the current context. By absoluteness if $\langle M, E\rangle$ is not generically $\theta+1$ iterable, it is not generically $\theta+1$ iterable in $M^{\mathrm{col}(\omega,<\delta)}$ for some $\delta$. Let $\langle\kappa_0,\eta_0,\gamma_0\rangle$ be the least tripe in the lexicographical order such that:

i. $\kappa < \omega_1^M$ is regular in $M$,

ii. $\eta_0 < \kappa_0$

iii. there is a $\delta$ and a putative iteration

$$\langle\langle M_i, E_i, \pi_{i,j}; i \leqslant j \leqslant \gamma_0\rangle, \langle G_i; i < \gamma_0\rangle\rangle$$

of $\langle H^M_{\kappa_0}; \in, E\rangle$ inside $M^{\mathrm{col}(\omega,<\delta)}$ such that $\pi_{0,\gamma_0}(\eta_0)$ is ill-founded.

Since $I$ is precipitous, $\gamma_0$ has to be a limit ordinal, $\eta_0$ has to be a limit ordinal in any case. Let $i^* < \gamma_0$ and $\eta^* < \pi_{i^*,\gamma_0}(\eta_0)$ be such that $\pi_{i^*,\gamma_0}(\eta^*)$ is ill-founded. Since $\kappa_0$ is regular we can consider

$$\langle\langle M_i, E_i, \pi_{i,j}; i^* \leqslant i \leqslant j \leqslant \gamma_0\rangle, \langle G_i; i^* \leqslant i < \gamma_0\rangle\rangle$$

as a putative iteration of $H^{M_{i^*}}_{\pi_{0,i^*}(\kappa_0)}$

By elementarity, $\langle\pi_{0,i^*}(\kappa_0), \pi_{0,i^*}(\eta_0), \pi_{0,i^*}(\gamma_0)\rangle$ is the least triple $\langle\kappa,\eta,\gamma\rangle$ such that condition i. to iii. holds with respect to $M_{i^*}$.

However as showed before the triple $\langle\pi_{0,i^*}(\kappa_0), \eta^*, \gamma_0 - i^*\rangle$ also fullfils i. to iii. and is lexicographically smaller than $\langle\pi_{0,i^*}(\kappa_0), \pi_{0,i^*}(\eta_0), \pi_{0,i^*}(\gamma_0)\rangle$, a contradiction! $\dashv$

### 4.2.4 The consistency strength of one ideally strong cardinal

**Lemma 4.13.** *Suppose $\neg(0^¶)$. Let $\kappa$ be ideally strong in $V$, then $\kappa$ is strong in the core model.*

Proof. Let $K = K^V$ be the core model below $(0^¶)$ as in [Jenc]. Let $\lambda \in \mathrm{OR}$. We have to show that there is an embedding $j : K \to M$ such that $K|\lambda \in M$.

Let $\lambda \in \mathrm{OR}$, by the ideal strongness of $\kappa$, there is an ideal extender $E$ such that if $G$ is $E$-generic over $V$:

$$K|\lambda \in \mathrm{Ult}(V, G) = M.$$

Claim 1. *In $V[G]$, $K$ iterates to $K^{\mathrm{Ult}(V,G)} = K^M = K^*$.*

$j$ exists in $V[G]$ and $K = K^{V[G]}$, hence by [Jenc, §5.3 Lemma 5 p. 7] $K^*$ is an iterate of $K$ and $j \restriction K$ is the iteration map.

Claim 2. $K|\lambda = K^*|\lambda$.

By the previous claim, we already know that $K|\nu = K^*|\nu$, where $\nu$ is the length of the first extender, $F$, of the iteration $j$. Since $F$ was used in the iteration, $F \notin K^*$. Suppose $K|\lambda \neq K^*|\lambda$, then $\mathrm{lh}(F) < \lambda$. Since $\mathrm{lh}(F) < \lambda$, $F \in K|\lambda \subseteq M$. By [Jenc, §5.2 Lemma 2 p. 3] we have that $\langle K|\mathrm{lh}(F), F\rangle$ is a generalized beaver for $K^*$ and hence $\mathrm{Ult}(K^*, F)$ is well-founded. Let us coiterate $K^*$ and $\mathrm{Ult}(K^*, F)$:

$$\begin{tikzcd}
K^* \arrow[r, "i_F"'] & \mathrm{Ult}(K^*,F) \arrow[dr, "k"] & \\
& & W \\
& K^* \arrow[ur, "i"'] &
\end{tikzcd}$$

Since $F \in M$, we can apply [Jenc, §5.3 Lemma 5 p. 7] to $k \circ i_F$ in $M$. We get that $k \circ i_F = i$ and thus $k = \mathrm{id}$, $i = i_F$. This shows that $F$ is on the $K^*$-sequence, a contradiction!

Thus we can assume that $\mathrm{lh}(F) \geqslant \lambda$ and so we have:

$$K|\lambda \triangleleft K^*.$$

Hence $j \restriction K : K \to K^*$ and $K|\lambda \in K^*$, which finishes the proof. ⊣

**Corollary 4.14.** *The existence of an ideally strong cardinal is equiconsistent to the existence of a strong cardinal.*

## 4.3 More ideally strong cardinals

As we have seen in the last section, lifting existing embeddings after forcing has been a very fruitful method to construct ideally strong cardinals. In this section, the lifting of various embeddings will be our main concern, especially when forcing "above" a large cardinal. In the last part we prove that such generic embeddings implies the existence of strong cardinals in the core model, giving a lower bound to our construction. Let us first put some light on the problems that arise, when constructing more than one ideally strong cardinal. The key problem is that, while forcing with so called "small forcings" preserves large cardinal properties, forcing above a strong cardinal $\kappa$ will, in general, destroy its strongness, even if we don't add a new subset of $\kappa$.

**Remark 4.15.** Let $\kappa$ be a strong cardinals and $\beta > 2^{(2^\kappa)^+}$, then in $V^{\mathrm{col}(\beta,\beta^+)}$ $\kappa$ is not necessarily $\beta^{++V}$-strong anymore.

Proof. Let $K = V$ be the minimal core model for one strong cardinal. Let $\kappa$ be strong in $V$ and $\beta$ as in the remark. Let $E$ be an extender witnessing the $\beta^{++V}$-strongness of $\kappa$, and $G$ a $\mathrm{col}(\beta, \beta^+)$-generic filter. Since $G$ does not add any $\omega$-sequence, $E$ is still an $\omega$-closed extender in the forcing extension. Let $M$ be the ultrapower of $V[G]$ by $E$ and $j$ the ultrapower map. Suppose $E$ is witnessing $\beta^{++V}$-strongness in $V[G]$. Then $G$ would be in $H^M_{\beta^{++V}}$ and thus $M$ believes that there is a $\mathrm{col}(\gamma, \gamma^{+K})$-generic filter over $K$ for some cardinal $\gamma \leq j(\kappa)$, hence $K$ believes that there is a $\mathrm{col}(\gamma, \gamma^{+K})$-generic filter over $K$ for some cardinal $\gamma \leq \kappa$, a contradiction! ⊣

### 4.3.1 Lifting of generic embeddings

With some more detailed analysis of the ultrapower by an extender we may lift the original issue.

**Lemma 4.16.** *Suppose GCH. Let $\kappa$ be a strong cardinal and $\lambda > \kappa$ a regular cardinal such that $2^{<\lambda} = \lambda$. Let $j : V \to M$ be an ultrapower by a $\langle \kappa, V_\lambda \rangle$-extender witnessing the $\lambda$-strongness of $\kappa$. Then for every $M$-sequence of ordinals $\lambda < \mu_i < \nu_i < \mu_{i+1} < j(\kappa)$ for $i < j(\kappa)$ such that $M \vDash$ "$\mu_i, \nu_i$ are regular cardinals", there is a $G \in V$ that is $\mathbb{P}$-generic over $M$, where $\mathbb{P}$ is the easton iteration of all* $\operatorname{col}(\mu_i, < \nu_i)^M$.

Proof. Remark that since each $\operatorname{col}(\mu_i, < \nu_i)^M$ is $\lambda$-closed in $M$, so is $\mathbb{P}$. Since $j$ is an ultrapower by a $\langle \kappa, V_\lambda \rangle$-extender[3], we have that:

i. $M$ is closed under sequence of length $\kappa$: ${}^{\kappa}M \cap V \subseteq M$,

ii. $H_\lambda \subseteq M$,

iii. $\lambda < j(\kappa) < \lambda^{+V}$.

Hence every dense set of $\mathbb{P}$ in $M$ is of the form $j(f)(a)$, for an $f : [V_\kappa]^{\overline{\overline{a}}} \to V_\kappa$ and some $a \in [V_\lambda^V]^{<\omega} \subseteq M$. By GCH we can count in $V$ all such $f$ in a sequence of order type $\kappa^+$. Let

$$\langle f_\xi ; \xi < \kappa^+ \rangle$$

be such a sequence. Moreover $V_\lambda^V$ has cardinality $\lambda$ in $M$ as well as in $V$. Using the fact that for any given $\xi$, $j(f_\xi) \in M$, in $M$ we can look at the set

$$X_\xi = \{ j(f_\xi)(a);\ a \in V_\lambda \wedge j(f_\xi)(a) \text{ is a dense set in } \mathbb{P} \}.$$

Since $M$ believes that the forcing iteration is an iteration of levy collapses of strong cardinals above $\lambda$, $\mathbb{P}$ is $\lambda$-closed in $M$. Now define the sequence $p_\xi$ for $\xi < \kappa^+$ as follows

i. $p_0$ be the empty condition,

ii. $p_{\xi+1}$ is a condition below $p_\xi$ and below each element of $X_\xi$,

iii. if $\nu < \kappa^+$ is a limit ordinal, let $p_\nu$ be some condition below each $p_\xi$ for $\xi < \nu$.

The successor steps works in $M$ because $X_\xi$ and $p_\xi$ are both in $M$ and $\mathbb{P}$ is $\lambda$-closed. For the limit steps: we can define the sequence $\langle p_\xi;\ \xi < \nu \rangle$ in $V$. Since $\nu < \kappa^+$ and $M$ is $\kappa$-closed, the sequence is in $M$ as well. Hence by the $\lambda$-closedness of $\mathbb{P}$ in $M$, there is a $p_\nu$ less than all the $p_\xi$ in $M$.

Now the sequence $\langle p_\xi ; \xi < \kappa^+ \rangle \subseteq M$ is definable in $V$, let $G \in V$ be the filter generated by all this points. $G$ is $\mathbb{P}$-generic over $M$. ⊣

[3] that is $E \subseteq \left\{ \langle a, x \rangle ; a \in [V_\lambda]^{<\omega} \text{ and } x \subseteq \mathcal{P}([\kappa]^{\overline{\overline{a}}}) \right\}$ for more on this type of extender see [MS89, p. 83 ff.]

**Lemma 4.17.** *Let $E$ be a $\langle\kappa, V_\lambda\rangle$-extender, where $\lambda$ is such that ${}^\kappa V_\lambda \subseteq V_\lambda$ and $\mathbb{P}$ a $\kappa$-distributive forcing. Let*

$$j : V \to M = \mathrm{Ult}(V, E)$$

*be the ultrapower map. Let $G$ be $\mathbb{P}$-generic over $V$, then $j$ can be lifted to an embedding*

$$\tilde{\jmath} : V[G] \to M[G'],$$

*where $G'$ is the completion of $j''G$ in $j(\mathbb{P})$.*

Proof. Let $E$ be as in the theorem and $j : V \to M = \mathrm{Ult}(V, E)$. We have that $M$ is closed under $\kappa$-sequences, that is ${}^\kappa M \cap V \subseteq M$. Let $\mathbb{P}$ be a $\kappa$-distributive forcing and $G$ be $\mathbb{P}$-generic over $V$. Let

$$G' = \{q \in j(\mathbb{P});\ \exists p \in G, j(p) \leq q\}.$$

We claim that $G'$ is already generic over $M$! Let $D = j(f)(a)$ be some dense open set in $M$. This implies

$$\left\{u \in [V_\kappa]^{\overline{a}}; f(u) \text{ is a dense open set of } \mathbb{P}\right\} \in E_a$$

but then the set

$$A = \left\{f(u);\ u \in [V_\kappa]^{\overline{\overline{a}}} \wedge f(u) \text{ is a dense open set of } \mathbb{P}\right\}$$

has only size $\kappa$. By $\kappa$-distributivity of $\mathbb{P}$, $\bigcap A$ is still dense. Let $p \in A \cap G$, we have that $j(p) \in D \cap G'$. ⊣

Notice that this lemma alone does not give the the desired result since $G$ itself might not be in $M[G']$. We want to combine this and the techniques developed in the measurable case to get the desired result. Sadly for the forcing we have in mind, using only strongness will not suffice. We will use the concept of $A$-strongness to bypass this problem.

### 4.3.2 Forcing two ideally strong cardinals

**Lemma 4.18.** *Let $A$ be the class of all strong cardinal. Suppose $V \models$ "GCH, $\kappa$ is an $A$-strong cardinal, $\delta > \kappa$ is the only strong cardinal above $\kappa$". Let $n : \mathrm{OR} \to \mathrm{OR}$ such that $n(\gamma) = \gamma^+$ and let $\gamma_\mu$ denote the smallest strong cardinal above $\mu$. For $\gamma$ strong, let $\mathbb{P}_\gamma$ be $\mathrm{col}(n(\gamma), < \mu_\gamma)$, the levy collaps of $\mu_\gamma$ to $n(\gamma)$ and let $\mathbb{P}$ be the easton support iteration of all $\mathbb{P}_\gamma$ for $\gamma$ strong such that $\mu_\gamma$ exists. Let $G$ be $\mathbb{P}$-generic over $V$. In $V[G]$, $\kappa$ is strong.*

Proof. We first follow the same strategy as in the measurable case. For some set of ordinals $I$, let $\mathbb{P} \restriction I$ be the easton forcing iteration of $\mathbb{P}_\gamma$ for all $\gamma \in I$. Let $G$ be $\mathbb{P}$-generic over $V$, let $\lambda > \delta$ be a large enough regular cardinal with ${}^{\kappa}V_\lambda \subseteq V_\lambda$, we have to show that there is an embedding

$$\tilde{\jmath} : V[G] \to \tilde{M}$$

with the property that $H_\lambda^{V[G]} \subseteq M$. Let $E$ be an $\langle \kappa, V_\lambda \rangle$-extender witnessing that $\kappa$ is $A$-$\lambda$-strong in V. We want to lift up the embedding $j : V \to M$ associated to $E$.

Let us recall the cardinal arithmetic setting. We have that

$$\lambda < \mathrm{card}^V(j(\kappa)) < \mathrm{card}^V(j(\delta)) < \mathrm{card}^V((2^{j(\delta)})^M) < \lambda^{+V}.$$

Moreover since we use $V_\lambda$ to index the extender, the ultrapower is closed under $\kappa$-sequences. Notice that since $E$ is a witness that $\kappa$ is $A$-$\lambda$-strong, we have that $\mathbb{P} \subseteq j(\mathbb{P})$. Since $G$ is $\mathbb{P}$-generic over $V$ and $(H_\lambda)^V = (H_\lambda)^M$, we thus have that $G$ is $\mathbb{P}^M \upharpoonright \kappa = \mathbb{P}$-generic over $M$.

If we can show that there is an $\tilde{G} \in V[G]$ such that $G * \tilde{G}$ is $j(\mathbb{P})$-generic over $M$ and $j"G = \tilde{G} \cap \mathrm{ran}(j)$, we will be able to lift the embedding $j$ to an embedding

$$\tilde{\jmath} : V[G] \to M[G \times \tilde{G}]$$

Let $\sigma_{\mathbb{P}^*}$ be an $M^{\mathbb{P}}$-name for $j(\mathbb{P}) \upharpoonright [\delta, j(\kappa)[$ and $\mathbb{P}^* = \sigma_{\mathbb{P}^*}^G$. Let further $\sigma_{\mathbb{P}^{**}}$ be a $M^{\mathbb{P} * \mathbb{P}^*}$-name for $\mathbb{P}_{j(\kappa)}$.

We want to find $G^*$, a $\mathbb{P}^*$-generic filter over $M[G]$ and $G^{**}$, a $\mathbb{P}_{j(\kappa)}$-generic filter over $M[G \times G^*]$. That way using the factor lemma [Jec03, Lemma 21.8 pp. 396], we will have that $G \times G^* \times G^{**}$ is a $j(\mathbb{P})$-generic filter over $M$. In order to produce a $\mathbb{P}^*$-generic filter, we'd like to use Lemma 4.16, sadly we need a filter generic over $M[G]$ rather than just $M$. Let us argue why the proof still holds true.

Claim 1. *There is a filter $G^* \in V[G]$ that is $\mathbb{P}^*$-generic over $M[G]$.*

Proof. We want to run the very same argument as in Lemma 4.16. Let us first show that $M[G]$ is still closed under $\kappa$-sequences. Let $\tau$ be the name for a $\kappa$-sequence in $V[G]$. Without loss of generality, we can assume that $\tau$ is a nice name, that is, it is of the form

$$\tau = \{\langle\langle \eta, \xi\rangle, q\rangle;\ \eta < \kappa \wedge q \in A^\eta \wedge q \Vdash \tau(\eta) = \xi\},$$

where $A^\eta$ is a maximal antichain. Since $A^\eta \in V_\lambda$, each $A^\eta$ is in $M$. Since $M$ is closed under $\kappa$-sequences. the sequence of all $A^\eta$ is in $M$ as well and thus $\tau$ is in $M$. Therefore $M[G]$ is closed under $\kappa$-sequences from $V[G]$. Every dense set of $\mathbb{P}^*$ in $M$ is of the form $j(f)(a)^G$, for an $f : [V_\kappa]^{\overline{\overline{a}}} \to V_\kappa^{\mathbb{P}}$ and some $a \in V_\lambda^V = V_\lambda^M$, as $j$ is the ultrapowermap generated by $E$. By $\mathsf{GCH}$ we can count in $V$ all such $f$ in a sequence of order type $\kappa^+$, $\langle f_\xi ; \xi < \kappa^+\rangle$. Also remark that $V_\lambda$ has cardinality $\lambda$ in $M[G]$ as well as in $V[G]$. Using the fact that for any given $\xi$, $j(f_\xi) \in M$, in $M[G]$ we can look at the set

$$X_\xi = \left\{ j(f_\xi)(a)^G;\ a \in V_\lambda \wedge j(f_\xi)(a) \text{ is a } \mathbb{P}\text{-name for a dense set in } \mathbb{P}^* \right\}.$$

Since $M[G]$ believes that the forcing iteration $\mathbb{P}^*$ is an iteration of levy collapses of strong cardinals above $\lambda$, $\mathbb{P}^*$ is $\lambda$-closed in $M$. Now define the sequence $p_\xi$ for $\xi < \kappa^+$ as follows

i. $p_0$ be the empty condition,

ii. $p_{\xi+1}$ is a condition below $p_\xi$ and below each element of $X_\xi$,

iii. if $\nu < \kappa^+$ is a limit ordinal, let $p_\nu$ be some condition below each $p_\xi$ for $\xi < \nu$.

The successor steps works in $M[G]$ because $X_\xi$ and $p_\xi$ are both in $M[G]$ and $\mathbb{P}$ is $\lambda$-closed, the limit steps works because they are definable sequences in $V[G]$ of length at most $\kappa$, hence by the $\kappa$-closedness of $M[G]$ the sequences are also in $M[G]$, hence by the $\lambda$-closedness of $\mathbb{P}$ in $M[G]$, $p_\nu$ is definable in $M[G]$.

Now the sequence $\langle p_\xi; \xi < \kappa^+\rangle \subseteq M$ is definable in $V[G]$, let $G^* \in V[G]$ be the filter generated by all this points. $G^*$ is $\mathbb{P}^*$-generic over $M[G]$. $\dashv$

Let $\mathbb{P}^{**} = \sigma^{G\times G^*}_{\mathbb{P}^{**}}$. Setting $G'$ and $G''$ sucht that $G' = G \cap H_\kappa$ and $G' \times G'' = G$, we can see that we are already able to lift $j$ to some $j_1 : V[G'] \to M[G' * G^*]$. As in the last step, we won't be able to directly use the appropriate lemma, in this case Lemma 4.17. But with some small modification, the main idea of the lemma carries on in our situation.

Notice that $\mathbb{P}$ is an iteration of successor length. Let $\tau \in M$ be a name for an open dense set of $\mathbb{P}^{**}$. Hence there is a $a \in [V_\lambda]^{<\omega}$ and a $f : [V_\kappa]^{\overline{\overline{a}}} \to V$ such that $\tau = j(f)(a)$ and

$$X = \left\{u \in [V_\kappa]^{\overline{\overline{a}}};\ f(u) \text{ is a } \mathbb{P} \upharpoonright \kappa\text{-name for an open dense set in } \mathbb{P}_\kappa\right\} \in E_a$$

We have that $\left\{(f(u))^{G'};\ u \in X\right\}$ is of cardinality $\kappa$ hence the intersection of all such sets

$$\dot{D}^{G'} = \bigcap \left\{(f(u))^{G'};\ u \in X\right\}$$

is still a dense set in $V[G']$, where $\dot{D}$ is a name such that there is a $q \in G'$ with

$$q \Vdash \text{“}\dot{D} = \bigcap\{(f(u));\ u \in X\} \text{ and } \dot{D} \text{ is a dense set”}.$$

Let $\sigma^{G'} \in \dot{D}^{G'} \cap G''$ and let $p \in G'$, $p < q$, with $p \Vdash$ “$\sigma \in \dot{D}$”.

We have that $p = j(p) \Vdash$ “$j(\sigma) \in \tau$”. Since $p \in G' \subseteq G * G^*$ it follows that $j(\sigma)^{G*G^*} \in \tau^{G*G^*}$. As the iteration has an easton support, it is is bounded below $\kappa$ at stage larger or equal to $\kappa$. This shows that:

$$j(\sigma)^{G*G^*} = j''\sigma^G \in j''G''.$$

Thus $G^{**}$, the closure of $j''G''$ in $M[G * G^*]$, is a $\mathbb{P}^{**}$-generic filter over $M[G * G^*]$. By the factor lemma $G * G^* * G^{**}$ is $j(\mathbb{P})$-generic over $M$. Setting $\tilde{G} = G^* * G^{**}$, we get the desired result by lifting $j$ using the classical definition:

$$j(\tau^G) = j(\tau)^{G*\tilde{G}}$$

for $\tau$ a $\mathbb{P}$-name in $V$. $\dashv$

**Corollary 4.19.** *Let $A$ be the class of all strong cardinal. Suppose $V \vDash$ "GCH, $\kappa$ is an $A$-strong cardinal, $\delta > \kappa$ is the only strong cardinal above $\kappa$". Then for every successor cardinal $\mu$ below the least strong cardinal, there is a forcing $\mathbb{Q}$ such that, whenever $G$ is $\mathbb{Q}$-generic over $V$, $\kappa$ and $\delta$ are ideally strong in $V[G]$, $\mu^{+V[G]} = \kappa$ and $n(\kappa)^{+V[G]} = \lambda$.*

Proof. First apply $\mathbb{P} \restriction \kappa$, where $\mathbb{P}$ is the forcing defined in the previous lemma. From the point of view of $\lambda$ this is a small forcing, hence if $G'$ is $\mathbb{P} \restriction \kappa$-generic over $V$, $\lambda$ is strong in $V[G']$. Now we can just do the levy collaps $\mathrm{col}(n(\kappa), \lambda)$. By results of the last section, $\lambda$ is ideally strong in $V[G', G'']$, where $G''$ is $\mathrm{col}(n(\kappa), \lambda)$-generic over $V[G']$. By the factor lemma, this is the same as forcing in one time with $\mathbb{P}$ as defined in the last lemma. Let $G_1 = G' \times G''$. In $V[G_1]$, $\kappa$ is still strong, hence we can force with $\mathrm{col}(\mu, < \kappa)$ for some regular cardinal $\mu$. Let $G_2$ be $\mathrm{col}(\mu, < \kappa)$-generic over $V[G_1]$ and set $G = G_1 \times G_2$. In $V[G]$ $\kappa$ is ideally strong. Let us show that $\lambda$ remains ideally strong in $V[G]$.

Remark that $\mathrm{col}(\mu, < \kappa) \cap V[G_1]$ and $\mathrm{col}(\mu, < \kappa) \cap V$ are forcing equivalent. Hence we can first force with $\mathrm{col}(\mu, < \kappa) \cap V$ and then force with $\mathbb{P}$. In the first extension $V[G_2]$, $\lambda$ remains strong hence by the previous theorem $\lambda$ is ideally strong in $V[G_2, G_1]$. ⊣

### 4.3.3 Forcing many generically strong cardinals

**Definition 4.20.** We say that a cardinal $\kappa \in V$ is *generically strong* if for all $A \in V$ there is a forcing $\mathbb{P}$ such that, if $G$ is $\mathbb{P}$-generic over $V$, in $V[G]$ there is a definable embedding $j : V \to M \subseteq V[G]$ with critical point $\kappa$ and $A \in M$.

Obviously if $\kappa$ is ideally strong it is generically strong.

**Lemma 4.21.** *Let $\kappa, \lambda$ be two strong cardinals and $\mu, \nu$ two successor cardinals such that $\mu < \kappa < \nu < \lambda$ . Let $\mathbb{P} = \mathrm{col}(\mu, < \kappa) \times \mathrm{col}(\nu, < \lambda)$ and let $G$ be $\mathbb{P}$-generic over $V$. In $V[G]$ $\kappa$ and $\lambda$ are generically strong.*

Proof. Let $\mu, \nu, \kappa, \lambda$ and $G$ be as in the theorem. After forcing with $\mathrm{col}(\mu, < \kappa)$, $\lambda$ remains strong, hence by Theorem 4.10, $\lambda$ is an ideally strong cardinal in $V[G]$. Let $X \in V[G]$ be some set. We only have to show that there is a forcing $\mathbb{P}$, such that if $H$ is $\mathbb{P}$-generic over $V[G]$, there is a definable embedding in $V[G, H]$,

$$\tilde{\jmath} : V[G] \to \tilde{M}$$

such that $G, X \in \tilde{M}$ and $\mathrm{cp}(\tilde{\jmath}) = \kappa$. Split $G$ into $G_\kappa$ $\mathrm{col}(\mu, < \kappa)$-generic over $V$ and $G_\lambda^\nu$, $\mathrm{col}(\nu, < \lambda) \cap V$-generic over $V[G_\kappa]$.

Let $\tau$ be a $\mathbb{P}$-name for $X$ and let $\theta$ be a large enough regular cardinal such that $\{\tau\} \cup (2^\lambda)^+ \subseteq H_\theta$ and ${}^\kappa V_\theta \subseteq V_\theta$. Since $\kappa$ is strong in $V$ there is a $\langle \kappa, V_\theta \rangle$-extender, say $E$, such that $H_\theta \subseteq \mathrm{Ult}(V, E)$. Let

$$j : V \to M = \mathrm{Ult}(V, E)$$

be the associated ultrapowermap. We have that $\theta < j(\kappa) < \theta^{+V}$. By Lemma 4.17, we know that we can lift $j$ to $\bar{\jmath}' : V[G_\lambda^\nu] \to M[G_\lambda^j]$, where

$$G_\lambda^j = \{q \in \mathrm{col}(j(\nu), j(\lambda)) \cap M;\ \exists p \in G_\lambda^\nu\ q < j(p)\}$$

is $\mathrm{col}(j(\nu), j(\lambda))$-generic over $M$.

Let $G_{\theta^+}$ be $\mathrm{col}(\mu, <\theta^+)$-generic over $V$, such that:

i. $G \in V[G_{\theta^+} \cap \mathrm{col}(\mu, <\lambda^{+V})]$,

ii. $G_\kappa = G_{\theta^+} \cap \mathrm{col}(\mu, <\kappa)$.

We can construct such an $G_{\theta^+}$, because by [Fuc08, lemma 2.2] $\mathrm{col}(\mu, <\lambda) \times \mathrm{col}(\mu, \lambda)$ is forcing equivalent to $\mathrm{col}(\mu, \{\lambda\})$. Let $G_{\theta+1} = G_{\theta^+} \cap \mathrm{col}(\mu, <\theta+1)$. We first want to create a $\mathrm{col}(\mu, ]\theta, j(\kappa)[)^{M[G_{\theta+1}]}$-generic filter over $M$, $G_1$.

Claim 1. *There is a $G_{j(\kappa)} \in V[G_{\theta+1}]$ such that:*

*i. $G_{j(\kappa)}$ is $\mathrm{col}(\mu, j(\kappa))$ generic over $M$,*

*ii. $G_{j(\kappa)} \cap (\mathrm{col}(\mu, <\theta))^V = G_{\theta+1} \cap (\mathrm{col}(\mu, <\theta))^V$,*

*iii. we can lift $j$ to some:*

$$j \subseteq \bar{\jmath} : V[G_\kappa] \to M[G_{j(\kappa)}].$$

Proof. Remark that since $H_\theta^V = H_\theta^M$, we have that

$$\mathrm{col}(\mu, <\theta+1) \cap M = \mathrm{col}(\mu, <\theta+1) \cap V$$

As $M \subseteq V$, we have that $G_{\theta+1}$ is $\mathrm{col}(\mu, <\theta+1)$-generic over $M$ as well. Now look at $\mathrm{col}(\mu, ]\theta, j(\kappa)[) \cap M$ in $V$; As $M$ is $\kappa$-closed it is a $<\mu$-closed forcing in $V$. $\mathrm{col}(\mu, ]\theta, j(\kappa)[) \cap M$ adds a surjective function from $\mu$ to $\theta$. By [Fuc08, lemma 2.2] it is forcing equivalent to $\mathrm{col}(\mu, \{\theta\})$, hence we can define a $\mathrm{col}(\mu, ]\theta, j(\kappa)[) \cap M$-generic filter $G_1$ over $V$ from $G_{\theta^+} \cap \mathrm{col}(\mu, \{\theta+1\})$. But since $M \subseteq V$, being a dense set is upward absolute between the two models, hence $G_1$ is also generic over $M$. Set

$$G_{j(\kappa)} = G_{\theta+1} \times G_1.$$

By the product lemma, $G_{j(\kappa)}$ is $\mathrm{col}(\mu, <j(\kappa)) \cap M$-generic over $M$. Remark that, as $j''G_\kappa \subseteq G_{j(\kappa)}$, we can lift $j$ to an embedding

$$\bar{\jmath} : V[G_\kappa] \to M[G_{j(\kappa)}].$$ ⊣

Remark that since $\mathrm{col}(j(\nu), j(\lambda))$ is $< j(\nu)$ closed, $G_{j(\kappa)}$ is $\mathrm{col}(\mu, <j(\kappa) \cap M$-generic over $V[G_\lambda^j]$ as well. Hence by the product forcing theorem $G_{j(\kappa)} \times G_\lambda^j$ is $\mathrm{col}(\mu, <j(\kappa)) \cap M \times \mathrm{col}(j(\nu), <j(\lambda)) \cap M$-generic over $M$.

Let $\tilde{\jmath} : V[G] \to M[G_{j(\kappa)} \times G_\lambda^j]$ be such that

$$\tilde{\jmath}(\tau^G) = j(\tau)^{G_{j(\kappa)} \times G_\lambda^j},$$

where $\tau$ is a $V^{\mathbb{P}}$-name.

Claim 2. *$\tilde{\jmath}$ is a fully elementary embedding that lifts $j$.*

Proof. Let $\varphi$ be some formula such that $V[G] \vDash \varphi(\tau^G)$ for some $\mathbb{P}$-name $\tau$. There is a $p \in G$ such that

$$p \Vdash \varphi(\tau)$$

Hence by the elementarity of $j$:

$$j(p) \Vdash \varphi(j(\tau))$$

But by construction $j(p) \in G_{j(\kappa)} \times G^j_\lambda$ and $j(\tau)^{G_{j(\kappa)} \times G^j_\lambda} = \tilde{\jmath}(\tau^G)$, hence

$$M[G_{j(\kappa)} \times G^j_\lambda] \vDash \varphi(\tilde{\jmath}(\tau^G)).$$

Let $x \in V$ then $x = \check{x}^G$ and thus $\tilde{\jmath}(x) = j(\check{x})^{G_{j(\kappa)} \times G^j_\lambda} = j(x)$, as $j(\check{x}) = \check{j(x)}$. Hence $j \subseteq \tilde{\jmath}$.⊣

Hence we can lift $j$ to

$$\tilde{\jmath} : V[G] \to M[G_{j(\kappa)} \times G^j_\lambda]$$

on the other hand $\tau, G \in M[G_{j(\kappa)} \times G^j_\lambda]$. As $\tilde{\jmath}$ is definable from $j$, $G$ and $G_{j(\kappa)} \times G^j_\lambda$, it is definable in $V[G_{\theta^+}]$. Hence $\kappa$ is generically strong in $V[G]$. ⊣

Notice that the proof actually showed:

**Theorem 4.22.** *Let $\kappa$ be strong in $V$ and $\mu < \kappa$ some cardinal. Let $G$ be* $\mathrm{col}(\mu, <\kappa)$*-generic over $V$ and $\mathbb{P}$ some $<\kappa^+$-closed forcing in $V$. Let $H$ be $\mathbb{P}$ -generic over $V[G]$. Then $\kappa$ is generically strong in $V[G,H]$.*

Proof. Let $\mathbb{P}$, $G$ and $H$ be as in the lemma. Let $\theta$ be some large cardinal, such that $\mathbb{P} \in H_\theta$ and ${}^\kappa V_\theta \subseteq V_\theta$. It suffices to prove that there is some embedding

$$\pi : V[G,H] \to M,$$

such that $H^V_\theta \subseteq M$. Let $E$ be a $\langle \kappa, V_\theta \rangle$-extender and $j$ the associated ultrapower. By Lemma 4.17, we can lift $j$ to some

$$\bar{\jmath} : V[H] \to M[H^j],$$

where $H^j$ is the $M$-closure of $j''H$ in $M$. The last proof showed that we can then lift $j$ to some

$$\tilde{\jmath} : V[G,H] \to V[G_{j(\kappa)}, H^j],$$

where $G_{j(\kappa)}$ is some $\mathrm{col}(\mu, j(\kappa))$-generic filter over $M$ such that $G = G_{j(\kappa)} \cap \mathrm{col}(\mu,\kappa)$ and $H \in M[G_{j(\kappa)}]$. ⊣

It is not hard to see that applying this theorem to the easton support forcing product of the levy collapse of strong cardinals, we get the following corollary:

**Corollary 4.23.** *$A$ is a set of strong cardinals such that* $\mathrm{otp}(A) < \min(A)$*, and let $f : A \to \mathrm{OR}$ a function such that for all $\mu \in A$, $f(\mu)$ is a successor cardinal and for all $\mu < \nu \in A$ $\mu < f(\nu)$. Then there is a forcing $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$:*

*i. every $f(\mu)$ is a successor cardinal, moreover $f(\mu)^{+V[G]} = \mu$*

*ii. every $\mu$ in $A$ is generically strong in $V[G]$.*

Proof. Let $A$, $f$ be as in the theorem and $\mathbb{P}$ the easton support forcing product of all $\mathbb{Q}_\kappa$ for $\kappa \in A$, where $\mathbb{Q}_\kappa = \mathrm{col}(f(\kappa), < \kappa)$. That is $p \in \mathbb{P}$ if $p \in \Pi_{\kappa \in A}\mathbb{Q}_\kappa$ and for all limit point $\lambda$ of $A$, the set of $i < \lambda$ such that $(p)_i \neq \mathbb{1}_{\mathbb{Q}_i}$ is bounded in $\lambda$. For every $\kappa \in A$ we can split the forcing $\mathbb{P}$ in three pieces $\mathbb{P}_\kappa$ the easton support product of all $\mathbb{Q}_i$ for $i \in A \cap \kappa$, $\mathbb{Q}_\kappa$ and $\mathbb{P}^\kappa$ the easton support product of all $\mathbb{Q}_i$ such that $i \in A \setminus \kappa + 1$. Notice that $\mathbb{P}^\kappa$ is $\kappa$-closed. For every filter $G$, $\mathbb{P}$-generic over $V$, let $G_\kappa = G \cap \mathbb{P}_\kappa$ and $G^\kappa = G \cap (\mathbb{Q}_\kappa \times \mathbb{P}^\kappa)$. $\mathbb{P}_\kappa$ is a small forcing, hence $\kappa$ is strong in $V[G_\kappa]$, by Theorem 4.22 $\kappa$ is generically strong in $V[G_\kappa, G^\kappa] = V[G]$. ⊣

Giving one concrete example of such a function $f$:

**Corollary 4.24.** *Suppose* ZFC+ *"there are $\omega$ strong cardinals" is consistent, then so is* ZFC+ *"every $\aleph_{2n+1}$ is generically strong for $n \in \omega$"*

### 4.3.4 The consistency strength of many generically strong cardinals

We have seen how to get many generically strong cardinals, starting with the same amount of strong cardinals. Let us now answer the reverse question, whether one gets the strong cardinals "back". Let $\Omega$ be some large measurable cardinal and $\mu_0$ a $< \Omega$-complete ultrafilter on $\Omega$. From now on we will work in $V_\Omega$.

**Theorem 4.25.** *Suppose there is no inner model with a Woodin cardinal. Let $\kappa$ be generically strong in $V$, then $\kappa$ is strong in the core model.*

Proof. Let $K = K^V$ be the core model as defined in [Ste96]. We work towards contradiction.

Claim 1. *Suppose $\kappa$ is not strong in $K$, there is a $\theta$ such that for every $\nu > \theta$ either $\mathrm{cp}(E^K_\nu) > \theta$ or $\mathrm{cp}(E^K_\nu) < \kappa$.*

Proof. Let $\theta$ be smallest cardinal strictly larger than the Mitchell order of $\kappa$. We claim that $\theta$ has already the desired properties. Suppose not and let $F$ be an extender on the $K$-sequence with critical point $\lambda < \theta$ and index $\nu > \theta$. Let $\mathcal{M} = \mathrm{Ult}(K\|\nu, F)$ and $j$ be the associated ultrapower map. We know that $K|\nu \models$ "$\kappa$ is $\lambda$-strong", hence $\mathcal{M} \models$ "$\kappa$ is $j(\lambda)$-strong". Let $E$ be some extender of the $\mathcal{M}$ sequence with critical point $\kappa$ and index larger than $\theta$. Since $\theta$ is a cardinal, there must be cofinally many $E$-generators below $\theta$. Let $\mu$ be such a generator. Then $E \restriction \mu + 1$ has natural length $\mu + 1$, hence by the initial segment condition either the completion of $E$ is on the $\mathcal{M}$ sequence or it is one ultrapower away. Since $\mu + 1$ is a successor ordinal, the second case can not occur.Thus, we have that there is some $\mu + 1 < \gamma < \nu$ such that $E^{\mathcal{M}}_\gamma$ is the trivial completion of $E \restriction \mu + 1$. Since $\gamma < \nu$ we have that $E^{\mathcal{M}}_\gamma = E^K_\gamma$ by coherency. Hence for every $\nu < \theta$ we can find a $\gamma > \nu$ such that $E_\gamma$ has critical point $\kappa$ and is on the $K$ sequence, a contradiction to the definition of $\theta$! ⊣

Let $\theta$ be as in the claim. Since $\kappa$ is generically strong, there is a forcing $\mathbb{P}$ such that for every $\mathbb{P}$-generic $G$ over $V$, there is an embedding in $V[G]$

$$j : V \to M$$

with $H^V_{\theta^+} \in M$. Let $K^M$ be the core model as computed in $M$. Notice that $K^V = K^{V[G]}$, we will drop the superscript and call it $K$ in what follows.

Claim 2. *$K^M$ is a universal weasel in $V[G]$.*

Proof. The same proof as Lemma 3.22 shows that $K^M$ is iterable in $V[G]$. The set of fixed point of $j$ is a club set in $\{\alpha;\ \mathrm{cf}(\alpha) \neq \kappa\}$, but since $\mathbb{P}$ has the $(2^{\mathrm{card}(\mathbb{P})})^{+V}$-c.c. for stationary many successor of some fixed point $\alpha$ of $j$, we have that $\alpha^{+V} = \alpha^{+K} \leq j(\alpha)^{+K^M} \leq \alpha^{+M} \leq \alpha^{+V[G]}$. For all $\alpha$ larger than $(2^{\mathrm{card}(\mathbb{P})})^{+V}$, $\alpha^{+V} = \alpha^{+V[G]}$. Hence weak covering is true for some thick class in $K^M$, hence it is a universal weasel in $V[G]$. $\dashv$

We would like to coiterate $K$ with $K^M$, but then the following diagram might not be commutative.

$$\begin{tikzcd} K \arrow[dr, "\pi^{\mathcal{T}}_{\Lambda}"] \arrow[dd, "j"'] & \\ & Q \\ K^M \arrow[ur, "\pi^{Q}_{\Lambda}"'] & \end{tikzcd}$$

By slightly modifying the iterations, we can get a common iterate in a way that makes the triangle commutative. We will use a variation of the technique from the proof of Lemma 7.13 of [Ste96]. By [Ste96, Lemma 8.3] there is a universal weasel $W$ such that $K|\theta \lhd W$ (in fact $W$ witness that $K|\theta$ is $A_0$-sound), $W$ has the hull property at all $\alpha$ and the definability property at all $\alpha < \theta$. By [Ste96, Lemma 8.2], $W$ is a simple iterate of $K$, actually the iteration $\mathcal{T}_0$ from $K$ to $W$ is linear and only uses measures, that is extenders with only one generator. Finally if $\pi^{\mathcal{T}_0}_{0,\infty}$ is the iteration map, by applying $j$ we get an iteration tree $j(\mathcal{T})$ on $K^M$ such that the whole commutes as in the following diagram:

$$\begin{tikzcd} K \arrow[r, "\pi^{\mathcal{T}_0}_{0,\infty}"'] \arrow[d, "j"] & W \arrow[d, "j"] \\ K^M \arrow[r, "\pi^{\mathcal{T}_0^j}_{0,\infty}"'] & j(W) \end{tikzcd}$$

We have that

$$\mathrm{Def}(W) = \pi^{\mathcal{T}_0}_{0,\infty}{}''K$$

and the iteration is above $\theta$. We can lift that iteration via $j$ to get an linear iteration of $K^M$. Since the class of fixed points of $j$ is thick in $W$, $\Omega$ is thick in $j(W)$ and

$$\mathrm{Def}(j(W)) = j''\,\mathrm{Def}(W).$$

Let us coiterate $W$ and $j(W)$ in $V[G]$ and let $\mathcal{T}_1$ and $\mathcal{U}$ be the respective trees of the coiteration. Since both $W$ and $j(W)$ are universal weasel in $V[G]$ there is no drop on both side of the iteration. The coiteration might not commute on the whole range, but it does commute on $\mathrm{ran}(\pi^{\mathcal{T}_0}_{0,\infty}) = \mathrm{Def}(W)$ since all the elements of $\mathrm{Def}(W)$ are definable with skolem terms and parameters in a thick class of fix points, see Remark 1.53. Hence if we set $\mathcal{T} = \mathcal{T}_0{}^\frown\mathcal{T}_1$ and $\mathcal{Q} = j(\mathcal{T}_0)^\frown\mathcal{U}$

$$\begin{tikzcd}
K \arrow[r, "\pi^{\mathcal{T}}"] \arrow[dd, "j"] & W \arrow[dd, "j"] \arrow[dr, "\pi^{\mathcal{T}_1}"] & \\
 & & Q \\
K^M \arrow[r, "\pi^{j(\mathcal{T})}"'] & j(W) \arrow[ur, "\pi^{\mathcal{U}}"] &
\end{tikzcd}$$

This shows that $K$ and $K^M$ iterate via $\langle\mathcal{T},\mathcal{U}\rangle$ to a common model $Q$ such that the iterations commute with $j$. Let $\pi^{\mathcal{T}}_{\Lambda} : K \to Q$ be the iteration map on the $K$ side and $\pi^{\mathcal{Q}}_{\Lambda} : K^M \to Q$ the iteration map on the $K^M$ side, where $\Lambda$ is the length of the iteration.

Claim 3. *There is no $\mu \leq \kappa$ such that the coiteration uses extenders with critical point $\mu$ on both side of the coiteration.*

Proof. Suppose not an let $E$ be the first extender with critical point $\mu \leq \kappa$ used on the $W$ side and $F$ the first extender with critical point $\mu$ used on the $j(W)$ side. Notice that $\mu$ has the same subsets in every model. Let $\Gamma$ be a thick class of fixed points of $\pi^{\mathcal{T}}_{\Lambda}$ and $\pi^{\mathcal{Q}}_{\Lambda} \circ j$. Suppose $\mathrm{lh}(E) < \mathrm{lh}(F)$, and let $X \in E_a$. Since $W$ has the hull and definability property at all $\alpha < \theta$, there are $\vec{\eta} \in \Gamma$ and a skolem term $\tau$ such that $X = \tau^W(\vec{\eta})$. Hence

$$X = \tau^{j(W)}(\vec{\eta}) \cap \kappa^{\overline{\overline{a}}}$$

Notice that we need to cut with $\kappa$ just for the case $\mu = \kappa$. As $\pi^{\mathcal{T}}_{\Lambda}(X) = \tau^Q(\vec{\eta})$ and

$$\pi^{\mathcal{Q}}_{\Lambda}(X) = \tau^Q(\vec{\eta}) \cap \pi^{\mathcal{Q}}_{\Lambda} \circ j(\kappa^{\overline{\overline{a}}}).$$

Since $\mathrm{cp}(F) \leq \kappa$ and $a \in [\mathrm{lh}(F)]^{<\omega}$, $a \in [\pi^{\mathcal{Q}}_{\Lambda} \circ j(\kappa)]^{<\omega}$. Thus we have the following equivalence:

$$\begin{aligned}
X \in E_a &\iff a \in \pi^{\mathcal{T}}_{\Lambda}(X) \\
&\iff a \in \tau^Q(\vec{\eta}) \\
&\iff a \in \tau^Q(\vec{\eta}) \cap \pi^{\mathcal{Q}}_{\Lambda} \circ j(\kappa^{\overline{\overline{a}}}) \\
&\iff a \in \pi^{\mathcal{Q}}_{\Lambda}(X) \\
&\iff X \in F_a
\end{aligned}$$

Hence $E$ and $F$ are compatible, a contradiction to Lemma 1.49! If $\mathrm{lh}(E) > \mathrm{lh}(F)$, we can argue the very same way. $\dashv$

Claim 4. $\mathrm{cp}(\pi^{\mathcal{T}}_{\Lambda}) = \kappa$ *and* $\mathrm{cp}(\pi^{\mathcal{Q}}_{\Lambda}) > \kappa$.

Proof. By construction the iteration $\mathcal{T}_0$ is above $\kappa$, we claim that $\mathcal{T}_1$ does not have critical points less than $\kappa$ on the main branch. Suppose not, and let $\mu$ be the smallest ordinal such that there is an extender with critical point $\mu$ used in the coiteration. By commutativity, $\mu$ is the smallest on the $j(W)$ side as well. Hence both side would have use an extender with identical critical point less than $\kappa$ a contradiction to the previous claim! Thus the critical point of $\pi^{\mathcal{T}}_{\Lambda}$ is at least $\kappa$. Since the diagram commutes and $j$ has critical point $\kappa$, $\pi^{\mathcal{T}}_{\Lambda}$ must have critical point $\kappa$ as well. By the previous claim, this implies that $\mathrm{cp}(\pi^{\mathcal{Q}}_{\Lambda}) > \kappa$. ⊣

The last claim shows that $\mathcal{P}(\kappa) \cap K = \mathcal{P}(\kappa) \cap K^M$, hence $\kappa^{+K} = \kappa^{+K^M}$. Since $K|\theta \in M$, we can coiterate $K|\theta$ with $K^M$ in $M$. The coiteration coincide with the coiteration of $K$ and $K^M$ in $V$. Let $\Delta$ be the length of the coiteration of $K\|\theta$ with $K^M$.

Claim 5.

$$\pi^{\mathcal{T}}_{0,\Lambda} \upharpoonright K\|\theta = \pi^{\mathcal{T}}_{0,\Delta} \upharpoonright K\|\theta,$$

*that is the main branch of the coiteration of $K\|\theta$ with $K^M$ is an initial segment of the main branch of the coiteration of $K$ with $K^M$.*

Proof. Suppose not, then there is an extender $E$ used on the main branch of the $K$ side of the coiteration with index higher than $\theta$ such that $\mathrm{cp}(E) < \theta$. But by the properties of $\theta$, this implies that $\mathrm{cp}(E) < \kappa$. As $E$ is on the main branch, we would have $\mathrm{cp}(\pi^{\mathcal{T}}_{\Lambda}) < \kappa$ a contradiction to the previous claim! ⊣

This shows that $\pi^{\mathcal{T}}_{\Lambda} \upharpoonright K|\theta \in M$. Hence the last model of the iteration $Q|\pi^{\mathcal{Q}}_{\Lambda}(j(\theta))$ is in $M$ as well and we can coiterate $Q|\pi^{\mathcal{Q}}_{\Lambda}(j(\theta))$ with $K^M$ in $M$. Since $Q|\pi^{\mathcal{Q}}_{\Lambda}(j(\theta))$ is an iterate of $K^M$, it does not move in the coiteration and the $K^M$ side is simply the normal iteration to $Q|\pi^{\mathcal{Q}}_{\Lambda}(j(\theta))$. Hence we have that $j \upharpoonright (K^M|j(\theta)) \in M$. Since the diagram commute, we can deduce $j \upharpoonright \mathcal{P}(\kappa) \cap K$ by

$$j(x) = y \iff (\pi^{\mathcal{T}}_{\Lambda} \upharpoonright K|\theta)(x) = (\pi^{\mathcal{Q}}_{\Lambda} \upharpoonright K^M|j(\theta))(y).$$

Let $\alpha < \theta$ and $F$ be the extender of length $\alpha$ derived from $j \upharpoonright \mathcal{P}(\kappa) \cap K$. $F$ coheres with $K^M$. We want to study the iterability of the phalanx $\langle K^M, \mathrm{Ult}(K^M, F), \mathrm{lh}(F)\rangle$.

Claim 6. *The phalanx $\langle K^M, \mathrm{Ult}(K^M, F), \mathrm{lh}(F)\rangle$ is iterable.*

Proof. The aim of the proof is to show that there is an embedding from $\mathrm{Ult}(K^M, F)$ to some $Q^*$, where $Q^*$ is an iterate of $K^M$ beyond $j(\mathrm{lh}(F))$. Let us first construct $Q^*$ and then show that we can embed $\mathrm{Ult}(K^M, F)$ in it. Let $\mathcal{T}^j$ be the iteration on $K^M$ copied from $\mathcal{T}$ via $j$. We claim that at each step we can factorize by taking an ultrapower with $F$:

$$\begin{tikzcd}
K \arrow[rr, "\pi^{\mathcal{T}}_{0,\xi}"] \arrow[dr, "i^K_F"] \arrow[dd, "j"'] & & \cdots \arrow[r, "\pi^{\mathcal{T}}_{\eta,\Lambda}"] & Q \arrow[dr, "i^Q_F"] \arrow[dd, "j_\Lambda"] & \\
& \mathrm{Ult}(K^M, F) \arrow[dl, "k_0"'] & \cdots & & \mathrm{Ult}(Q, F) \arrow[dl, "k_\Lambda"'] \\
K^M \arrow[rr, "\pi^{\mathcal{T}^j}_{0,\xi}"] & & \cdots \arrow[r, "\pi^{\mathcal{T}^j}_{\eta,\Lambda}"] & Q^* &
\end{tikzcd}$$

Figure 1: Copying $\mathcal{T}$

The $j_\xi : \mathcal{M}^{\mathcal{T}}_\xi \to \mathcal{M}^{\mathcal{T}^j}_\xi$'s are the usual copy maps, hence we have that whenever $\eta \leq \xi < \mathrm{lh}(\mathcal{T})$,

$$j_\xi \restriction \mathrm{lh}(E^{\mathcal{T}}_\eta) = j_\eta \restriction \mathrm{lh}(E^{\mathcal{T}}_\eta).$$

By the previous claim we know that $\mathcal{T}$ is above $\kappa$. Moreover there are no truncations in $\mathcal{T}$ and thus in $\mathcal{T}^j$. Hence for every $X \in \mathcal{P}(\kappa) \cap K$,

$$\pi^{\mathcal{T}}_{0,\xi}(X) \cap \kappa = X.$$

Since $j(\kappa) \geqslant \mathrm{lh}(F)$, the iteration $\mathcal{T}^j$ is above $\mathrm{lh}(F)$, hence if $a \in [\mathrm{lh}(F)]^{<\omega}$ $\pi^{\mathcal{T}^j}_{\xi,\eta}(a) = a$. Using the commutativity of the diagram:

$$\begin{tikzcd}
K \arrow[r, "\pi^{\mathcal{T}}_{0,\xi}"'] \arrow[d, "j"] & \mathcal{M}^{\mathcal{T}}_\xi \arrow[d, "j_\xi"] \\
K^M \arrow[r, "\pi^{\mathcal{T}^j}_{0,\xi}"'] & \mathcal{M}^{\mathcal{T}^j}_\xi
\end{tikzcd}$$

we have that:

$$j_\xi(\pi^{\mathcal{T}}_{0,\xi}(X)) = \pi^{\mathcal{T}^j}_{0,\xi}(j(X)).$$

Thus for $a \in [\mathrm{lh}(F)]^{<\omega}$ and $X \in \mathcal{P}([\kappa]^{\overline{\overline{a}}}) \cap K$:

$$a \in j_\xi(X) \iff a \in j_\xi(\pi^{\mathcal{T}}_{0,\xi}(X)) \iff \pi^{\mathcal{T}^j}_{0,\xi}(a) \in \pi^{\mathcal{T}^j}_{0,\xi}(j(X)) \iff a \in j(X)$$

Hence the $\langle \kappa, \mathrm{lh}(F) \rangle$-extender derived by $j_\xi$ is nothing else than $F$ and thus we can factorize $j_\xi$ by $i^{\mathcal{M}^{\mathcal{T}}_\xi}_F$ with some map $k_\xi$ such that the diagram below commutes:

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_\xi \arrow[dr, "i^{\mathcal{M}^{\mathcal{T}}_\xi}_F"] \arrow[dd, "j_\xi"] & \\
& \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\xi, F) \arrow[dl, "k_\xi"'] \\
\mathcal{M}^{\mathcal{T}^j}_\xi &
\end{tikzcd}$$

Let $i_F^{K^M} : K^M \to \mathrm{Ult}(K^M, F)$ and $i_F^Q : Q \to \mathrm{Ult}(Q, F)$ be the ultrapower maps. Then defining $k : \mathrm{Ult}(K^M, F) \to \mathrm{Ult}(Q, F)$ such that

$$i_F^{K^M}(f)(a) \mapsto i_F^Q(\pi_\Lambda^{\mathcal{U}}(f) \upharpoonright \kappa)(a),$$

where $a \in [\mathrm{lh}(F)]^{<\omega}$ and $f : \kappa^{\overline{\overline{a}}} \to K^M$, $f \in K^M$. Let us show that this map is an embedding. Let $\varphi$ be a formula.

$$\begin{aligned}
\mathrm{Ult}(Q, F) \vDash \varphi(k(i_F^{K^M}(f)(a))) &\iff \mathrm{Ult}(Q, F) \vDash \varphi(i_F^Q(\pi_\Lambda^{\mathcal{Q}}(f) \upharpoonright \kappa)(a)) \\
&\iff \big\{u; Q \vDash \varphi(\pi_\Lambda^{\mathcal{U}}(f)(u))\big\} \cap \kappa \in F_a \\
&\iff \pi_\Lambda^{\mathcal{Q}}(\big\{u; K^M \vDash \varphi(f(u))\big\}) \cap \kappa \in F_a) \\
&\iff \big\{u; K^M \vDash \varphi(f(u))\big\} \in F_a \\
&\iff \mathrm{Ult}(K^M, F) \vDash \varphi(i_F^{K^M}(f)(a))
\end{aligned}$$

The first equivalence holds by definition of $k$, the third because $\mathrm{cp}(\pi_\Lambda^{\mathcal{Q}}) \geqslant \kappa$, the second and fourth is Łoś theorem for ultrapower. Putting everything together we get the following diagram:

$$\begin{tikzcd}
 & K \arrow[rr, "j"] \arrow[dl, "j"'] \arrow[d, "i_F^K"] \arrow[ddr, "\pi_\Lambda^{\mathcal{T}}"] & & K^M \arrow[d, "i_F^{K^M}"] \arrow[ddl, "\pi_\Lambda^{\mathcal{Q}}"'] \\
K^M \arrow[ddr, "\pi_\Lambda^{\mathcal{T}^j}"] & \mathrm{Ult}(K, F) \arrow[l, "k_0"'] & & \mathrm{Ult}(K^M, F) \arrow[ddl, "k"] \\
 & & Q \arrow[dl, "j_\Lambda"'] \arrow[d, "i_F^Q"] & \\
 & Q^* & \mathrm{Ult}(Q, F) \arrow[l, "k_\Lambda"] &
\end{tikzcd}$$

Figure 2: The complete diagram

Hence we can embed $\mathrm{Ult}(K^M, F)$ into $Q^*$ by $k_\Lambda \circ k$. Since $k_\Lambda \circ k$ has critical point strictly larger than $\mathrm{lh}(F)$, the map

$$\langle \mathrm{id}, k_\Lambda \circ k \rangle : \langle K^M, \mathrm{Ult}(K^M, F), \mathrm{lh}(F) \rangle \to \langle K^M, Q^*, \mathrm{lh}(F) \rangle,$$

is an embedding as well. Moreover since $\mathcal{T}$ was above $\kappa$, we have that $\mathcal{T}^j$ the iteration from $K^M$ to $Q^*$ is above $j(\kappa) > \mathrm{lh}(F)$. Thus we can embed $\langle K^M, Q^*, \mathrm{lh}(F) \rangle$the following way:

$$(\pi^{\mathcal{T}^j}, \mathrm{id}) : \langle K^M, Q^*, \mathrm{lh}(F) \rangle \to \langle Q^*, Q^*, \mathrm{lh}(F) \rangle.$$

$\langle Q^*, Q^*, \mathrm{lh}(F) \rangle$ is clearly iterable since $Q^*$ is an iterate of an universal weasel. This finishes the proof of the claim. $\dashv$

By [Ste96, Lemma 8.6 p. 77] this is, in fact equivalent to $F$ being on the $K^M$ sequence. Hence every initial segment of $j \upharpoonright \mathcal{P}(\kappa) \cap K$ is on the $K^M$ sequence. But this implies that $\kappa$ is Shelah in $K^M$, a contradiction! $\dashv$

Using Theorem 4.25 this gives us an immediate consistency strength result:

**Theorem 4.26.** *For $i \leq \omega$ the following two theories are equiconsistent:*

*i.* ZFC+ *"there are $\alpha$ generically strong cardinals, where $\alpha$ is less than the least generically strong cardinal"*

*ii.* ZFC+ *"there are $\alpha$ strong cardinals, where $\alpha$ is less than the least strong cardinal"*

## 4.4 Supercompactness

In this section we want to show that we can apply some of the forcing techniques developed to force generically strong cardinals from strong cardinals to supercompact cardinals.

**Definition 4.27.** Let $\kappa$ be a cardinal and $\gamma$ some ordinal. $\kappa$ is called *$\gamma$-supercompact* if and only if there is an embedding $j : V \to M$ such that ${}^{\gamma}M \cap V \subseteq M$. $\kappa$ is called *supercompact* if it is $\gamma$-supercompact for all $\gamma$.

**Remark 4.28.** Let $\kappa$ be $\gamma$-supercompact and $j : V \to M$ an embedding witnessing the $\gamma$-supercompactness. For any cardinal $\nu < \gamma$, $j''\nu \in M$.

**Theorem 4.29.** *Suppose* ZFC+ *"there exist $\omega$ many supercompact cardinals" is consistent, then so is* ZFC+ *"each $\aleph_{n+1}$ is generically strong".*

Proof. Let $\kappa_0 = \omega$ and $\langle \kappa_{n+1};\ n \in \omega \rangle$ be a monotone enumeration of all supercompact cardinals. Let $\mathbb{P}$ be the easton support forcing iteration of $\mathrm{col}(\kappa_n, < \kappa_{n+1})$ for $n \in \omega$. We want to show that if $G$ is $\mathbb{P}$-generic over $V$ then in $V[G]$ every $\aleph_{n+1}$ is generically strong. Let $\kappa = \kappa_{n+1}$ be supercompact in $V$ and $A \in V[G]$ be some subset of the ordinals. Let $\theta$ regular be bigger than $\sup(A)^{\overline{\overline{\mathbb{P}}}}$ and $2^{\mu}$, where $\mu = (2^{\sup\{\kappa_i;\ i<\omega\}})^{+}$. Let $j : V \to M$ be the embedding witnessing the $\theta$-supercompactness of $\kappa$. Let $G^{\theta}$ be $\mathrm{col}(\omega, < \theta^{++})$-generic over $V$ such that $G \in V[G^{\theta}]$.

It suffices to construct a $\tilde{G}$ with the properties that: $j''G \subseteq \tilde{G}$ and $\tilde{G}$ is $j(\mathbb{P})$-generic over $M$. We split the forcing $j(\mathbb{P})$ in three parts:[4]

$$\mathbb{P}_n = \prod_{i<n} \mathrm{col}(\kappa_i,\ < \kappa_{i+1}),\ \ \mathbb{Q}_n^j = \mathrm{col}(\kappa_n, < j(\kappa_{n+1})),$$

and finally

$$\mathbb{P}^{j,n} = \prod_{n<i<\omega} \mathrm{col}(j(\kappa_i), < j(\kappa_{i+1})).$$

We will choose generics over $V$ for $\mathbb{P}_0$ and $\mathrm{col}(\kappa_n, < \mu)$, this is equivalent to choosing generics over $M$ since $H_{\theta}^{V} \subseteq M$. We will construct the generic for $\mathbb{P}^{j,n}$ defining some master condition. Similarly we can split $\mathbb{P}$ in three forcings $\mathbb{P} = \mathbb{P}_n * \mathbb{Q}_n * \mathbb{P}^n$:

$$\mathbb{P}_n = \prod_{i<n} \mathrm{col}(\kappa_i, < \kappa_{i+1}), \mathbb{Q}_n = \mathrm{col}(\kappa_n, < \kappa)$$

[4]in a slight abuse of notation, we use the symbol $\prod$ to denote the easton support product

and
$$\mathbb{P}^n = \prod_{n<i<\omega} \mathrm{col}(\kappa_i, < \kappa_{i+1}).$$

Set $G_n = G \cap \mathbb{P}_n$. Looking at $\mathbb{Q}^* = \mathrm{col}(\kappa_n, \sup\{\kappa_n;\ n < \omega\})$, by [Fuc08, lemma 2.2] we have that $(\mathbb{Q}_n * \mathbb{P}^n) \times \mathbb{Q}^*$ and $\mathbb{Q}^*$ are forcing equivalent, hence there is a filter $G^*$, $\mathbb{Q}^*$-generic over $M$, such that $G \in M[G_n \times G^*]$. Notice that $G^*$ is $\mathbb{Q}^*$-generic over $V$ as well. Using the general theory about Levy collapse, as found in [Kan03, p.127 ff], there is a $\mathrm{col}(\kappa_n, < \mu)$-generic filter over $M$, say $G_1$, that is also generic over $V$ with $G^* \in M[G_n \times G_1]$.

Hence there is a filter $G_n \times G_1$, $\mathbb{P}_n * \mathrm{col}(\kappa_n, < \mu)$-generic over $V$ and $M$, such that $G \in M[G_n, G_1]$. Let $H^*$ be $\mathrm{col}(\kappa_n, [\mu, j(\kappa)[) \cap M[G_n]$-generic over $M[G_n, G_1]$ and set $H_n = G_1 \times H^*$, by the product lemma $H_n$ is $col(\kappa_n, j(\kappa)) \cap M[G_n]$-generic over $M[G_n]$. Notice that we can choose $H^n \in V[G^\theta]$, as all forcings we saw so far are in $H_{\theta^+}$ and hence are in a countable model in $V[G^\theta]$.

We now want to construct a generic filter $G^n$, $\mathbb{P}^{j,n}$-generic over $M[G_n, H_n]$, such that $j''(G \upharpoonright \mathbb{P}^n) \subseteq G^n$. Remember that
$$\mathbb{P}^{j,n} = \prod_{n<i<\omega} \mathrm{col}(j(\kappa_i), < j(\kappa_{i+1})).$$
and $\mathbb{P}^{j,n} = j(\mathbb{P}^n)$.

Let us now work in $M[G_n, H_n]$. Since $j$ was witnessing the $\theta$ compactness of $\kappa$ we have that $j''\kappa_i \in M$ for all $i$, moreover $G$ is in $M[G_n, H_n]$. Hence can compute $q_i = j''(G \upharpoonright \mathrm{col}(\kappa_i, \kappa_{i+1}))$ in $M[G_n, H_n]$. As $q_i$ has size $\kappa_i$ in $M[G_n, H_n]$, it is a condition of $\mathrm{col}(j(\kappa_i), < j(\kappa_{i+1}))$ for $i > n$. Hence $\dot{q} = \langle \check{q}_i; i < \omega \rangle$ is a condition of the forcing $\mathbb{P}^{j,n}$. Let $G^n$ be $\mathbb{P}^{j,n}$-generic over $M[G_n, H_n]$ with $\dot{q} \in G_2$. As we have seen, we can lift $j$ to $\tilde{j} : V[G] \to M[G_n, H_n, G^n]$.

$A$ was in $V[G]$ hence, by choice of $\theta$, there is a nice name $\tau$ with $\tau \in H_\theta$, thus $\tau \in M$ and $A \in M[G_n, H_n, G^n]$. Remark again that $\mathcal{P}(\mathbb{P}^{j,n}) \cap M[G_n, H_n]$ is countable in $V[G^\theta]$ hence we can choose $G^n \in V[G^\theta]$, thus we can define the embedding $\tilde{j}$ inside $V[G^\theta]$ and $\kappa$ is generically strong. $\dashv$

# Bibliography

[Bag00] Joan Bagaria. Bounded forcing axioms as principles of generic absoluteness. *Arch. Math. Logic*, 39(6):393–401, 2000.

[BS09] Daniel Busche and Ralf Schindler. The strength of choiceless patterns of singular and weakly compact cardinals. *Ann. Pure Appl. Logic*, 159(1-2):198–248, 2009.

[CS09] Benjamin Claverie and Ralf Schindler. Increasing $u_2$ by a stationary set preserving forcing. *J. Symbolic Logic*, 74(1):187–200, 2009.

[FMS88] M. Foreman, M. Magidor, and S. Shelah. Martin's maximum, saturated ideals, and nonregular ultrafilters. I. *Ann. of Math. (2)*, 127(1):1–47, 1988.

[Fuc08] Gunter Fuchs. Closed maximality principles: implications, separations and combinations. *J. Symbolic Logic*, 73(1):276–308, 2008.

[Jec03] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[Jena] Ronald Jensen. Making cardinals $\omega$-cofinal. `http://www.mathematik.hu-berlin.de/~raesch/org/jensen.html`. Handwritten notes.

[Jenb] Ronald Jensen. Measures of order zero. `http://www.mathematik.hu-berlin.de/~raesch/org/jensen.html`. Handwritten notes.

[Jenc] Ronald Jensen. Non-overlapping extenders sequences. Handwritten notes.

[Jend] Ronald Jensen. On some problems of Mitchell, Welch and Vickers. `http://www.mathematik.hu-berlin.de/~raesch/org/jensen/pdf/Some\_Problems.pdf`. Handwritten notes.

[Jen72] R.B. Jensen. The fine structure of the constructible hierarchy. With an appendix of J. Silver. *Ann. Math. Logic*, 4:229–308, 1972.

[JSSS09] Ronald Jensen, Ernest Schimmerling, Ralf Schindler, and John Steel. Stacking mice. *J. Symbolic Logic*, 74(1):315–335, 2009.

[Kan03] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.

[KLZ07] Richard Ketchersid, Paul Larson, and Jindřich Zapletal. Increasing $\delta^1_2$ and Namba-style forcing. *J. Symbolic Logic*, 72(4):1372–1378, 2007.

[KMS83] Alexander S. Kechris, Donald A. Martin, and Robert M. Solovay. Introduction to $Q$-theory. In *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Math.*, pages 199–282. Springer, Berlin, 1983.

[Mos09] Yiannis N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.

[MS89] Donald A. Martin and John R. Steel. A proof of projective determinacy. *J. Amer. Math. Soc.*, 2(1):71–125, 1989.

[MS94] William J. Mitchell and John R. Steel. *Fine structure and iteration trees*, volume 3 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1994.

[MS95] W. J. Mitchell and E. Schimmerling. Weak covering without countable closure. *Math. Res. Lett.*, 2(5):595–609, 1995.

[Sch] Ralf Schindler. Woodin's axiom $(*)$, bounded forcing axioms, and precipitous ideals on $\omega_1$. preprint.

[Sch95] Ernest Schimmerling. Combinatorial principles in the core model for one Woodin cardinal. *Ann. Pure Appl. Logic*, 74(2):153–201, 1995.

[Sch04] Ralf Schindler. Semi-proper forcing, remarkable cardinals, and bounded Martin's maximum. *MLQ Math. Log. Q.*, 50(6):527–532, 2004.

[SS] Ralf Schindler and John Steel. The core model induction. `http://wwwmath.uni-muenster.de/logik/Personen/rds/core\_model\_induction.pdf`. preprint.

[Ste] J.R. Steel. An outline of inner model theory. `http://math.berkeley.edu/%7Esteel/papers/steel1.pdf`. To appear in Handbook of Set Theory.

[Ste95] J. R. Steel. Projectively well-ordered inner models. *Ann. Pure Appl. Logic*, 74(1):77–104, 1995.

[Ste96] John R. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1996.

[SVW82] John R. Steel and Robert Van Wesep. Two consequences of determinacy consistent with choice. *Trans. Amer. Math. Soc.*, 272(1):67–85, 1982.

[SZ] Ralf Schindler and Martin Zeman. Fine structure. `http://wwwmath.uni-muenster.de/logik/Personen/rds/finestructure.pdf`. To appear in Handbook of Set Theory.

[SZ01] Ernest Schimmerling and Martin Zeman. Square in core models. *Bull. Symbolic Logic*, 7(3):305–314, 2001.

[SZ04] Ernest Schimmerling and Martin Zeman. Characterization of $\square_\kappa$ in core models. *J. Math. Log.*, 4(1):1–72, 2004.

[Woo99] W. Hugh Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, volume 1 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 1999.

[Zem02] Martin Zeman. *Inner models and large cardinals*, volume 5 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2002.

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