Representations of C*-dynamical systems implemented by Cuntz families

Evgenios T.A. Kakariadis and Justin R. Peters

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Dedicated to the memory of William B. Arveson

Abstract. Given a dynamical system (A, α) where A is a unital C*-algebra and α is a (possibly nonunital) *-endomorphism of A, we examine families $(\pi, \{T_i\})$ such that π is a representation of A, $\{T_i\}$ is a Toeplitz–Cuntz family and a covariance relation holds. We compute a variety of nonselfadjoint operator algebras that depend on the choice of the covariance relation, along with the smallest C*-algebra they generate, namely the C*-envelope. We then relate each occurrence of the C*-envelope to (a full corner of) an appropriate twisted crossed product. We provide a counterexample to show the extent of this variety. In the context of C*-algebras, these results can be interpreted as analogues of Stacey's famous result, for nonautomorphic systems and $n > 1$.

Our study involves also the one variable generalized crossed products of Stacey and Exel. In particular, we refine a result that appears in the pioneering paper of Exel on (what is now known as) Exel systems.

1. Introduction

For this paper, a dynamical system (A, α) consists of a unital C*-algebra A and a (possibly non unital) $\alpha \in \text{End}(A)$. We study the representation theory of universal objects that satisfy (more or less) the covariance relation

$$\pi(\alpha(x)) = \sum_{i=1}^{n} T_i \pi(x) T_i^*, \text{ for all } x \in A,$$

where $\{T_i\}$ is a Toeplitz–Cuntz family and $n \geq 1$. We were initially motivated by [9, 10, 11, 30, 36, 39]. Stacey's *multiplicity n crossed product* is the universal C*-algebra $A \times_{\alpha}^{n} \mathbb{N}$, when π is additionally assumed to be nondegenerate [39],

and a family of such representations is inherited by Cuntz's *twisted crossed product* $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ [10]. Therefore a canonical $*$-epimorphism $A \times_\alpha^n \mathbb{N} \to p(A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n)p$ is induced, where $p = [1_A] \in A_\infty$. Stacey shows that this $*$-epimorphism is injective when $n = 1$ or α is an automorphism [39, Prop. 3.3 and 3.4], and one can ask for an extension of this result. The counterexample provided in Subsection 4.32, shows that this is hopeless, and another approach should be considered.

This approach is given in the context of operator algebras both selfadjoint and nonselfadjoint, influenced by the ideas of Arveson [1, 3] and the work of many researchers in dilation theory (see [8, 12, 13, 14, 19, 25, 31, 35], to mention but a few). Instead of examining *a large* universal C*-algebra $\mathcal{T}_\mathcal{S}$ generated by a system of relations $\mathcal{S}$, one considers the noninvolutive analogue $\mathcal{A}_\mathcal{S}$ subject to $\mathcal{S}$. The goal is to associate *the smallest* C*-algebra $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_\mathcal{S})$ that is generated by $\mathcal{A}_\mathcal{S}$, namely the C*-envelope of $\mathcal{A}_\mathcal{S}$, to C*-algebras generated by an invertible system $\mathcal{S}'$ that dilates $\mathcal{S}$. We coin this as *Arveson's program on the* C*-*envelope.*[1]

There are some interesting consequences derived from this approach. First of all the C*-envelope possesses a ring theoretical injective property. Thus in the following short exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{T}_\mathcal{S} \longrightarrow \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_\mathcal{S}) \longrightarrow 0,$$

the algebra $\mathcal{A}_\mathcal{S}$ is preserved isometrically by the unique $*$-epimorphism $\mathcal{T}_\mathcal{S} \to \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_\mathcal{S})$. The same scheme holds by substituting $\mathcal{T}_\mathcal{S}$ with any C*-algebra generated by an isometric copy of $\mathcal{A}_\mathcal{S}$. Taking into account that $\mathcal{S}$ is reflected isometrically inside $\mathcal{T}_\mathcal{S}$, hence in $\mathcal{A}_\mathcal{S}$, this gives a picture on how narrow one can be on defining universal C*-algebras related to $\mathcal{S}$. One cannot go beyond the minimal $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_\mathcal{S})$. Additionally, the ideal $\mathcal{J}$ in $\mathcal{T}_\mathcal{S}$, namely *the Šilov ideal* of $\mathcal{A}_\mathcal{S}$, plays the role of the noncommutative Šilov boundary and it is equipped with the analogous properties.

The motivating and intriguing part of this theory is that, even though $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_\mathcal{S})$ always exists as a minimal object [19, 13, 2, 23], the interest lies exactly on connecting it to a natural C*-algebra of invertible $\mathcal{S}$-like relations. Nevertheless, the possibility of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_\mathcal{S})$ being $\mathcal{T}_\mathcal{S}$ is not excluded. In this paper we are interested in both such cases for a fixed system $\mathcal{S}$.

To this end we define the nonselfadjoint operator algebras $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$ and $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+$ in analogy with Cuntz's twisted crossed product and Stacey's crossed product respectively, where the use of the superscript "t" is self-explanatory. In Theorem 4.26 we show that $\mathrm{C}^*_{\mathrm{env}}(A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+)$ is a full corner of the twisted crossed product $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$. This is the analogue of [39, Prop. 3.3 and 3.4] for nonautomorphic systems and $n > 1$. On the other hand $\mathrm{C}^*_{\mathrm{env}}(A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+)$ is simply $A \times_\alpha^n \mathbb{N}$ when α is unital (see Theorem 4.17). In fact the given counterexample shows that $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$ and $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+$ are

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[1] We emphasize that the term C*-envelope must not be confused with the notion of an *enveloping* C*-*algebra*. The first one is a minimal C*-algebra associated to $\mathcal{A}_\mathcal{S}$, whereas the second one is the maximal C*-algebra $\mathcal{T}_\mathcal{S}$. See Subsection 2.1 for the pertinent definitions.

not canonically isomorphic unless α is an automorphism. From one point of view [28, 37], $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+$ resembles a *maximal twisted product*, in contrast to $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$ which resembles a *minimal twisted product*; they coincide when α is an automorphism, but not in general.

The notation A_{nd} emphasizes on that A is represented nondegenerately inside the universal objects. One can define $A \times_\alpha^t \mathcal{T}_n^+$ and/or $A \times_\alpha \mathcal{T}_n^+$, without imposing this requirement. By appropriately weighting the generators, we show that $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$ embeds in $A \times_\alpha^t \mathcal{T}_n^+$, and in Theorem 4.23 we give the connection of $\mathrm{C}_{\mathrm{env}}^*(A \times_\alpha^t \mathcal{T}_n^+)$ with the twisted crossed product. This is the stepping stone to get the results explained in the previous paragraph.

We give applications of our results in cases that involve *a transfer operator L* following [16, Sec. 4] and [11]. To do so we use the language of C*-correspondences. In Theorem 4.30 we show that the Cuntz–Pimsner algebra of the correspondence $\mathcal{M}_L \otimes X_n$ is a full corner of the twisted crossed product and has a rather special form. Here $\mathcal{M}_L$ is Exel's correspondence [16] and X_n is the Hawaiian earring graph on n edges.

Exel systems (A, α, L) were introduced in the pioneering paper of Exel [16] and have been under considerable investigation in a series of papers [4, 5, 20, 33]. In these papers a similar representation theory is used, when $n = 1$. For this reason, we present the one-variable case separately, as it may be of independent interest. This allows us also to simplify proofs for $n > 1$. A third reason for following this presentation of our results is that we correct an error in [16, Thm. 4.7]: it is claimed that, when α is injective and $\alpha(A)$ is hereditary, then $A \rtimes_{\alpha,L} \mathbb{N}$ [16, Def. 3.7] is *-isomorphic to $\mathfrak{A}(A, \alpha)$ [16, Def. 4.4]. However, in Theorem 3.14 we show that $A \rtimes_{\alpha,L} \mathbb{N}$ is actually *-isomorphic to $A \times_\alpha^1 \mathbb{N}$, while in Proposition 3.4 we show that $\mathfrak{A}(A, \alpha)$ coincides with $A \times_\alpha^1 \mathbb{N}$ exactly when $\alpha(1_A) = 1_A$. A fourth important reason is that $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+$ behaves in a different way when $n = 1$, since $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_1^+ \simeq A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_1^+$ (cp. Theorem 3.22, Theorem 4.16, Subsection 4.32).

In Proposition 3.9 we give necessary and sufficient conditions for A to be embedded injectively in $A \rtimes_{\alpha,L} \mathbb{N}$, adding to results obtained by Brownlowe and Raeburn [4]. In fact, when α is unital or (A, α, L) is as in [16, Sec. 4], then $A \rtimes_{\alpha,L} \mathbb{N}$ is the Cuntz–Pimsner algebra of $\mathcal{M}_L$, which we prove is a Hilbert bimodule (see Proposition 3.10 and Theorem 3.14). This is quite surprising. We remark that Brownlowe, Raeburn and Vitadello [5] examine Exel systems arising from commutative C*-algebras, for which they show that $A \rtimes_{\alpha,L} \mathbb{N}$ is again the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{M}_L}$. We wonder whether this is true in general; yet this lies outside the purview of this paper and is to be pursued elsewhere.

The paper is divided into three parts. In Section 2 we present notation and give the constructions that are used in the sequel. In Section 3 we present the results concerning the one variable case. In Section 4 we proceed to the examination of the multivariable case. We conclude by giving the counterexample mentioned earlier.

We emphasize that in general the dynamical systems (A, α) are neither injective nor unital. Noninjectivity is treated by combining [38, 39, 22]: the objects related to (A, α) are identified with the objects related to the injective system $(A/\mathcal{R}_\alpha, \dot{\alpha})$. (Therefore the copy of A inside these objects is $A/\mathcal{R}_\alpha$, and it is A exactly when α is injective.) On the other hand, the trick of "weighting" the generators makes dilation theory far more tractable for our purposes and helps us treat the nonunital cases. We hope that the reader will discover applications of these tricks to other constructions.

Remarks on Notation and Conventions. The noninvolutive part $\mathcal{T}_n^+$ of the Toeplitz–Cuntz algebra $\mathcal{T}_n$ is the linear span of elements s_μ, for a word μ in $\mathbb{F}_n^+$. Whereas it would be convenient to use the semigroup $\mathbb{F}_n^+$ (resp. $\mathbb{N}$) in our notations when possible, we avoid doing so since the group $\mathbb{F}_n$ (resp. $\mathbb{Z}$) generated by $\mathbb{F}_n^+$ (resp. $\mathbb{N}$) does not contain the analogous relations that hold in $\mathcal{T}_n$. Therefore $\mathcal{T}_n^+$ and $\mathbb{F}_n^+$ differ based on their representation theory and we want to emphasize that distinction in our notation. (Note that in this way we deliberately avoid any confusion with the notation of [12] and [34].) The isometric copy of $\mathcal{T}_n^+$ in $\mathcal{O}_n$ will be denoted by $\mathcal{O}_n^+$.

We will say that a family $\{T_i\}_{i=1}^n$ of operators in a $\mathcal{B}(H)$ is a *Toeplitz–Cuntz family* (resp. a *Cuntz family*) if the T_i are isometries with orthogonal ranges and $\sum_{i=1}^n T_i \leq I_H$ (resp. $\sum_{i=1}^n T_i = I_H$). For sake of simplicity we will often write $T_i \in \mathcal{T}_n^+$ (resp. $T_i \in \mathcal{O}_n^+$), meaning that $\{T_i\}_{i=1}^n$ is a Toeplitz–Cuntz family (resp. a Cuntz family).

Throughout the paper a number of different operator algebras are defined, and for this reason we include an Index at the end of this paper. However, even if the established notation may not be self-contained, we do not proceed to change it. For example, Stacey's crossed product $A \times_\alpha^n \mathbb{N}$ could be denoted alternatively by $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n$, as follows by its representation theory.

Finally, we make the convention that a *canonical representation* is a representation that maps generators to generators of the same index.

2. Operator algebras and constructions

2.1. **Operator algebras and the C*-envelope.** An operator algebra $\mathcal{A}$ is a closed subalgebra of some $\mathcal{B}(H)$, where H is a Hilbert space. When $\mathcal{A}$ is closed under the involution inherited by $\mathcal{B}(H)$ then it is a C*-algebra. The representation theory of an operator algebra consists of *completely contractive* (*resp. isometric*) *homomorphisms* $\rho : \mathcal{A} \to \mathcal{B}(H)$, i.e., every homomorphism $\rho_k \equiv \mathrm{id}_k \otimes \rho : M_k(\mathcal{A}) \to \mathcal{B}(H^{(k)})$ is contractive (resp. isometric).

A representation $\nu : \mathcal{A} \to \mathcal{B}(K)$ is called a *dilation* of ρ if $\rho(a) = P_H \nu(a)|_H$ for all $a \in \mathcal{A}$. A complete isometry ρ is called *maximal* if any dilation ν of ρ is trivial. By [13] every complete isometry has a dilation that is maximal. Therefore ρ is maximal if and only if, given a maximal dilation ν of ρ, there is a representation σ such that $\nu = \rho \oplus \sigma$.

A pair $(\mathcal{C}, \rho)$ such that $\mathcal{C}$ is a C*-algebra, $\rho : \mathcal{A} \to \mathcal{C}$ is a complete isometry and $\mathcal{C} = \mathrm{C}^*(\rho(\mathcal{A}))$, is called a C*-*cover of* $\mathcal{A}$. An ideal J in $\mathcal{C}$ is called *boundary*

if the restriction of the quotient map q_J to $\rho(\mathcal{A})$ is a complete isometry; *the Šilov ideal $\mathcal{J}$ is the largest boundary ideal.* Consequently, the pair $(\mathcal{C}/\mathcal{J}, q_{\mathcal{J}} \circ \rho)$, called *the C*-envelope of $\mathcal{A}$* and denoted simply by $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$, is the smallest C*-cover generated by $\mathcal{A}$. (It is clear that $\mathcal{C}/\mathcal{J}$ contains no nontrivial boundary ideals.) In fact the C*-envelope has a universal property: for every C*-cover $(\mathcal{C}, \rho)$ of $\mathcal{A}$ there is a (necessarily) unique $*$- epimorphism $\Phi : \mathcal{C} \to \mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$ such that $\Phi(\rho(a)) = a$ for all $a \in \mathcal{A}$.

Lemma 2.2. *Let $\pi : \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) \to \mathcal{B}(H)$ be a $*$-homomorphism. Then π is faithful if and only if $\pi|_{\mathcal{A}}$ is a complete isometry.*

Proof. If π is faithful then it is a complete isometry, hence $\pi|_{\mathcal{A}}$ is a complete isometry. For the converse, the $*$-homomorphism $\widetilde{\pi} : \mathrm{C}^*_{\mathrm{env}}(\mathcal{A})/\ker \pi \to \mathcal{B}(H)$ is faithful. Hence $\ker \pi = (0)$, as a boundary ideal of $\mathcal{A}$ in $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$. $\qquad\square$

Unlike [3], a representation π of a C*-cover $\mathcal{C}$ of $\mathcal{A}$ is called *boundary* if the restriction of π to $\mathcal{A}$ is maximal [13]. We say that $\mathcal{A}$ has *the unique extension property* if any faithful representation $\pi : \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) \to \mathcal{B}(H)$ is a boundary representation [14]. In particular the free atomic representation of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$ is boundary, therefore the irreducible representations of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$ are boundary as direct summands of a boundary representation. Therefore, if $\mathcal{A}$ has the unique extension property, then it admits a *Choquet boundary* in the sense of [3], i.e., the existence of sufficiently many irreducible boundary representations of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$.

The existence of the C*-envelope was first proved by Arveson in the case where there were enough boundary representations [1]. The first proof for the general case was given by Hamana [19]. Twenty five years later Dritschel and McCullough [13] gave an independent proof for the existence of the C*-envelope, simplified later by Arveson [2]. The first author gives an independent proof of Hamana's Theorem in [23]. The existence of the Choquet boundary for separable operator systems (or operator algebras) was proved by Arveson [3] and it is still an open problem for the nonseparable cases. Recently, Kleski [29] has proved that, for the separable case, the supremum can be replaced by the pointwise maximum.

We remark that the notion of the C*-envelope should not be confused with the notion of the enveloping C*-algebra of an involutive Banach algebra. The universal property of the C*-envelope suggests that it is the smallest C*-cover of X. One might want to rename the C*-envelope as the C*-*minimal cover*, though it seems impractical to try to change established terminology.

Below we describe the construction of universal/enveloping (in general non-selfadjoint) operator algebras.

2.3. Universal operator algebras.
An effective way to create a universal operator algebra $\mathcal{B}$ with respect to generators and relations is, first to form the corresponding universal C*-algebra A in the sense of Blackadar [6], and

then consider the appropriate operator subalgebra $\mathcal{B}$ of A. The universal representation of $\mathcal{B}$ is then the restriction of the universal representation of A to $\mathcal{B}$.

A second way is to construct an operator algebra $\mathcal{B}$ relative to (a certain family of) contractive representations of a Banach algebra B with the property that every contractive homomorphism (in this family) of B acting on a Hilbert space *lifts* to a completely contractive homomorphism of $\mathcal{B}$. (If in addition B has an involution to take in account, then $\mathcal{B}$ is the enveloping C*-algebra of B.) Below is a brief discussion of the construction of such a universal object.

Let B be a Banach algebra, $\mathcal{B}$ be an operator algebra and $\phi : B \to \mathcal{B}$ be a contractive algebraic homomorphism. We say that ρ lifts to a completely contractive homomorphism $\widetilde{\rho}$ if the following diagram is commutative

$$
\begin{array}{ccc}
B & \xrightarrow{\ \phi\ } & \mathcal{B} \\
 & \searrow{\scriptstyle \rho} & \Big| {\scriptstyle \widetilde{\rho}} \\
 & & \mathcal{B}(H).
\end{array}
$$

Suppose that the cardinality of B is less or equal to a cardinal β that we choose such that $\beta^{\aleph_0} = \beta$. First let $\mathcal{F}$ be the set of contractive representations (H_i, ρ_i) of B such that $\dim(H) \leq \beta$. Let $\mathcal{H} = \oplus_{i \in \mathcal{F}} H_i$, $\phi := \oplus_{i \in \mathcal{F}} \rho_i$, and $\mathcal{B} := \overline{\phi(B)}^{\mathcal{B}(\mathcal{H})}$. Then $\mathcal{B}$ is an operator algebra with the operator structure inherited from $\mathcal{B}(\mathcal{H})$. Equivalently, let the seminorms on $M_k(B)$

$$
\omega_k([a_{ij}]) = \sup\{\|[\rho(a_{ij})]\|_{\mathcal{B}(H_\rho^{(k)})} \mid (H_\rho, \rho) \in \mathcal{F}\}.
$$

If $\mathcal{N} = \ker \omega_1 (= \ker \phi)$, then $M_k(\mathcal{N}) = \ker \omega_k$. Hence the family of the induced norms $\|\cdot\|_k$ on $M_k(B/\mathcal{N})$ with $\|[b_{ij} + \mathcal{N}]\|_k = \omega_k([b_{ij}])$ is defined. Then $\mathcal{B}$ is the completion of the image of $B/\mathcal{N}$ in $\mathcal{B}(\mathcal{H})$ (cp. [7, Subsec. 2.4.6]). Every $(H_i, \rho_i) \in \mathcal{F}$ lifts to the completely contractive homomorphism $\widetilde{\rho_i}(\cdot) := P_{H_i} \cdot |_{H_i}$ of $\mathcal{B}$. We will refer to $\mathrm{id}_{\mathcal{B}} \equiv \oplus_{i \in \mathcal{F}} \widetilde{\rho_i}$ as *the universal representation of $\mathcal{B}$* (the representations $(H_i, \widetilde{\rho_i})$ are the building blocks, as shown below).

Let (H, ρ) be a contractive representation of B where H has arbitrary dimension, say $J_0 = \dim(H)$. Let the set $\mathcal{S}$ consist of pairs $(J, \{K_j\}_{j \in J})$ such that $J \subseteq J_0$, the K_j are mutually orthogonal subspaces of H with $\dim(K_j) \leq \beta$ and every K_j is reducing for $\rho(B)$, i.e., every K_j is $\rho(B)$-invariant and $\rho(B)^*$-invariant. The set $\mathcal{S}$ is nonempty. Indeed, let C be the C*-algebra generated by $\rho(B)$ inside $\mathcal{B}(H)$. Then the cardinality of C is less or equal to β, since it is the closure of the span of monomials of the form

$$
(\rho(a_1)^*)^{\epsilon_1} \rho(b_1) \rho(a_2)^* \cdots (\rho(a_n)^*)^{\epsilon_2}, \text{ with } \epsilon_1, \epsilon_2 = 0, 1, \text{ and } a_l, b_l \in B.
$$

Then the Hilbert subspace $[C\xi]$, for $\xi \in H$, is reducing for C (and consequently for $\rho(B)$), and has cardinality less or equal to β. Choosing an $\eta \in H$ that is orthogonal to $[C\xi]$ we can form the subspace $[C\eta]$ that is orthogonal to $[C\xi]$ and so on.

We define a partial order in $\mathcal{S}$ by the rule $(J, \{K_j\}_{j \in J}) \leq (J', \{K_{j'}\}_{j' \in J'})$, if $J \subseteq J'$ and $K_{j'} = K_j$ when $j' = j \in J$. If $\mathcal{I}$ is a chain of such pairs then $(I, \{K_k\}_{k \in I})$, with $I = \cup_{l \in \mathcal{I}} J_l$, and $K_k = K_{j_l}$, when $k = j_l \in J_l$, is a maximal element of $\mathcal{I}$ in $\mathcal{S}$. Applying Zorn's Lemma we obtain a maximal element in $\mathcal{S}$, say $(J, \{K_j\}_{j \in J})$.

If there is a nonzero $\xi \in H$ such that $\xi \perp K$, then $[C\xi]$ is reducing for $\rho(B)$ and has dimension less or equal that β, which leads to a contradiction. Thus $H = \oplus_{j \in J} K_j$. Therefore $\rho = \oplus_{j \in J} \rho_j$, where $\rho_j = \rho|_{K_j}$. But every ρ_j is in $\mathcal{F}$, hence lifts to a completely contractive representation of $\mathcal{B}$, say $\widetilde{\rho}_j$. Thus $\widetilde{\rho} = \oplus_{j \in J} \widetilde{\rho}_j$ is a completely contractive representation of $\mathcal{B}$, which is a lifting of ρ.

Similarly one can start with a family $\mathcal{F}'$ of contractive representations of B. Starting with representations in $\mathcal{F}'$ that act on Hilbert space with the upper bound on the dimension, one can construct an operator algebra $\mathfrak{A}(B, \mathcal{F}')$ that has the universal property for representations in $\mathcal{F}'$ (acting on Hilbert spaces of arbitrary dimension). We will refer to $\mathfrak{A}(B, \mathcal{F}')$ as *the universal operator algebra relative to $\mathcal{F}'$*.

2.4. **Radical and direct limits.** Given a dynamical system (A, α), let the *radical ideal* $\mathcal{R}_\alpha$ be the closure of $\cup_n \ker \alpha^n$ [38]. Since $x \in \mathcal{R}_\alpha$ if and only if $\lim_n \alpha^n(x) = 0$, an injective $*$-homomorphism is defined by

$$\dot{\alpha} : A/\mathcal{R}_\alpha \to A/\mathcal{R}_\alpha : x + \mathcal{R}_\alpha \mapsto \alpha(x) + \mathcal{R}_\alpha.$$

The *direct limit dynamical system* $(A_\infty, \alpha_\infty)$ *associated to* (A, α) [39] is

If (A, α) is unital, then $A_\infty \neq (0)$. It may be the case that A_∞ is trivial, but α_∞ is always an automorphism of A_∞. The image of A in A_∞ is $A/\mathcal{R}_\alpha$, thus $(A_\infty, \alpha_\infty)$ coincides with the extension $((A/\mathcal{R}_\alpha)_\infty, (\dot{\alpha})_\infty)$ of $(A/\mathcal{R}_\alpha, \dot{\alpha})$.

2.5. **Exel systems.** An *Exel system* (A, α, L) consists of a dynamical system (A, α) with a *transfer operator* $L : A \to A$, i.e., L is a continuous positive linear map such that $L(\alpha(a)b) = aL(b)$, for all $a, b \in A$.

By definition, the range of L is an ideal of A. By [16, Prop. 2.3] L is called *nondegenerate* if one of the following equivalent conditions holds:

(1) the mapping $\alpha \circ L$ is a conditional expectation onto $\alpha(A)$;
(2) $\alpha \circ L \circ \alpha = \alpha$;
(3) $\alpha(L(1)) = \alpha(1)$.

In that case $A = \ker \alpha \oplus \operatorname{Im} L$ as an orthogonal sum of ideals. When $L(1) = 1$ then L is nondegenerate and onto A. Thus α is injective.

2.6. **C*-correspondences.** We use [27, 32] as a general reference. A C*-*correspondence X over A* is a right Hilbert A-module together with a $*$-homomorphism $\phi_X : A \to \mathcal{L}(X)$. A (*Toeplitz*) *representation* (π, t) of X consists of a $*$-homomorphism $\pi : A \to \mathcal{B}(H)$ and a linear map $t : X \to \mathcal{B}(H)$, such that $\pi(a)t(\xi) = t(\phi_X(a)(\xi))$ and $t(\xi)^* t(\eta) = \pi(\langle \xi, \eta \rangle_X)$, for all $a \in A$ and $\xi, \eta \in X$; by the C*-identity we get also that $t(\xi)\pi(a) = t(\xi a)$. A representation (π, t) is said to be *injective* if π is injective; then t is an isometry. The C*-algebra generated by (π, t) is the closed linear span of $t(\xi_1) \cdots t(\xi_n) t(\eta_m)^* \cdots t(\eta_1)^*$. Every pair (π, t) defines a $*$-homomorphism $\psi_t : \mathcal{K}(X) \to B$, such that $\psi_t(\Theta_{\xi,\eta}^X) = t(\xi)t(\eta)^*$ [21, Lemma 2.2].

Let K be an ideal in $\phi_X^{-1}(\mathcal{K}(X))$; we say that (π, t) is K-*coisometric* if $\psi_t(\phi_X(a)) = \pi(a)$, for $a \in K$. Following [27], the representations (π, t) that are J_X-coisometric, where $J_X = \ker \phi_X^\perp \cap \phi_X^{-1}(\mathcal{K}(X))$, are called *covariant representations*. The ideal J_X is the largest ideal on which the restriction of ϕ_X takes values in the compacts and is injective. When $\phi_X|_{J_X}$ induces an isomorphism onto $\mathcal{K}(X)$ then X is (called) *a Hilbert bimodule* [26].

The *Toeplitz–Cuntz–Pimsner* algebra $\mathcal{T}_X$ is the universal C*-algebra for "all" representations of X, and the *Cuntz–Pimsner algebra* $\mathcal{O}_X$ is the universal C*-algebra for "all" covariant representations of X. The *tensor algebra* $\mathcal{T}_X^+$ is the norm-closed algebra generated by the copies of A and X in $\mathcal{T}_X$. For more details see [17, 27, 35]. There is an important connection between $\mathcal{T}_X$ and $\mathcal{O}_X$ established in full generality by Katsoulis and Kribs [25, Thm. 3.7]: the C*-envelope of $\mathcal{T}_X^+$ is $\mathcal{O}_X$. As a consequence, the ideal $\mathcal{K}(F(X)J_X)$ is the Šilov ideal of $\mathcal{T}_X^+$.

For an ideal $K \subseteq \phi_X^{-1}(\mathcal{K}(X))$ we denote by $\mathcal{O}(K, X)$ the universal C*-algebra for "all" K-coisometric representations of X. It is easy to deduce that $\mathcal{O}(K, X) \simeq \mathcal{T}_X / \mathcal{I}$, where $\mathcal{I}$ is the ideal in $\mathcal{T}_X$ generated by $\pi_u(a) - \psi_{t_u}(k)$, with $a \in K$, $\phi_X(a) = k \in \mathcal{K}(X)$, and (π_u, t_u) is the universal representation of $\mathcal{T}_X$. The ideal $K \subseteq \phi^{-1}(\mathcal{K}(X))$ may not be contained in J_X. Nevertheless, there are necessary and sufficient conditions that guarantee this. Note that $\mathcal{O}_X$ is the minimal such C*-algebra containing A.

Lemma 2.7. *Let X be a correspondence over A and K an ideal of A contained in $\phi^{-1}(\mathcal{K}(X))$. Then the following are equivalent:*

(1) $A \hookrightarrow \mathcal{O}(K, X)$;
(2) $\phi_X|_K$ *is injective;*
(3) $K \subseteq J_X$;
(4) *every J_X-covariant representation is K-covariant;*
(5) $\mathcal{T}_X^+ \hookrightarrow \mathcal{O}(K, X)$.

Proof. For [(1) $\Leftrightarrow$ (2)] see [4, Lemma 2.2]. If item (2) above holds, then $K \subseteq \ker \phi_X^\perp$, hence $K \subseteq J_X$. The implications [(3) $\Rightarrow$ (4)] and [(5) $\Rightarrow$ (1)] are

obvious. Finally assume that item (4) holds. Then the diagram

$$
\begin{array}{ccc}
\mathcal{T}_X & \xrightarrow{\quad q_1 \quad} & \mathcal{O}_X \\
& \searrow{\scriptstyle q_2} \quad \nearrow{\scriptstyle q_3} & \\
& \mathcal{O}(K, X) &
\end{array}
$$

of canonical $*$-epimorphisms commutes. By [25], the restriction of q_1 to the tensor algebra is a complete isometry, hence $\mathcal{T}_X^+ \hookrightarrow \mathcal{O}(K, X)$. $\square$

3. One-variable case

The reader is referred to [4, 5, 11, 16, 20, 33, 39] for examples that arise naturally in the context, that we won't repeat. We will write $1_A \equiv 1$.

3.1. **The crossed products $\mathfrak{A}(A, \alpha)$ and $A \times_\alpha^1 \mathbb{N}$.** Exel [16] defines the universal C*-algebra $\mathfrak{A}(A, \alpha)$ relative to the class

$$
\mathcal{F}_E = \{(\pi, T) \mid T \text{ isometry, and } \pi(\alpha(x)) = T\pi(x)T^*, \text{ for all } x \in A\},
$$

that is generated by $\pi(A)$ and T. Since T is an isometry, then

$$
T\pi(x) = \pi(\alpha(x))T, \text{ for all } x \in A.
$$

Moreover, $\pi(\alpha(1)) = T\pi(1)T^* = \pi(\alpha(1))TT^* = TT^*\pi(\alpha(1))$.

A variation was given earlier by Stacey [39]: for a dynamical system (A, α) with $A_\infty \neq (0)$, let $A \times_\alpha^1 \mathbb{N}$ be the universal C*-algebra relative to the class

$$
\mathcal{F}_S = \{(\pi, T) \in \mathcal{F}_E \mid \pi \text{ is nondegenerate}\},
$$

generated by $\pi(x)T^n(T^*)^m$ for $n, m \in \mathbb{Z}_+$. Since π is nondegenerate the C*-algebra $A \times_\alpha^1 \mathbb{N}$ is generated by $\pi(A)$ and T.

Nondegeneracy of A is not assumed in the definition of $\mathfrak{A}(A, \alpha)$, otherwise $\mathfrak{A}(A, \alpha)$ would coincide with $A \times_\alpha^1 \mathbb{N}$. Thus $1 \in A$ is a projection in $\mathfrak{A}(A, \alpha)$. The connection between $\mathfrak{A}(A, \alpha)$ and $A \times_\alpha^1 \mathbb{N}$ is established below.

Proposition 3.2. *The C*-algebra $A \times_\alpha^1 \mathbb{N}$ is $*$-isomorphic to the C*- subalgebra of $\mathfrak{A}(A, \alpha)$ generated by $\pi_u(A)$ and $T_u\pi_u(1)$, where (π_u, T_u) is the universal representation of $\mathfrak{A}(A, \alpha)$. Moreover, $\pi_u(1)$ is a unit for $A \times_\alpha^1 \mathbb{N}$.*

Proof. First note that $\pi_u(1)$ is a unit for C$^*(\pi_u(A), T_u\pi_u(1))$, since

$$
\pi_u(1) \cdot T_u\pi_u(1) = \pi_u(1)\pi_u(\alpha(1))T_u = \pi_u(\alpha(1))T_u = T_u\pi_u(1).
$$

Hence the restriction of the identity representation on $K = \pi_u(1)H_u$ defines a faithful representation of C$^*(\pi_u(A), T_u\pi_u(1))$. Moreover $\pi_u|_K$ is nondegenerate for A and $T_u\pi_u(1)|_K$ is an isometry. Therefore there is a canonical $*$-epimorphism $\Phi : A \times_\alpha^1 \mathbb{N} \to \mathrm{C}^*(\pi_u(A), T_u\pi_u(1))$.

On the other hand, let (π, T) be a pair for $A \times_\alpha^1 \mathbb{N}$, where π is nondegenerate and T is an isometry. Then (π, T) defines a pair also for $\mathfrak{A}(A, \alpha)$. Thus there is a canonical $*$-epimorphism $\Psi : \mathfrak{A}(A, \alpha) \to A \times_\alpha^1 \mathbb{N}$. Therefore $\Psi(T_u\pi_u(1)) = T\pi(1) = T$, since π is assumed nondegenerate. Thus the restriction of Ψ to C$^*(\pi_u(A), T_u\pi_u(1))$ is still onto $A \times_\alpha^1 \mathbb{N}$.

It is straightforward that $\Psi \circ \Phi = \mathrm{id}_{A \times_\alpha^1 \mathbb{N}}$, which completes the proof. $\square$

By the following Proposition, we can restrict to injective dynamical systems. Recall that $\dot{\alpha}(x + \mathcal{R}_\alpha) = \alpha(x) + \mathcal{R}_\alpha$.

Proposition 3.3. *The* C^**-algebras* $\mathfrak{A}(A, \alpha)$ *and* $\mathfrak{A}(A/\mathcal{R}_\alpha, \dot{\alpha})$ *are* $*$*- isomorphic; analogously,* $A \times_\alpha^1 \mathbb{N} \simeq (A/\mathcal{R}_\alpha) \times_{\dot{\alpha}}^1 \mathbb{N}$. *Moreover,* $A/\mathcal{R}_\alpha$ *embeds in* $(A/\mathcal{R}_\alpha) \times_{\dot{\alpha}}^1 \mathbb{N}$ *and* $\mathfrak{A}(A/\mathcal{R}_\alpha, \dot{\alpha})$.

Proof. The first part goes as in the proof of [22, Prop. 3.14]; in short, if $x \in \ker \alpha^n$, then $T^n \pi(x) = \pi(\alpha^n(x))T^n = 0$, hence $\pi(x) = 0$ since T is an isometry. A limit argument then shows that $\pi|_{\mathcal{R}_\alpha} = 0$; thus $(\pi \circ q_{\mathcal{R}_\alpha}, T)$ is a pair for $\mathfrak{A}(A, \alpha)$ if an only if (π, T) is a pair for $\mathfrak{A}(A/\mathcal{R}_\alpha, \dot{\alpha})$.

For the second part, it suffices to prove it in the case where α is injective. To this end let (π, U) be a representation of the usual crossed product $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$. Then (π, U) defines a pair for $\mathfrak{A}(A, \alpha)$. Hence there is a canonical $*$-epimorphism $\mathfrak{A}(A, \alpha) \to A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$, that fixes A. Thus we obtain a canonical factorization of $A \hookrightarrow A_\infty \hookrightarrow A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$ by

$$ A \to A \times_\alpha^1 \mathbb{N} \subseteq \mathfrak{A}(A, \alpha) \to A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}. $$

Therefore A embeds in $A \times_\alpha^1 \mathbb{N}$. $\square$

Proposition 3.4. *Let* (A, α) *be a dynamical system with* A *unital and* α *injective. Then the following are equivalent*

    (1) $\alpha(1) = 1$;
    (2) $A \times_\alpha^1 \mathbb{N} \simeq \mathfrak{A}(A, \alpha)$;
    (3) $1 \in A$ *is a unit for* $\mathfrak{A}(A, \alpha)$.

Proof. Let (π_u, T_u) be the universal representation of $\mathfrak{A}(A, \alpha)$. If item (1) above holds then $\pi_u(1)$ is a unit for $\mathfrak{A}(A, \alpha)$, hence π_u can be assumed nondegenerate and item (2) is implied. It is obvious that item (2) implies (3). To end the proof, assume that item (3) holds. Then for $T_u \in \mathfrak{A}(A, \alpha)$ we obtain $\pi_u(1)T_u = T_u\pi_u(1)$. Since T_u is an isometry, then

$$ \pi_u(\alpha(1)) = \pi_u(\alpha(1))T_u^*T_u = T_u^*\pi_u(1)T_u = T_u^*T_u\pi_u(1) = \pi_u(1). $$

Since α is injective, π_u is faithful, hence $\alpha(1) = 1$. $\square$

3.5. The Toeplitz algebra $\mathcal{T}(A, \alpha, L) \simeq \mathcal{T}_{\mathcal{M}_L}$.

In his pioneering paper Exel [16] examines operator algebras related to a system (A, α, L). In his original definition, $\mathcal{T}(A, \alpha, L)$ is the universal C^*-algebra relative to the class

$$ \{(\pi, S) \mid S\pi(x) = \pi(\alpha(x))S, \ \pi(L(x)) = S^*\pi(x)S, \ \text{for all } x \in A\}, $$

generated by $\pi(A)$ and S. A priori $\mathcal{T}(A, \alpha, L)$ seems to differ from the universal C^*-algebra subject to the same representation theory, but generated by $\pi(A)$ and $\pi(A)S$. These two objects are equivalent as π can be chosen to be unital [4, 33].

The existence of at least one such pair (π, S) comes from a representation of the following C*-correspondence. Let A be the semi-A-inner product with

$$x \cdot y = x\alpha(y), \quad \langle x, y \rangle = L(x^*y), \text{ for all } x, y \in A,$$

and let $\mathcal{M}_L$ be the Hilbert A-module associated to it. That is, $\mathcal{M}_L$ is the completion of the quotient $A/\mathcal{N}_1$, where $\mathcal{N}_1 = \{x \in A \mid L(y^*x) = 0, \text{ for all } y \in A\}$. Along with the $*$-homomorphism $\phi_{\mathcal{M}_L} : A \to \mathcal{L}(\mathcal{M}_L)$, such that $\phi(y)(x + \mathcal{N}_1) = yx + \mathcal{N}_1$, the module $\mathcal{M}_L$ becomes the C*- correspondence over A associated to (A, α, L). By definition, every pair (π, S) of $\mathcal{T}(A, \alpha, L)$ defines a representation of $\mathcal{M}_L$.

Lemma 3.6. *If $\xi \in A$ then $\xi + \mathcal{N}_1 = \xi\alpha(1) + \mathcal{N}_1$.*

Proof. By the computation

$$L(\eta^*(\xi - \xi\alpha(1))) = L(\eta^*\xi) - L(\eta^*\xi\alpha(1)) = L(\eta^*\xi) - L(\eta^*\xi) \cdot 1 = 0,$$

we obtain that $\xi - \xi\alpha(1) \in \mathcal{N}_1$. $\square$

Let $\mathcal{M}_{L^n}$ be the C*-correspondence associated to (A, α^n, L^n). For $n = 0$, we identify $\mathcal{M}_{L^0}$ with the trivial C*-correspondence A over A. Let $\mathcal{M}_\infty$ be the direct sum $\oplus_{n \geq 0} \mathcal{M}_{L^n}$. For every $n \in \mathbb{Z}_+$, the mappings

$$\gamma : \mathcal{M}_{L^n} \to \mathcal{M}_{L^{n+1}} : x + \mathcal{N}_n \mapsto \alpha(x) + \mathcal{N}_{n+1},$$

are adjointable with $\gamma_n^*(x + \mathcal{N}_{n+1}) = L(x) + \mathcal{N}_n$. Define the pair (ρ, S) by

$$S : \mathcal{M}_\infty \to \mathcal{M}_\infty : (x_0, x_1 + \mathcal{N}_1, \dots) \mapsto (0, \gamma_0(x_0), \gamma_1(x_1 + \mathcal{N}_1), \dots),$$

$$\rho(y) : \mathcal{M}_\infty \to \mathcal{M}_\infty : (x_0, x_1 + \mathcal{N}_1, \dots) \mapsto (yx_0, yx_1 + \mathcal{N}_1, \dots).$$

We record for further use that

$$S^* : \mathcal{M}_\infty \to \mathcal{M}_\infty : (x_0, x_1 + \mathcal{N}_1, \dots) \mapsto (L(x_1), L(x_2) + \mathcal{N}_1, \dots).$$

Then the pair (ρ, S) defines a pair for $\mathcal{T}(A, \alpha, L)$. Moreover ρ is injective and $\rho(1)$ is the identity of $\mathcal{L}(\mathcal{M}_\infty)$. Thus C*$(\rho(A), \rho(A)S) = $ C*$(\rho(A), S)$.

The following is proved by Brownlowe and Raeburn [4] and Larsen [33], independently. We give an alternative short proof.

Theorem 3.7. [4, 33] *The C*-algebras $\mathcal{T}(A, \alpha, L), \mathcal{T}_{\mathcal{M}_L}$ and C*(ρ, S) are $*$-isomorphic. Consequently, $1 \in A$ acts as a unit on $\mathcal{T}(A, \alpha, L)$.*

Proof. By universality there are canonical $*$-epimorphisms Φ_1 and Φ_2 such that

$$\mathcal{T}_{\mathcal{M}_L} \xrightarrow{\Phi_1} \mathcal{T}(A, \alpha, L) \xrightarrow{\Phi_2} \text{C}^*(\rho(A), \rho(A)S) = \text{C}^*(\rho, S),$$

where we use that $\rho(1) = I$. Then the $*$-epimorphism $\Phi = \Phi_2 \circ \Phi_1$ defines a pair (ρ, t) for $\mathcal{M}_L$ with

$$\rho(x) = \Phi(x), \quad t(x + \mathcal{N}_1) := \Phi(x + \mathcal{N}_1) = \rho(x)S, \text{ for all } x \in A.$$

Note that ρ is injective and (ρ, t) admits the gauge action $\beta_z = \mathrm{ad}_{u_z}$, for $z \in \mathbb{T}$, where $u_z(0, \ldots, 0, x + \mathcal{N}_n, 0, \ldots) = z^n(0, \ldots, 0, x + \mathcal{N}_n, 0, \ldots)$. Moreover, for a compact operator $\Theta_{x+\mathcal{N}_1, y+\mathcal{N}_1} \in \mathcal{K}(\mathcal{M}_L)$ observe that

$$\psi_t(\Theta_{x+\mathcal{N}_1, y+\mathcal{N}_1})(a, 0, \ldots) = \rho(x) S S^*(ya, 0, \ldots) = 0,$$

for all $a \in A$. Hence $\psi_t(\mathcal{K}(\mathcal{M}_L))|_A = 0$. Therefore, if there was an $a \in A$ such that $\rho(a) = \psi_t(k)$ for some $k \in \mathcal{K}(\mathcal{M}_L)$, then

$$(aa^*, 0, \ldots) = \rho(a)(a^*, 0, \ldots) = \psi_t(k)(a^*, 0, \ldots) = 0,$$

hence $a = 0$. Thus the ideal $\{x \in A \mid \rho(x) \in \psi_t(\mathcal{K}(\mathcal{M}_L))\}$ is trivial. Then Φ is a $*$-isomorphism, in view of the gauge invariance theorem for Toeplitz–Cuntz–Pimsner algebras [17, 27]. Consequently, Φ_1 and Φ_2 are $*$-isomorphisms. $\square$

3.8. Exel's crossed product $A \rtimes_{\alpha, L} \mathbb{N} \simeq \mathcal{O}(K_\alpha, \mathcal{M}_L)$. In what follows we identify $\mathcal{T}(A, \alpha, L), \mathcal{T}_{\mathcal{M}_L}$ with $\mathrm{C}^*(\rho, S)$ and we will omit the symbol ρ.

A *redundancy* (a, k) is a pair in $A \times \overline{ASS^*A}$ such that

$$abS = kbS, \text{ for all } b \in A.$$

Equivalently, $\phi_{\mathcal{M}_L}(a) = k \in \mathcal{K}(\mathcal{M}_L)$ and $\rho(a)t(b + \mathcal{N}_1) = \psi_t(k)t(b + \mathcal{N}_1)$. *Exel's crossed product $A \rtimes_{\alpha, L} \mathbb{N}$ is the quotient of $\mathcal{T}(A, \alpha, L)$ by the ideal $\mathcal{I}$ generated by $a - k$, where (a, k) is a redundancy and $a \in \overline{A\alpha(A)A}$* [16].

Brownlowe and Raeburn [4, Cor. 3.6] show that $A \rtimes_{\alpha, L} \mathbb{N}$ is $*$-isomorphic to the relative Cuntz–Pimsner algebra $\mathcal{O}(K_\alpha, \mathcal{M}_L)$ where

$$K_\alpha := \overline{A\alpha(A)A} \cap \phi_{\mathcal{M}_L}^{-1}(\mathcal{K}(\mathcal{M}_L)).$$

(cp. remarks preceding Lemma 2.7). In [4, Thm. 4.2] it is proved that $A \hookrightarrow A \rtimes_{\alpha, L} \mathbb{N}$ if and only if L is *almost faithful on K_α*, i.e., if $x \in K_\alpha$ and $L((xy)^*xy) = 0$ for all $y \in A$, then $x = 0$. This is equivalent to letting $\phi_{\mathcal{M}_L}|_{K_\alpha}$ be injective. In view of Lemma 2.7, which extends [4, Lemma 2.2], we obtain a list of other sufficient and necessary conditions. We gather all these in the following Proposition.

Proposition 3.9. *Let $\mathcal{M}_L$ be the C^*-correspondence associated with an Exel system (A, α, L). Then the following are equivalent:*

  (1) $A \hookrightarrow \mathcal{O}(K_\alpha, \mathcal{M}_L)$;
  (2) $\phi_{\mathcal{M}_L}|_{K_\alpha}$ is injective;
  (3) $K_\alpha \subseteq J_{\mathcal{M}_L}$;
  (4) there is a canonical $*$-epimorphism $\mathcal{O}(K_\alpha, \mathcal{M}_L) \to \mathcal{O}_{\mathcal{M}_L}$;
  (5) $\mathcal{T}_{\mathcal{M}_L}^+ \hookrightarrow \mathcal{O}(K_\alpha, \mathcal{M}_L)$;
  (6) L is almost faithful on K_α.

In particular, for unital systems the picture is further simplified.

Proposition 3.10. *Let (A, α, L) be an Exel system such that $\alpha(1) = 1$. Then the following are equivalent:*

  (1) $A \hookrightarrow \mathcal{O}(K_\alpha, \mathcal{M}_L)$;
  (2) $\mathcal{O}(K_\alpha, \mathcal{M}_L) \simeq \mathcal{O}_{\mathcal{M}_L}$.

If any of the above holds, then $\mathcal{M}_L$ is a Hilbert bimodule.

Proof. Since $A \hookrightarrow \mathcal{O}_{\mathcal{M}_L}$ it suffices to show $[(1) \Rightarrow (2)]$. By $\alpha(1) = 1$ we obtain $\overline{A\alpha(A)A} = A$. Hence $K_\alpha = \phi^{-1}(\mathcal{K}(\mathcal{M}_L))$, thus $J_{\mathcal{M}_L} \subseteq K_\alpha$. If item (1) holds then $K_\alpha \subseteq J_{\mathcal{M}_L}$, therefore $J_{\mathcal{M}_L} = K_\alpha = \phi^{-1}(\mathcal{K}(\mathcal{M}_L))$. $\qquad\square$

Remark 3.11. In [16, Sec. 4] Exel shows that for the systems (A, α, L) that arise (uniquely) by a $*$-monomorphism with hereditary range, the C*-algebras $\mathfrak{A}(A, \alpha)$ and $A \rtimes_{\alpha, L} \mathbb{N}$ coincide. But $A \rtimes_{\alpha, L} \mathbb{N}$ is a quotient of the unital C*-algebra $\mathcal{T}(A, \alpha, L)$, hence $\mathfrak{A}(A, \alpha)$ is unital. In view of Proposition 3.4 this implies that α is unital, which is not the general case as stated in [16, Thm. 4.7]. The error appears in the computations of the proof of [16, Prop. 4.6], where $1 \in A$ is treated as a unit of $\mathfrak{A}(A, \alpha)$.

3.12. **Revisiting Exel's example.** We correct the error that appears in [16, Thm. 4.7]. In fact we do more; by following an alternate route, we shed light also on the noninjective case (see Theorem 3.16). To simplify notation, we identify $A \rtimes_{\alpha, L} \mathbb{N}$ with $\mathcal{O}(K_\alpha, \mathcal{M}_L)$, and $\mathcal{T}(A, \alpha, L), \mathcal{T}_{\mathcal{M}_L}$ with C$^*(\rho, S)$, where we will omit writing ρ.

Our basic assumption is that (A, α, L) satisfies

$$(\dagger) \qquad \alpha \circ L(x) = \alpha(1)x\alpha(1), \text{ for all } x \in A.$$

In this case, $\alpha(L(1)) = \alpha(1)$ and L is *nondegenerate* in the sense of [16].

The choice of $(\dagger)$ enables us to identify K_α with $\overline{A\alpha(A)A}$.

Proposition 3.13. *Let an Exel system (A, α, L) that satisfies $(\dagger)$. Then we obtain $\phi_{\mathcal{M}_L}^{-1}(\mathcal{K}(\mathcal{M}_L)) = \overline{A\alpha(A)A} + \ker \phi_{\mathcal{M}_L}$. Therefore $J_{\mathcal{M}_L} \subseteq K_\alpha = \overline{A\alpha(A)A}$.*

Proof. Trivially $\ker \phi_{\mathcal{M}_L} \subseteq \phi_{\mathcal{M}_L}^{-1}(\mathcal{K}(\mathcal{M}_L))$. For all $x, \xi \in A$, we obtain

$$\Theta_{\alpha(x)+\mathcal{N}_1, 1+\mathcal{N}_1}(\xi + \mathcal{N}_1) = \alpha(x)\alpha(L(\xi)) + \mathcal{N}_1 = \alpha(x)\xi\alpha(1) + \mathcal{N}_1$$
$$= \alpha(x)\xi + \mathcal{N}_1 = \phi_{\mathcal{M}_L}(\alpha(x))(\xi + \mathcal{N}_1),$$

where we have used Lemma 3.6. Therefore $\phi_{\mathcal{M}_L}(\alpha(A)) \subseteq \mathcal{K}(\mathcal{M}_L)$, hence $\overline{A\alpha(A)A} \subseteq \phi_{\mathcal{M}_L}^{-1}(\mathcal{K}(\mathcal{M}_L))$. Now, let $\xi, \eta, \zeta \in A$; then

$$\Theta_{\eta+\mathcal{N}_1, \zeta+\mathcal{N}_1}(\xi + \mathcal{N}_1) = \eta\alpha(L(\zeta^*\xi)) + \mathcal{N}_1 = \eta\alpha(1)\zeta^*\xi\alpha(1) + \mathcal{N}_1$$
$$= \eta\alpha(1)\zeta^*\xi + \mathcal{N}_1 = \phi_{\mathcal{M}_L}(\eta\alpha(1)\zeta^*)(\xi + \mathcal{N}_1).$$

By passing to limits of linear combinations, we obtain that for every $k \in \mathcal{K}(\mathcal{M}_L)$ there is an $x \in \overline{A\alpha(A)A}$ such that $\phi_{\mathcal{M}_L}(x) = k$. Thus, $\phi_{\mathcal{M}_L}^{-1}(\mathcal{K}(\mathcal{M}_L)) \subseteq \overline{A\alpha(A)A} + \ker \phi_{\mathcal{M}_L}$. To end the proof, recall that $J_{\mathcal{M}_L} = \phi_{\mathcal{M}_L}^{-1}(\mathcal{K}(\mathcal{M}_L)) \cap \ker \phi_{\mathcal{M}_L}^\perp$. Therefore

$$J_{\mathcal{M}_L} = \phi^{-1}(\mathcal{K}(\mathcal{M}_L)) \cap \ker \phi_{\mathcal{M}_L}^\perp \cap (\overline{A\alpha(A)A} + \ker \phi_{\mathcal{M}_L})$$
$$= \phi^{-1}(\mathcal{K}(\mathcal{M}_L)) \cap \ker \phi_{\mathcal{M}_L}^\perp \cap \overline{A\alpha(A)A} \subseteq K_\alpha,$$

and $\overline{A\alpha(A)A} \subseteq \overline{A\alpha(A)A} \cap (\overline{A\alpha(A)A} + \ker \phi_{\mathcal{M}_L}) = K_\alpha \subseteq \overline{A\alpha(A)A}$. $\qquad\square$

Theorem 3.14. *Let an Exel system (A, α, L) that satisfies (†). Then the following are equivalent*

  (1) $A \hookrightarrow \mathcal{O}(K_\alpha, \mathcal{M}_L)$;
  (2) $\mathcal{O}(K_\alpha, \mathcal{M}_L) \simeq \mathcal{O}_{\mathcal{M}_L}$.

If any of the above holds, then $\mathcal{M}_L$ is a Hilbert bimodule.

Proof. In view of Propositions 3.9 and 3.13, if any of items (1) or (2) holds, then $J_{\mathcal{M}_L} = K_\alpha = \overline{A\alpha(A)A}$ which implies that they are equivalent. Then $\phi_{\mathcal{M}_L}|_{J_{\mathcal{M}_L}}$ is onto $\mathcal{K}(\mathcal{M}_L)$, hence $\mathcal{M}_L$ is a Hilbert bimodule. $\square$

We want to examine the connection with $A \times_\alpha^1 \mathbb{N}$. Towards this, the first step is the following Proposition.

Proposition 3.15. *Let (A, α, L) be an Exel system that satisfies (†). Then there are canonical $*$-epimorphisms such that the diagram*

$$
\begin{array}{ccc}
\mathcal{T}(A, \alpha, L) & \xrightarrow{\quad \phi \quad} & \mathcal{O}(K_\alpha, \mathcal{M}_L) \\
 & \tilde{\phi} \searrow \qquad \swarrow \hat{\phi} & \\
 & A \times_\alpha^1 \mathbb{N} &
\end{array}
$$

commutes.

Proof. Recall that $A \times_\alpha^1 \mathbb{N} \hookrightarrow \mathfrak{A}(A, \alpha)$ by Proposition 3.2. Let (π, T) be a pair for $\mathfrak{A}(A, \alpha)$ and define $S' = \pi(\alpha(1))T = T\pi(1)$. Then,

$$S'\pi(x) = T\pi(x) = \pi(\alpha(x))T = \pi(\alpha(x))S'$$

for all $x \in A$. Moreover,

$$
\begin{aligned}
(S')^*\pi(x)S' &= T^*\pi(\alpha(1))\pi(x)\pi(\alpha(1))T = T^*\pi(\alpha(1)x\alpha(1))T \\
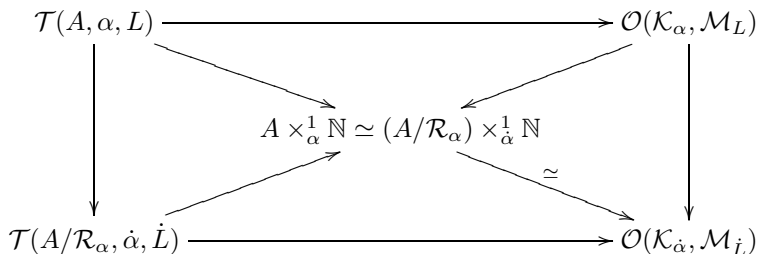&= T^*\pi(\alpha \circ L(x))T = T^*T\pi(L(x)) = \pi(L(x)),
\end{aligned}
$$

for all $x \in A$. Therefore (π, S') defines a representation of $\mathcal{T}(A, \alpha, L)$, thus the $*$-epimorphism $\tilde{\phi}$ is defined by the universal property. The rest of the proof follows similar arguments with those of [16, Thm. 4.7]. $\square$

Given (A, α, L) satisfying (†), one can define a new system $(A/\mathcal{R}_\alpha, \dot{\alpha}, \dot{L})$ satisfying (†), such that $\dot{\alpha}$ is injective and $\dot{L}$ is a well defined (positive) operator. Indeed, for $x \in \mathcal{R}(A)$ we obtain

$$\left\| \alpha^{n+1}(L(x)) \right\| = \|\alpha^n(\alpha(1)x\alpha(1))\| = \left\| \alpha^{n+1}(1)\alpha^n(x)\alpha^{n+1}(1) \right\| \le \|\alpha^n(x)\|,$$

thus $L(\mathcal{R}_\alpha) \subseteq \mathcal{R}_\alpha$ and $\dot{L}$ is well defined.

Theorem 3.16. *Let (A, α, L) be an Exel system that satisfies* (†). *Then the following diagram commutes*

$$
\begin{array}{ccc}
\mathcal{T}(A, \alpha, L) & \xrightarrow{\hspace{4cm}} & \mathcal{O}(\mathcal{K}_\alpha, \mathcal{M}_L) \\
& A \times^1_\alpha \mathbb{N} \simeq (A/\mathcal{R}_\alpha) \times^1_{\dot{\alpha}} \mathbb{N} & \\
\mathcal{T}(A/\mathcal{R}_\alpha, \dot{\alpha}, \dot{L}) & \xrightarrow{\hspace{4cm}} & \mathcal{O}(\mathcal{K}_{\dot{\alpha}}, \mathcal{M}_{\dot{L}})
\end{array}
$$

and consequently, $\mathcal{M}_{\dot{L}}$ is a Hilbert bimodule.

Proof. In view of the previous remarks, Propositions 3.3 and 3.15, and Theorem 3.14, it suffices to show that $A \times^1_\alpha \mathbb{N} \simeq \mathcal{O}(K_\alpha, \mathcal{M}_L)$ when α is injective. In this case, relation (†) for $x = 1$ gives that $L(1) = 1$. Therefore the operator S of Theorem 3.7 becomes an isometry.

Let (π, S') be the induced representation of $\mathcal{O}(K_\alpha, \mathcal{M}_L)$ by (ρ, S). Then S' is an isometry, as the range of the isometry S. By Proposition 3.13, we obtain that $(\rho(\alpha(x)), S\rho(x)S^*)$ is a redundancy, hence $\pi(\alpha(x)) = S'\pi(x)S'^*$ for all $x \in A$. Thus (π, S') defines a representation for $\mathcal{O}(K_\alpha, \mathcal{M}_L)$, hence there is a canonical $*$-epimorphism $\mathcal{O}(K_\alpha, \mathcal{M}_L) \to A \times^1_\alpha \mathbb{N}$, that is the inverse for $\hat{\phi}$ of Proposition 3.15. $\qquad\square$

Remark 3.17. Let the sets $Q = \{[\text{relation (†)}], [\alpha(A) \text{ is hereditary}]\}$ and $W = \{[L(1) = 1], [\alpha \text{ is injective}]\}$. It is easy to check that whenever an item from Q and an item from W hold, then all items in Q and W hold. The implication that concerns us at this point is that, when $\alpha(A)$ is hereditary and α is injective then the system (A, α, L) satisfies (†) and $L(1) = 1$. This follows readily by the remarks before [16, Prop. 4.3], which completes the correction of [16, Thm. 4.7] mentioned earlier. In any of these cases, the transfer operator is uniquely determined, since α is injective.

Moreover, we have showed that in this case there is a canonical $*$-isomorphism $\mathcal{O}_{\mathcal{M}_L} \to A \times^1_\alpha \mathbb{N}$. Surprisingly, the converse also holds and we leave the proof of this metamathematical remark to the reader.

Remark 3.18. The systems that we examine may not be unital. In fact this case is trivial: when $1 = \alpha(1) = L(1)$ and (†) holds, then α is injective and $\alpha \circ L = \mathrm{id}$; therefore α is an automorphism with $\alpha^{-1} = L$. Hence $\mathcal{T}(A, \alpha, L)$ is the C*-algebra generated by the universal representation of $\mathfrak{A}(A, \alpha, \mathrm{is})_r$ [22], and $A \times_{\alpha,L} \mathbb{N}$ is the usual crossed product $A \rtimes_\alpha \mathbb{Z}$.

Remark 3.19. Brownlowe, Raeburn and Vitadello [5] examine a second class of examples of Exel systems induced by classical dynamical systems (T, ϕ). For this class they show that the set of redundancies is the Katsura ideal, moreover that $C_0(T) \times_{\alpha,L} \mathbb{N}$ is again the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{M}_L}$. This

identification is obtained in our case and we wonder whether this is true in general.

3.20. **C*-envelopes.** In this subsection we examine the C*-envelope of the nonselfadjoint parts of $\mathfrak{A}(A,\alpha)$ and $A \times^1_\alpha \mathbb{N}$, and the form of $\mathcal{O}_{\mathcal{M}_L}$.

3.20.1. *The nonselfadjoint part of $\mathfrak{A}(A,\alpha)$ and $A \times^1_\alpha \mathbb{N}$.* We remark that the automorphic extension of the (unital) automorphism α_∞ of A_∞ to the multiplier algebra $\mathcal{M}(A_\infty)$ will be denoted by the same symbol.

Theorem 3.21. *Let $A \times_\alpha \mathcal{T}_1^+ := \text{alg}\{\pi_u(A), T_u\} \subseteq \mathfrak{A}(A,\alpha)$. Then the C*-envelope of $A \times_\alpha \mathcal{T}_1^+$ is the C*-subalgebra $\text{C}^*(B,U)$ of $\mathcal{M}(B) \rtimes_\beta \mathbb{Z}$, where $B = (A/\mathcal{R}_\alpha)_\infty$ and $\beta = (\dot\alpha)_\infty$.*

Proof. It suffices to show it for α injective, i.e., $\mathcal{R}_\alpha = (0)$. By [39] the pair (π_u, T_u) extends to a covariant unitary pair (π, U) of $(A_\infty, \alpha_\infty)$. In turn a pair (π, U) defines a pair for $\mathfrak{A}(A,\alpha)$. Therefore the canonical embedding $\mathfrak{A}(A,\alpha) \to \mathcal{M}(A_\infty) \rtimes_{\alpha_\infty} \mathbb{Z}$ defines a completely isometric homomorphism of $A \times_\alpha \mathcal{T}_1^+$. It is immediate that $\text{C}^*(\pi(A), U) = \text{C}^*(\pi(A_\infty), U)$ since the powers $U^{-n}\pi(x)U^n$ generate $\pi(A_\infty)$. Thus $\text{C}^*(\pi(A_\infty), U)$ is a C*-cover of $A \times_\alpha \mathcal{T}_1^+$.

Let $\{\beta_z\}$ be the gauge action defined on $\mathcal{M}(A_\infty) \rtimes_{\alpha_\infty} \mathbb{Z}$; then $A \times_\alpha \mathcal{T}_1^+$ is β_z-invariant. Thus, if $\mathcal{J}$ is the Šilov ideal in $\text{C}^*(\pi(A_\infty), U)$ and it is nontrivial then it is also β_z-invariant. Therefore $\mathcal{J}$ intersects non trivially the fixed point algebra $\text{C}^*(A_\infty, I)$ of $\text{C}^*(\pi(A_\infty), U)$. Since A_∞ is essential in $\mathcal{M}(A_\infty)$, then $\mathcal{J}$ intersects A_∞. Since A_∞ is a direct limit there is an $x \in A_\infty$ such that $x \in \mathcal{J}$ and $\alpha_\infty^n(x) \in A$, for some n. But then $\pi(\alpha_\infty^n(x)) = U^n\pi(x)U^{-n} \in \mathcal{J}$. Therefore $\mathcal{J}$ intersects with A, hence with $A \times_\alpha \mathcal{T}_1^+$, which is a contradiction. $\square$

Recall that if p is a projection in a C*-algebra $\mathfrak{A}$ then the corner $p\mathfrak{A}p$ is called *full* if the only ideal of $\mathfrak{A}$ containing $p\mathfrak{A}p$ is $\mathfrak{A}$.

Theorem 3.22. *Let $A_{\text{nd}} \times_\alpha \mathcal{T}_1^+ := \text{alg}\{\pi_u(A), T_u\pi_u(1)\} \subseteq A \times^1_\alpha \mathbb{N}$. Then the C*-envelope of $A_{\text{nd}} \times_\alpha \mathcal{T}_1^+$ is a full corner of $B \rtimes_\beta \mathbb{Z}$, where $B = (A/\mathcal{R}_\alpha)_\infty$, $\beta = (\dot\alpha)_\infty$.*

Proof. It suffices to show it for α injective, i.e., $\mathcal{R}_\alpha = (0)$. By [39, Prop. 3.3] (or usual dilation theory) we obtain that $p(A_\infty \rtimes_{\alpha_\infty} \mathbb{Z})p \simeq A \times^1_\alpha \mathbb{N}$ which we show is $\text{C}^*_{\text{env}}(A_{\text{nd}} \times_\alpha \mathcal{T}_1^+)$. To this end, fix a pair (π, U) that integrates to a faithful representation of $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$.

As in the proof of Theorem 3.21, we can use the same gauge-invariance argument to show that the Šilov ideal intersects with the fixed point algebra $pA_\infty p$ and consequently intersects with $pAp = A$, by moving backwards with $\text{ad}_{(U^n p)^*}$, which leads to a contradiction.

To prove that it is full, let an ideal $\mathcal{I}$ of $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$ containing the corner. Then $\mathcal{I}$ contains a copy of A. If (e_i) is a c.a.i. of A_∞ then

$$\pi[0, x, \alpha(x), \dots] = \lim_i U^*\pi(e_i) \cdot \pi[x, \alpha(x), \dots] \cdot \pi(e_i)U \in \mathcal{I}.$$

Inductively $A_\infty \subseteq \mathcal{I}$. Therefore $\pi(x)U^n = \lim_i \pi(x) \cdot \pi(e_i)U^n \in \mathcal{I}$, for all $x \in A_\infty$. Thus $\mathcal{I}$ contains the generators of $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$, hence it is trivial. $\square$

3.22.1. *The tensor algebra* $\mathcal{T}^+_{\mathcal{M}_L}$. By [25] the C*-envelope of any tensor algebra is the Cuntz–Pimsner algebra. Under our assumptions for (A, α, L) we show that $\mathcal{O}_{\mathcal{M}_L}$ is a corner of a usual crossed product. In addition it has a special form, which we will use in the multivariable case.

Theorem 3.23. *Let an Exel system (A, α, L) that satisfies* (†) *and $L(1) = 1$. Then $\mathcal{O}_{\mathcal{M}_L}$ is a full corner of $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$. In particular $p(A_\infty \rtimes_{\alpha_\infty} \mathbb{Z})p$ is the closed linear span of*

$$\pi(x_0) + \sum_{n=0}^{k} \pi(x_n)U^n p + \sum_{n=1}^{k} pU^{-n}\pi(y_n), \text{ for } x_0, x_n, y_n \in A.$$

Proof. The canonical identification $\mathcal{O}_{\mathcal{M}_L} \simeq A \times^1_\alpha \mathbb{N}$ gives that $\mathcal{T}^+_{\mathcal{M}_L}$ is completely isometrically isomorphic to $A_{\mathrm{nd}} \times_\alpha \mathcal{T}^+_1$. Hence the first part of the theorem follows by Theorem 3.22 and [25]. The second part follows readily by Theorem 3.14, since $\mathcal{M}_L$ is a Hilbert bimodule. Indeed, in this case $\mathcal{O}_{\mathcal{M}_L} = \overline{\mathcal{T}^+_{\mathcal{M}_L} + (\mathcal{T}^+_{\mathcal{M}_L})^*}$. $\square$

Remark 3.24. When (A, α, L) satisfies (†) and $L(1) = 1$, then $\alpha^n \circ L^n(x) = \alpha^n(1)x\alpha^n(1)$, for all $x \in A$, hence $pA_\infty p = A$. Therefore the monomials $p\pi(x)U^n p$ and $pU^{-n}\pi(x)p$, for $x \in A_\infty$, that span $p(A_\infty \rtimes_{\alpha_\infty} \mathbb{Z})p$ are written as $\pi(x')\pi(\alpha^n(1))U^n$ and $U^{-n}\pi(\alpha^n(1))\pi(x')$, for $x' \in A$. This gives an ad-hoc proof of Theorem 3.23 and we record it for further use in Theorem 4.30.

4. MULTIVARIABLE CASE

We wish to explore universal nonselfadjoint operator algebras that arise from dynamical systems, subject to covariant relations induced by a Cuntz family. The representation theory of such objects was exploited by Cuntz [9], Murphy [36], Stacey [39], and Laca [30] for *the twisted crossed product* and *the crossed product of multiplicity n*.

4.1. **The twisted crossed product.** Let $\alpha \in \mathrm{Aut}(A)$; then the usual crossed product $A \rtimes_\alpha \mathbb{Z}$ is defined, and we simply write $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{O}_n$ for the unique tensor product C*-algebra. If (π_u, U_u) is the universal unitary covariant pair for (A, α), then the *twisted crossed product of A by α*, denoted by $A \rtimes_\alpha \mathcal{O}_n$, is the C*-subalgebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{O}_n$ generated by $\pi_u(A)$ and the Cuntz family $\{U_u \otimes S_i\}$. By convention C*(π_u, U_u) commutes with $\mathcal{O}_n$. As a consequence, the embedding of $\mathcal{O}_n$ in $A \rtimes_\alpha \mathcal{O}_n$ is not multiplicity free.

Let (π, U) acting on H_1 such that C*$(\pi, U) \simeq A \rtimes_\alpha \mathbb{Z}$ and let $\mathcal{O}_n$ acting on H_2. Then $A \rtimes_\alpha \mathcal{O}_n$ is $*$-isomorphic to the C*-algebra in $\mathcal{B}(H_1 \otimes H_2)$ generated by $\pi(A) \otimes I_{H_1}$ and $\{U \otimes S_i\}$. Therefore the twisted crossed product does not depend on the choice of the covariant pair (π, U), but on the dynamical system (A, α), in accordance with the original definition [9]. We remark that the notation $A \rtimes_\alpha \mathcal{O}_n$ is due to Stacey [39] because of this fact.

4.2. **Multiplicity n crossed products.** The *multiplicity n crossed product* is the universal C*-algebra generated by $\pi(A)T_\nu T_\mu^*$ such that

  (1) π is a nondegenerate representation of A,
  (2) $\{T_i\}$ is a Toeplitz–Cuntz family, and
  (3) $\pi(\alpha(x)) = \sum_{i=1}^n T_i\pi(x)T_i^*$, for all $x \in A$,

and it is denoted by $A \times_\alpha^n \mathbb{N}$ [39]. It is immediate, due to nondegeneracy, that $A \times_\alpha^n \mathbb{N}$ is generated by $\pi(A)$ and $\{T_i\}$. Analogous to the one variable case we remove nondegeneracy by "weighting" the isometries T_i.

Proposition 4.3. *The multiplicity n crossed product $A \times_\alpha^n \mathbb{N}$ is $*$-isomorphic to the* C*-subalgebra C*$(\pi_u(A), \{T_{u,i}\pi_u(1)\})$ *of the universal* C*-algebra C*$(\pi_u(A), \{T_{u,i}\})$ *relative to* $(\pi, \{T_i\})$ *such that*

  (1) π is a representation of A,
  (2) $\{T_i\}$ is a Toeplitz–Cuntz family, and
  (3) $\pi(\alpha(x)) = \sum_{i=1}^n T_i\pi(x)T_i^*$, for all $x \in A$.

In particular, $A \times_\alpha^n \mathbb{N}$ coincides with C*$(\pi_u(A), \{T_{u,i}\})$ *if and only if $\alpha(1) = 1$.*

Proof. If $\pi_u(\alpha(x)) = \sum_{i=1}^n T_{u,i}\pi_u(x)T_{u,i}^*$, then $\pi_u(\alpha(x))T_{u,i} = T_{u,i}\pi_u(x)$, for all $x \in A$ and $i = 1, \ldots, n$. Hence $\pi_u(1)T_{u,i}\pi(1) = T_{u,i}\pi_u(1)$ for all $i = 1, \ldots, n$. The proof is completed by arguments similar to those of Propositions 3.2 and 3.4. $\square$

Remark 4.4. The set of representations of $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ are sufficient to obtain an injective embedding of A in $A \times_\alpha^n \mathbb{N}$. However, they may not be enough to obtain the norm of $A \times_\alpha^n \mathbb{N}$. In other words the canonical $*$-homomorphism $A \times_\alpha^n \mathbb{N} \to A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ may not be injective, as we show in the counterexample in Subsection 4.32. Therefore there is no analogue of [39, Prop. 3.3] for $n > 1$.

4.5. **The semicrossed product $A \times_\alpha \mathcal{T}_n^+$.** In what follows we construct an operator algebra $A \times_\alpha \mathcal{T}_n^+$ starting with a Banach algebra $\ell^1(A, \alpha, \mathbb{F}_n^+)$. This universal operator algebra should not be mistaken with the objects examined in [12, 15, 18].

Let the semigroup $\mathbb{F}_n^+$ and the linear tensor product $A \otimes c_{00}(\mathbb{F}_n^+) = \operatorname{span}\{x \otimes \delta_\mu \mid x \in A, \mu \in \mathbb{F}_n^+\}$. The multiplication rule that turns $A \otimes c_{00}(\mathbb{F}_n^+)$ into an algebra is

$$(x \otimes \delta_i) \cdot (y \otimes \delta_j) = (x\alpha(y)) \otimes \delta_{ij},$$

for $i, j \in \{1, \ldots, n\}$. Define the ℓ^1-norm

$$\left| \sum_\mu x_\mu \otimes \delta_\mu \right|_1 = \sum_\mu \|x_\mu\|_A,$$

where μ is taken over a finite subset of finite paths in $\mathbb{F}_n^+$, and let $\ell^1(A, \alpha, \mathbb{F}_n^+)$ be the Banach algebra that comes from the ℓ^1-completion of $A \otimes c_{00}(\mathbb{F}_n^+)$. Note that $\ell^1(A, \alpha, \mathbb{F}_n^+) = \overline{\operatorname{span}\{x \otimes \delta_\mu \mid x \in A, \mu \in \mathbb{F}_n^+\}}^{|\cdot|_1}$. When $\alpha(1) = 1$, then

$1 \otimes \delta_\varnothing$ is the unit for $\ell^1(A, \alpha, \mathbb{F}_n^+)$, and $\ell^1(A, \alpha, \mathbb{F}_n^+) = \overline{\mathrm{alg}}^{|\cdot|_1} \{x \otimes \delta_\varnothing, 1 \otimes \delta_\mu \mid x \in A, \mu \in \mathbb{F}_n^+\}$.

Consider the class

$$\mathcal{F} = \big\{ (\pi, \{T_i\}_{i=1}^n) \mid \pi : A \to \mathcal{B}(H), \, T_i \in \mathcal{T}_n^+, \, \pi(\alpha(x)) = \sum_{i=1}^n T_i \pi(x) T_i^* \big\},$$

If $(\pi, \{T_i\}) \in \mathcal{F}$ then $T_\mu \pi(x) = \pi(\alpha^{|\mu|}(x)) T_\mu$, for all $x \in A$, and $\mu \in \mathbb{F}_n^+$. Moreover $(\pi, \{T_i\}_{i=1}^n)$ defines a representation $\rho : \ell^1(A, \alpha, \mathbb{F}_n^+) \to \mathcal{B}(H)$, such that $\rho(x \otimes \delta_\mu) = \pi(x) T_\mu$. Indeed, ρ is $|\cdot|_1$-contractive on $A \otimes c_{00}(\mathbb{F}_n^+)$ (and extends to $\ell^1(A, \alpha, \mathbb{F}_n^+)$) since

$$\left\| \rho(\sum_\mu x_\mu \otimes \delta_\mu) \right\| \le \sum_\mu \|\rho(x_\mu) T_\mu\| \le \sum_\mu \|x_\mu\|_A = \left| \sum_\mu x_\mu \otimes \delta_\mu \right|_1.$$

In this case we say that ρ *integrates* $(\pi, \{T_i\})$.

Definition 4.6. We denote by $A \times_\alpha \mathcal{T}_n^+$ the universal operator algebra relative to the class $\mathcal{F}$ that is generated by $\pi(x)$ and T_μ, for $x \in A$ and $\mu \in \mathbb{F}_n^+$.

Remark 4.7. The semicrossed product $A \times_\alpha \mathcal{T}_n^+$ constructed here differs in general from the tensor algebra relative to n $*$-endomorphisms α_i of A, examined in [35, 12, 24]. Nevertheless, when α_i coincide with an automorphism α, then they are completely isometrically isomorphic.

The class $\mathcal{F}$ of representations is nonempty. Moreover, the existence of such representations is equivalent to $A_\infty \ne (0)$ [39, Prop. 2]. We present such an example, which we prove that has Fourier coefficients.

Example 4.8. Let (π, U) be the left regular representation of the dynamical system $(A_\infty, \alpha_\infty)$ acting on H. Let $\mathcal{K} = H \otimes K$, where K is a Hilbert space on which $\mathcal{O}_n$ acts (faithfully). If $\{S_i\}$ is a Cuntz family on K, let $T_i = U \otimes S_i$ and $\widehat{\pi} = \pi \otimes I_K$. Then $\{T_i\}$ is also a Cuntz family. Moreover if $q : A \to A_\infty$ is the canonical $*$-homomorphism, then

$$\sum_{i=1}^n T_i \widehat{\pi} \circ q(c) T_i^* = \sum_{i=1}^n (U \pi[c, \alpha(c), \dots] U^*) \otimes S_i S_i^*$$

$$= \pi \circ \alpha_\infty[c, \alpha(c), \dots] \otimes I = \widehat{\pi} \circ q(\alpha(c)),$$

for $i = 1, \dots, n$. Hence $(\widehat{\pi} \circ q, \{T_i\})$ defines a pair in $\mathcal{F}$. Note that $I_H \otimes S_i \in \widehat{\pi}(A_\infty)'$; therefore, for a finite sum $\sum_{\mu \in F} \widehat{\pi}(x_\mu) T_\mu$, we define

$$P_\nu(\sum_{\mu \in F} \widehat{\pi} \circ q(x_\mu) T_\mu) := (I \otimes S_\nu^*) \cdot \left(\sum_{\mu \in F} \widehat{\pi} \circ q(x_\mu) T_\mu \right) \cdot (U^{-|\nu|} \otimes I_K)$$

$$= \sum_{\mu \in F} \widehat{\pi} \circ q(x_\mu) \cdot (I \otimes S_\nu^*) T_\mu (U^{-|\nu|} \otimes I_K)$$

$$= \widehat{\pi} \circ q(x_\nu) \cdot (U^{|\nu|} \otimes I_K)(U^{-|\nu|} \otimes I_K) = \widehat{\pi} \circ q(x_\nu),$$

since $T_\mu = U^{|\mu|} \otimes S_\mu$. Thus $P_\nu(\sum_{\mu \in F} \widehat{\pi} \circ q(x_\mu) T_\mu) = 0$, for every $\nu \in F$, if and only if $\widehat{\pi} \circ q(x_\nu) = 0$ for every $\nu \in F$, if and only if $x_\nu \in \mathcal{R}_\alpha$ for every $\nu \in F$, if and only if $\sum_{\mu \in F} \widehat{\pi} \circ q(x_\mu) T_\mu = 0$.

Proposition 4.9. *The kernel $\mathcal{N}$ of the seminorm ω_1 relative to $\mathcal{F}$ coincides with $\ell^1(\mathcal{R}_\alpha, \alpha, \mathbb{F}_n^+)$.*

Proof. First observe that

$$\ell^1(\mathcal{R}_\alpha, \alpha, \mathbb{F}_n^+) = \overline{\mathrm{span}}^{|\cdot|_1}\{x \otimes \delta_\mu \mid x \in \mathcal{R}_\alpha, \mu \in \mathbb{F}_n^+\}$$

can be embedded as an ideal in $\ell^1(A, \alpha, \mathbb{F}_n^+)$ and that it is contained in $\mathcal{N}$.

For the converse, assume that $X = \sum_{\mu \in F} x_\mu \otimes \delta_\mu$ is in $\mathcal{N}$, where F is finite, and the sum is written in reduced form with respect to the words μ. Let ρ be the representation of Example 4.8. By definition of $\mathcal{N}$ we have that $\rho(X) = \sum_{\mu \in F} \widehat{\pi} \circ q(x_\mu) T_\mu = 0$; then $x_\nu \in \mathcal{R}_\alpha$, by Example 4.8. Hence $X \in \ell^1(\mathcal{R}_\alpha, \alpha, \mathbb{F}_n^+)$. To end the proof, note that an $X \in \mathcal{N}$ is the ℓ^1-sum of such elements and that $\ell^1(\mathcal{R}_\alpha, \alpha, \mathbb{F}_n^+)$ is ℓ^1-closed. $\qquad\square$

Analogous to the one variable case [22, Prop. 3.4], we can always assume that (A, α) is injective.

Proposition 4.10. *The operator algebras $A \times_\alpha \mathcal{T}_n^+$ and $(A/\mathcal{R}_\alpha) \times_{\dot{\alpha}} \mathcal{T}_n^+$ are completely isometrically isomorphic.*

We mention that the unit $1 \in A$ may not be the unit of $A \times_\alpha \mathcal{T}_n^+$, since A is not represented nondegenerately. Below we give the exact nonselfadjoint analogue of $A \times_\alpha^n \mathbb{N}$.

Definition 4.11. Let $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+$ be the universal operator algebra relative to the class

$$\mathcal{F}_{\mathrm{nd}} = \big\{(\pi, \{T_i\}_{i=1}^n) \in \mathcal{F} \mid \pi \text{ is nondegenerate}\big\},$$

generated by $\pi(x) T_\mu$, for $x \in A, \mu \in \mathbb{F}_n^+$.

Due to nondegeneracy $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+$ is generated separately by a copy of A and a copy of $\mathcal{T}_n^+$. Moreover, the C*-algebra generated by the universal representation of $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+$ is $A \times_\alpha^n \mathbb{N}$. Analogous to the C*-case for $n = 1$ we obtain the following.

Proposition 4.12. *The operator algebras $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+$ and $(A/\mathcal{R}_\alpha)_{\mathrm{nd}} \times_{\dot{\alpha}} \mathcal{T}_n^+$ are unital completely isometrically isomorphic. Moreover, $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+$ is unital completely isometrically isomorphic to the subalgebra generated by $\pi_u(A)$ and $\{T_{u,i} \pi_u(1)\}$ in $A \times_\alpha \mathcal{T}_n^+$.*

There is a considerable difference between the unital and nonunital dynamical systems. One such difference is highlighted below.

Remark 4.13. Fix $(\pi, \{T_i\}_{i=1}^n) \in \mathcal{F}$. When $\alpha(1) = 1$, then $\pi(1)$ is a unit for $C^*(\pi(A), T_i)$, hence π can be chosen unital. Moreover $\{T_i\}$ is automatically

a Cuntz family. Thus $(\pi, \{T_i\}) \in \mathcal{F}$ if and only if $T_i \in \mathcal{O}_n^+$ and $\pi(\alpha(x)T_i = T_i\pi(x)$. Therefore, $A \times_\alpha \mathcal{T}_n^+$ is generated by $A/\mathcal{R}_\alpha$ and $\mathcal{O}_n^+$.

Additionally, the algebras $A \times_\alpha \mathcal{T}_n^+$ and $A_{\mathrm{nd}} \times_\alpha \mathcal{T}_n^+$ coincide. Hence the C*-algebra generated by the universal representation of $A \times_\alpha \mathcal{T}_n^+$ coincides with $A \times_\alpha^n \mathbb{N}$.

Proposition 4.14. *Let (A, α) be a unital dynamical system. Then every completely isometric representation ρ that integrates a family $(\pi, \{S_i\}) \in \mathcal{F}$ is maximal. Therefore $A \times_\alpha \mathcal{T}_n^+$ has the unique extension property.*

Proof. Without loss of generality we can assume that α is injective. Hence by the previous remarks the algebra $A \times_\alpha \mathcal{T}_n^+$ is generated by A and $\mathcal{O}_n^+$. Let $\rho : A \times_\alpha \mathcal{T}_n^+ \to \mathcal{B}(H)$ be a representation that integrates a pair $(\pi, \{S_i\})$ in $\mathcal{F}$ and let $\nu : A \times_\alpha \mathcal{T}_n^+ \to \mathcal{B}(K)$ be a dilation of ρ. Then $\nu|_A$ is a dilation of the $*$-representation $\rho|_A$, hence $\nu|_A$ is trivial. That is H is $\nu(A)$-reducing. It suffices to show that $\rho|_{\mathcal{O}_n^+}$ is also maximal, for then H will be $\nu(\mathcal{O}_n^+)$-reducing. This will imply that H is $\nu(A \times_\alpha \mathcal{T}_n^+)$-reducing, hence ρ is maximal.

Claim. The tensor algebra $\mathcal{T}_n^+$ has the unique extension property.
Proof of the Claim. Let $\nu : \mathcal{O}_n^+ \to \mathcal{B}(K)$ be a maximal dilation of ρ, which is a unital completely isometric map. Then ν extends to a unique (faithful) $*$-representation of $\mathcal{O}_n$, which we will again denote by ν. Hence $\{\nu(S_i)\}$ is also a Cuntz family. For $i = 1, \ldots, n$, let

$$\nu(S_i) = \left[\begin{array}{cc} \rho(S_i) & a_i \\ 0 & b_i \end{array} \right].$$

Note that the $(2, 1)$-entry must be zero since $\rho(S_i)$ is an isometry and $\nu(S_i)$ is a contraction. Then

$$\nu(S_i)\nu(S_i)^* = \left[\begin{array}{cc} \rho(S_i)\rho(S_i)^* + a_i a_i & * \\ * & * \end{array} \right]$$

and $\sum_{i=1}^n \nu(S_i)\nu(S_i)^* = I_K$, since $\{\nu(S_i)\}$ is a Cuntz family. Thus, by equating the $(1, 1)$-entries

$$\sum_{i=1}^n \rho(S_i)\rho(S_i)^* + a_i a_i^* = I_H.$$

But $\{\rho(S_i)\}$ is in turn a Cuntz family on H, thus $0 \le a_i a_i^* \le \sum_{i=1}^n a_i a_i^* = 0$. Hence $a_i = 0$ for every $i = 1, \ldots, n$, and ν is a trivial dilation of ρ. $\square$

The following Corollary is immediate. Note that the operator algebra we have constructed may not be separable (cp. Subsection 2.1).

Corollary 4.15. *Let (A, α) be a unital dynamical system. Then the Choquet boundary exists for the operator algebra $A \times_\alpha \mathcal{T}_n^+$.*

Theorem 4.16. *Let (A, α) be a unital dynamical system. Then $\mathrm{C}^*_{\mathrm{env}}(A \times_\alpha \mathcal{T}_n^+) \simeq A \times_\alpha^n \mathbb{N}$.*

Proof. In view of Corollary 4.15 and Lemma 2.2, we have that the identity representation of $A \times_\alpha^n \mathbb{N}$ is faithful and $A \times_\alpha \mathcal{T}_n^+$ has the unique extension property, thus $A \times_\alpha^n \mathbb{N} \simeq \mathrm{C}_{\mathrm{env}}^*(A \times_\alpha \mathcal{T}_n^+)$.   □

Theorem 4.17. *Let (A, α) be an automorphic dynamical system. Then the C^*-envelope of $A \times_\alpha \mathcal{T}_n^+$ is the twisted crossed product $A \rtimes_\alpha \mathcal{O}_n$.*

Proof. By assumption $\alpha(1) = 1$, thus the C^*-algebra generated by the universal representation of $A \times_\alpha \mathcal{T}_n^+$ is $A \times_\alpha^n \mathbb{N}$. Then [39, Prop. 3.4] completes the proof.   □

Remark 4.18. Following the results for $n = 1$ of [22], and the previous Theorem 4.17, one might speculate that the C^*-envelope of $A \times_\alpha \mathcal{T}_n^+$ is (a corner of) the twisted crossed product $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ (at least when α is injective). The counterexample constructed in Subsection 4.32 shows that this is not true in general, even when α is unital.

Remark 4.19. Alternatively one could define the universal object associated to $\ell^1(A, \alpha, \mathbb{F}_n^+)$ relative to the class

$$\left\{ (\pi, \{T_i\}_{i=1}^n) \mid \pi : A \to \mathcal{B}(H), \, T_i \in \mathcal{T}_n^+, \, \pi(\alpha(x))T_i = T_i \pi(x) \right\}.$$

It is easy to check that the previous analysis applies to this case also.

4.20. **The semicrossed product $A \times_\alpha^t \mathcal{T}_n^+$.** We present the nonselfadjoint analogue of the twisted crossed product. In the next Subsection we give Exel systems (A, α, L) for which the class $\mathcal{F}^t$, that follows, is nonempty.

Definition 4.21. Let $\mathcal{F}^t$ be the subclass of

$$\left\{ (\pi, S, \{T_i\}) \mid S \text{ isometry}, \, S\pi(x) = \pi(\alpha(x))S, \, T_i \in \mathcal{T}_n^+ \right\}.$$

such that $(\pi, S, \{S_i\})$ integrates to a representation ρ of $\ell^1(A, \alpha, \mathbb{F}_n^+)$ with

$$\rho(x \otimes \delta_i) = \pi(x)S \otimes S_i, \text{ for all } x \in A, i = 1, \ldots, n.$$

We denote by $A \times_\alpha^t \mathcal{T}_n^+$ the universal operator algebra relative to the class $\mathcal{F}^t$, generated by $\pi(x) \otimes I$ and $S^{|\mu|} \otimes T_\mu$, for $x \in A$ and $\mu \in \mathbb{F}_n^+$.

Remark 4.22. We mention that the canonical map $A \times_\alpha \mathcal{T}_n^+ \to A \times_\alpha^t \mathcal{T}_n^+$ may not be a complete isometry, as we show within the counterexample of Subsection 4.32.

Theorem 4.23. *The C^*-envelope of the operator algebra $A \times_\alpha^t \mathcal{T}_n^+$ is the C^*-subalgebra $\mathrm{C}^*(B, \{U \otimes S_i\})$ of the twisted tensor product $\mathcal{M}(B) \rtimes_\beta \mathcal{O}_n$, where $B = (A/\mathcal{R}_\alpha)_\infty$ and $\beta = (\dot\alpha)_\infty$.*

Proof. It is easy to deduce that $A \times_\alpha^t \mathcal{T}_n^+$ is unital completely isometrically isomorphic to $A/\mathcal{R}_\alpha \rtimes_{\dot\alpha} \mathcal{T}_n^+$. Therefore we can always assume that the endomorphism α is injective.

Claim 1. The C^*-algebra $\mathrm{C}^*(A_\infty, \{U \otimes S_i\})$ is a C^*-cover for $A \times_\alpha^t \mathcal{T}_n^+$.
Proof of Claim 1. Let $(\pi, S, \{T_i\}) \in \mathcal{F}^t$ acting on $H_1 \otimes H_2$; then we can extend the pair (π, S) of (A, α) to a covariant unitary pair (Π, U) (acting on

an H_3) of the dynamical system $(A_\infty, \alpha_\infty)$ by [39, Prop. 2.3] and dilate $\{T_i\}$ to a Cuntz family $\{S_i\}$ separately. Then $(\Pi, U, \{S_i\})$ defines a representation of $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$. Conversely, if (Π, U) is a covariant unitary pair of $(A_\infty, \alpha_\infty)$ and $\{S_i\}$ a Cuntz family, then $(\Pi|_A, U, \{S_i\})$ defines a representation of $A \times_\alpha^t \mathcal{T}_n^+$. Hence, there is a canonical completely isometric embedding $A \times_\alpha^t \mathcal{T}_n^+ \hookrightarrow \mathcal{M}(A_\infty) \rtimes_{\alpha_\infty} \mathcal{O}_n$.

To end the proof of Claim 1, note that $\pi(A) \otimes I$ and $U \otimes S_i$ generate the twisted product. Indeed, it suffices to show that they generate $\pi(A_\infty) \otimes I$; this is proved by noting that

$$\pi[0, x, \alpha(x), \dots] \otimes I = (U^* \pi(x) U) \otimes I = (U^* \otimes S_i^*)(\pi(x) \otimes I)(U \otimes S_i),$$

for all $x \in A$, and then using induction, since $\pi(A_\infty) = \overline{\cup_n U^{-n} \pi(A) U^n}$.

Claim 2. The representations of $\mathcal{M}(A_\infty) \rtimes_{\alpha_\infty} \mathcal{O}_n$ are boundary for $A \times_\alpha^t \mathcal{T}_n^+$. *Proof of Claim 2.* Let $\rho \equiv (\pi, \{U \otimes S_i\})$ be a representation of $\mathcal{M}(A_\infty) \rtimes_{\alpha_\infty} \mathcal{O}_n$ acting on a Hilbert space H, and let ν be a dilation of the restriction of ρ to $A \times_\alpha^t \mathcal{T}_n^+$ acting on a Hilbert space K. Since $\rho|_A$ is a $*$-representation we get that H is $\nu(A)$-reducing. Also, $\{U \otimes S_i\}$ is a Cuntz family, hence $\mathrm{C}^*(U \otimes S_i)$ is isomorphic to $\mathcal{O}_n$. Therefore, ρ defines a unital completely isometric representation of $\mathcal{O}_n^+$ thus, as in Proposition 4.14, it is maximal. Hence H is reducing also for $\nu(\mathcal{O}_n^+)$. Therefore ν is a trivial dilation, thus ρ is maximal. $\qquad\square$

Remark 4.24. In view of the theory developed in [22] one could define the universal nonselfadjoint object relative to the class

$$\mathcal{F}_{\mathrm{un}}^t = \left\{ (\pi, U, \{T_i\}) \in \mathcal{F}^t \mid U \text{ unitary} \right\}.$$

The proof of Theorem 4.23 shows then that it is completely isometrically isomorphic to $A \times_\alpha^t \mathcal{T}_n^+$, thus they share the same C*-envelope. When α is injective, U can be considered alternatively a contraction [22].

If in addition α is assumed to be an automorphism they are unital completely isometrically isomorphic to $A \times_\alpha \mathcal{T}_n^+$. Indeed, the latter is true because $\mathrm{C}_{\mathrm{env}}^*(A \times_\alpha \mathcal{T}_n^+) = A \rtimes_\alpha \mathcal{O}_n = \mathrm{C}_{\mathrm{env}}^*(A \times_\alpha^t \mathcal{T}_n^+)$, via a C*-isomorphism that fixes the nonselfadjoint parts [39, proof of Prop. 3.4].

In the definition of $A \times_\alpha^t \mathcal{T}_n^+$ we imposed that it is generated by $\pi_u(A)$ and $\{S_u \otimes S_{u,i}\}$. This is not the algebra generated by $\pi_u(x) S_u \otimes S_{u,i}$, unless α is unital. Following the generalized C*-crossed products' theory, another way to impose this is the following.

Definition 4.25. Let $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$ be the universal operator algebra of $\ell^1(A, \alpha, \mathbb{F}_n^+)$ relative to the class

$$\mathcal{F}_{\mathrm{nd}}^t = \left\{ (\pi, S, \{T_i\}) \in \mathcal{F}^t \mid \pi \text{ is nondegenerate} \right\}.$$

As a consequence $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$ is generated by $\pi(x) \otimes I$ and $S^{|\mu|} \otimes T_\mu$, for $x \in A$ and $\mu \in \mathbb{F}_n^+$.

As in Proposition 3.2 $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$ is unital completely isometrically isomorphic to the subalgebra of $A \times_\alpha^t \mathcal{T}_n^+$ generated by $\pi_u(A) \otimes I$ and the family $\{S_u \pi_u(1) \otimes T_{u,i}\}$. Indeed, this is true for the C*-algebras generated by the universal representations, whose restriction implies the above identification. Therefore, if $(\pi, U, \{S_i\})$ is a faithful representation of $\mathcal{M}(B) \rtimes_\beta \mathcal{O}_n$ as in Theorem 4.23, then $(\pi, U\pi(1), \{S_i\})$ integrates to a maximal completely isometric representation of $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$.

Theorem 4.26. *The C*-envelope of $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$ is a full corner of $B \rtimes_\beta \mathcal{O}_n$, where $B = (A/\mathcal{R}_\alpha)_\infty$, $\beta = (\dot\alpha)_\infty$.*

Proof. For the first part, in view of the previous remarks (and the—by now—familiar trick), it suffices to show that, when α is injective, then $\mathrm{C}^*(\pi(A) \otimes I, \{U\pi(1) \otimes S_i\})$ coincides with $p(A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n)p$, where $p = \pi(1) \otimes I$.

Since $U\pi(1) \otimes S_i = p(U \otimes S_i)p$ we obtain that $\mathrm{C}^*(\pi(A) \otimes I, \{U\pi(1) \otimes S_i\})$ is a C*-subalgebra of $p(A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n)p$.

For the converse, first recall that $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ is generated by $\pi(A_\infty) \otimes I$ and $\pi(x)U \otimes S_i$ for $x \in A_\infty$ and $i = 1, \dots, n$. Hence the generators of $p(A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n)p$ are $p(\pi(A_\infty) \otimes I)p$ and

$$p(\pi(x)U \otimes S_i)p = (\pi(1)\pi(x)U\pi(1)) \otimes S_i = p(\pi(x) \otimes I)p \cdot (U\pi(1) \otimes S_i).$$

Due to that equation, it suffices to show that $p(\pi(A_\infty) \otimes I)p$ is in $\mathrm{C}^*(\pi_u(A) \otimes I, \{U\pi(1) \otimes S_i\})$, which follows as in the proof of Claim 1 of Theorem 4.23.

To prove that $p(A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n)p$ is a full corner, let an ideal $\mathcal{I}$ of $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ containing $p(A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n)p$. Then $\pi(A) \otimes I \subseteq \mathcal{I}$ and for an approximate unit (e_i) of A_∞ and $x \in A$ we obtain

$$\pi[0, x, \alpha(x), \dots] \otimes I = \lim_i (U^*\pi(e_i) \otimes S_i^*) \cdot (\pi(x) \otimes I) \cdot (\pi(e_i)U \otimes S_i) \in \mathcal{I}.$$

Inductively, we get that $\pi(A_\infty) \otimes I \subseteq \mathcal{I}$. Therefore,

$$\pi(x)U^{|\mu|} \otimes S_\mu = \lim_i \pi(x) \otimes I \cdot \pi(e_i)U^{|\mu|} \otimes S_\mu,$$

for every $x \in A_\infty$ and $\mu \in \mathbb{F}_n^+$. Thus the nonzero ideal $\mathcal{I}$ contains the generators of $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$, hence it is a trivial ideal. $\qquad\square$

Remark 4.27. The previous theorem asserts that a generalization of [39, Prop. 3.3] for $n > 1$ holds for the C*-envelope of $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$. This agrees also with Theorem 3.22 for $n = 1$.

As in the one variable case, it is immediate that $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+ = A \times_\alpha^t \mathcal{T}_n^+$ if and only if $\alpha(1_A) = 1_A$. In the following subsection we examine a class of nonunital systems, so that $A \times_\alpha^t \mathcal{T}_n^+$ and $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$ differ.

4.28. **Tensor products with Exel systems.** Let an Exel system (A, α, L) and X_n be the C*-correspondence of the Hawaiian earring graph on n edges, over $\mathbb{C}$. Let $\mathcal{X} = \mathcal{M}_L \otimes X_n$ be the exterior tensor product, which becomes a C*-correspondence over $A \otimes \mathbb{C} \simeq A$ in the obvious way, with $\phi_{\mathcal{X}} = \phi_{\mathcal{M}_L} \otimes \mathrm{id}$.

If (ρ, s) is a representation of X_n and (π, t) is a representation of $\mathcal{M}_L$ then $(\pi \otimes \rho, t \otimes s)$ defines a representation of $\mathcal{M}_L \otimes X_n$, which is injective when π

is injective. Moreover, when identifying $\mathcal{K}(\mathcal{X})$ with the spatial tensor product $\mathcal{K}(\mathcal{M}_L) \otimes \mathcal{K}(X_n)$ by the $*$-isomorphism j, with

$$j\left(\Theta_{y+\mathcal{N}_1, x+\mathcal{N}_1} \otimes \Theta_{\delta_i, \delta_j}\right) = \Theta_{(y+\mathcal{N}_1) \otimes \delta_i, (x+\mathcal{N}_1) \otimes \delta_j},$$

then $\psi_{t \otimes s} \circ j = \psi_t \otimes \psi_s$. Thus, if (π, S) is a representation of $\mathcal{M}_L$ and $\{T_i\}$ is a Toeplitz–Cuntz family then $(\hat{\pi}, \{\hat{T}_i\})$ defines a representation of $\mathcal{X}$, where $\hat{\pi} = \pi \otimes I$ and $\hat{T}_i = S \otimes T_i$. Hence,

$$\begin{aligned}
\hat{T}_i \hat{\pi}(x) &= (S \otimes T_i)(\pi(x) \otimes I) = (S\pi(x)) \otimes T_i \\
&= (\pi(\alpha(x))S) \otimes T_i = (\pi(\alpha(x)) \otimes I)(S \otimes T_i) = \hat{\pi}(\alpha(x))\hat{T}_i.
\end{aligned}$$

Conversely, if $(\hat{\pi}, t)$ is a representation of $\mathcal{X}$, let $\hat{T}_i = t((1 + \mathcal{N}_1) \otimes \delta_i)$. Then $\hat{T}_i \hat{\pi}(x) = \hat{\pi}(\alpha(x))\hat{T}_i$, for all $i = 1, \dots, n$ and $x \in A$.

Lemma 4.29. *With the above notations $J_{\mathcal{X}} = J_{\mathcal{M}_L}$, where we identify $A \otimes \mathbb{C}$ with A.*

Proof. It is immediate that $\ker \phi_{\mathcal{X}} = \ker \phi_{\mathcal{M}_L}$. To show that $\phi^{-1}(\mathcal{K}(\mathcal{X})) = \phi_{\mathcal{M}_L}^{-1}(\mathcal{K}(\mathcal{M}_L))$, first let $x \in A$, such that $\phi_{\mathcal{M}_L}(x) = k \in \mathcal{K}(\mathcal{M}_L)$. Then

$$\phi_{\mathcal{X}}(x) = \phi_{\mathcal{M}_L}(x) \otimes \mathrm{id} = k \otimes \left(\sum_{i=1}^n \Theta_{\delta_i, \delta_i}^{X_n}\right) \in \mathcal{K}(\mathcal{M}_L) \otimes \mathcal{K}(X_n) = \mathcal{K}(\mathcal{X}),$$

for the spatial tensor product [32, Chap. 4]. For $x \in A$ such that $\phi_{\mathcal{X}}(x) = \phi_{\mathcal{M}_L}(x) \otimes \mathrm{id} \in \mathcal{K}(\mathcal{M}_L) \otimes \mathcal{K}(X_n)$, let (e_i) be a c.a.i. of $\mathcal{K}(\mathcal{M}_L)$. Then $(e_i \otimes \mathrm{id})$ is a c.a.i. for $\mathcal{K}(\mathcal{M}_L) \otimes \mathcal{K}(X_n)$. Since,

$$\begin{aligned}
\lim_i \|e_i \phi_{\mathcal{M}_L}(x) - \phi_{\mathcal{M}_L}(x)\| &= \lim_i \|(e_i \phi_{\mathcal{M}_L}(x) - \phi_{\mathcal{M}_L}(x)) \otimes \mathrm{id}\| \\
&= \lim_i \|e_i \phi_{\mathcal{M}_L}(x) \otimes \mathrm{id} - \phi_{\mathcal{M}_L}(x) \otimes \mathrm{id}\| \\
&= \lim_i \|(e_i \otimes \mathrm{id})\phi_{\mathcal{X}}(x) - \phi_{\mathcal{X}}(x)\| = 0,
\end{aligned}$$

we obtain that $\phi_{\mathcal{M}_L}(x) \in \mathcal{K}(\mathcal{X})$, and the proof is complete. $\square$

Consequently, if (π, t) is a covariant representation of $\mathcal{M}_L$ and (ρ, s) is a covariant representation for X_n, i.e., $\{s(\delta_i)\}$ is a Cuntz–Krieger family, then $(\pi \otimes \rho, t \otimes s)$ is a covariant representation of $\mathcal{X}$. Indeed, let $x \in J_{\mathcal{X}}$; then $x \in J_{\mathcal{M}_L}$ by Lemma 4.29. Thus

$$\begin{aligned}
(\pi \otimes \rho)(x) &= \pi(x) \otimes I = \psi_t\left(\phi_{\mathcal{M}_L}(x)\right) \otimes \psi_s\left(\sum_{i=1}^n \Theta_{\delta_i, \delta_i}^{X_n}\right) \\
&= \psi_{t \otimes s} \circ j\left(\phi_{\mathcal{M}_L}(x) \otimes \mathrm{id}\right) = \psi_{t \otimes s}\left(\phi_{\mathcal{X}}(x)\right).
\end{aligned}$$

From now on the Exel system (A, α, L) will be assumed that satisfies the conditions of Subsection 3.12, i.e.,

$$L(1) = 1 \text{ and } \alpha \circ L(x) = \alpha(1)x\alpha(1), \text{ for all } x \in A.$$

Let (π_u, T_u) be the universal representation of $\mathfrak{A}(A, \alpha)$. By Proposition 3.2 and Theorem 3.16,

$$\mathrm{C}^*(\pi_u(A), T_u\pi_u(1)) \simeq A \times_\alpha^1 \mathbb{N} \simeq A \rtimes_{\alpha, L} \mathbb{N} \simeq \mathcal{O}_{\mathcal{M}_L}$$

via a canonical mapping, and the pair $(\pi_u, T_u\pi_u(1))$ defines a faithful covariant representation of $\mathcal{M}_L$ that admits a gauge action $\{\beta_z\}$. Thus, if $\{S_i\}$ is a Cuntz family, then $(\pi_u \otimes \mathrm{id}, \{(T_u\pi_u(1)) \otimes S_i\})$ defines a covariant representation of $\mathcal{X}$ that inherits the gauge action $\{\beta_z \otimes I\}$ and is faithful on A. Therefore by the gauge invariance theorem [27], the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{X}}$ is $*$-isomorphic to the C*-algebra generated by $(\pi_u \otimes \mathrm{id}, \{(T_u\pi_u(1)) \otimes S_i\})$. Note that it is the C*-subalgebra $\mathrm{C}^*(A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+)$ of $\mathrm{C}^*(A \times_\alpha^t \mathcal{T}_n^+)$.

The next result follows readily from the previous analysis and the results we established in Subsection 4.20.

Theorem 4.30. *Let an Exel system that satisfies* (†) *and* $L(1) = 1$. *Then:*

   (1) $\mathcal{T}_{\mathcal{X}} \simeq A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$;
   (2) *The* C*-envelope $\mathcal{O}_{\mathcal{X}}$ *of* $\mathcal{T}_{\mathcal{X}}^+$ *is a full corner of* $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$;
   (3) *If* $(\pi, U, \{S_i\})$ *integrates to an injective representation of* $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ *and* $p = \pi(1) \otimes I$, *then* $p(A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n)p$ *is the closed linear span of the monomials* $(\pi(x)U^{|\mu|-|\nu|}\pi(1)) \otimes S_\mu S_\nu^*$, *when* $|\mu| \geq |\nu|$, *and* $(\pi(1)U^{|\mu|-|\nu|}\pi(x)) \otimes S_\mu S_\nu^*$, *when* $|\mu| \leq |\nu|$, *for* $x \in A$.

Proof. It suffices to comment on item (3). Its proof follows in the same way as the second part of Theorem 3.23, as explained in Remark 3.24. (Recall that $p(A_\infty)p = A$ in this case.) $\qquad\square$

Remark 4.31. We mention that in the representation theory of $A \times_\alpha^t \mathcal{T}_n^+$, $A_{\mathrm{nd}} \times_\alpha^t \mathcal{T}_n^+$ and $\mathcal{T}_{\mathcal{X}}$, the commutant of $\mathcal{O}_n$ is nontrivial. This provided an isometry S that commutes with the isometries S_i, hence we could extend each (π, S) to a unitary pair (Π, U) without losing control on the isometries S_i. The counterexample that follows shows why this step is crucial in order to prove a connection with the twisted tensor product of $(A_\infty, \alpha_\infty)$.

4.32. **Counterexample.** Let $A = \mathcal{O}_n$ acting faithfully on some H and let us denote by S_i its generators. Define the $*$-endomorphism α of $\mathcal{O}_n$ by

$$\alpha(x) = \sum_{i=1}^n S_i x S_i^*, \text{ for all } x \in A.$$

This dynamical system is unital and injective. Therefore $A \times_\alpha^n \mathbb{N}$ is exactly $\mathrm{C}^*(\pi_u(A), \{S_{u,i}\})$ of Proposition 4.3.

First we show that the canonical $*$-epimorphism $q : A \times_\alpha^n \mathbb{N} \to A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ is not injective (cp. Remark 4.4). To this end, fix a family $(\sigma \otimes I, \{U \otimes T_i\})$ that integrates to a faithful representation of $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$. Then the representation implies a number of new relations; for example

$$(U \otimes T_1)^*(\sigma(x) \otimes I)(U \otimes T_2) = (U^*\sigma(x)U) \otimes (T_1^*T_2) = 0, \text{ for all } x \in A.$$

Thus, if q were a $*$-isomorphism, then for all families $(\pi, \{\hat{T}_i\})$ of $A \times_\alpha^n \mathbb{N}$ we would have that $\hat{T}_1^* \pi(x) \hat{T}_2 = 0$. In particular this should be true for the family $(\mathrm{id}, \{S_i\})$. But

$$S_1^* \mathrm{id}(S_1) S_2 = S_1^* S_1 S_2 = S_2 \neq 0.$$

In turn the representation that integrates $(\mathrm{id}, \{S_i\})$ is also not faithful. If it were, then $A \times_\alpha^n \mathbb{N} \simeq \mathrm{C}^*(\mathrm{id}, \{S_i\})$, and the latter is $\mathcal{O}_n$ since it contains and it is contained in $\mathcal{O}_n$. Thus, the $*$-epimorphism q above should be faithful, being a representation of the simple C*-algebra $\mathcal{O}_n$, which leads to a contradiction.

By Proposition 4.14 every representation of $A \times_\alpha \mathcal{T}_n^+$ is maximal. Therefore the C*-algebra $A \times_\alpha^n \mathbb{N}$ generated by the universal representation of $A \times_\alpha \mathcal{T}_n^+$ is $\mathrm{C}_{\mathrm{env}}^*(A \times_\alpha \mathcal{T}_n^+) \simeq A \times_\alpha^n \mathbb{N}$.

Thus the canonical embedding $A \times_\alpha \mathcal{T}_n^+ \to A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ is completely contractive, but not completely isometric. Otherwise, by Proposition 4.14 it would extend to a $*$-isomorphism $A \times_\alpha^n \mathbb{N} \to A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$, which is a contradiction. This settles Remark 4.18.

Finally, referring to Remark 4.22, if $A \times_\alpha^t \mathcal{T}_n^+$ were completely isometrically isomorphic to $A \times_\alpha \mathcal{T}_n^+$ via the canonical embedding, then they would have $A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ as the same C*-envelope. Therefore $A \times_\alpha^n \mathbb{N} \simeq A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$, which again is a contradiction.

Another consequence of this counterexample is described in the following Remark.

Remark 4.33. For (A, α) as above, consider $\ell^1(A, \alpha, \mathbb{F}_n^+)$ as a $\|\cdot\|_1$-dense subalgebra of $A \times_\alpha \mathcal{T}_n^+$. Then the homomorphism $\rho: \ell^1(A, \alpha, \mathbb{F}_n^+) \to A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ is completely contractive. Moreover, by Example 4.8 it admits a Fourier transform, which implies that ρ is also injective.

However, the homomorphism $\rho: \left(\ell^1(A, \alpha, \mathbb{F}_n^+), \|\cdot\|_1\right) \to A_\infty \rtimes_{\alpha_\infty} \mathcal{O}_n$ is not completely isometric. If it were, then it would extend to a completely isometry to its $\|\cdot\|_1$-closure $A \times_\alpha \mathcal{T}_n^+$, and we have already argued that this leads to a contradiction.

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References

[1] W. B. Arveson, Subalgebras of C*-algebras, Acta Math. **123** (1969), 141–224. MR0253059 (40 #6274)

[2] W. B. Arveson, *Notes on the unique extension property*, 2006, `http://math.berkeley.edu/~arveson/Dvi/unExt.pdf`

[3] W. B. Arveson, The noncommutative Choquet boundary, J. Amer. Math. Soc. **21** (2008), no. 4, 1065–1084. MR2425180 (2009g:46108)

[4] N. Brownlowe and I. Raeburn, Exel's crossed product and relative Cuntz-Pimsner algebras, Math. Proc. Cambridge Philos. Soc. **141** (2006), no. 3, 497–508. MR2281412 (2007i:46067)

[5] N. Brownlowe, I. Raeburn and S. T. Vittadello, Exel's crossed product for non-unital C*-algebras, Math. Proc. Cambridge Philos. Soc. **149** (2010), no. 3, 423–444. MR2726727 (2011k:46092)

[6] B. Blackadar, Shape theory for C*-algebras, Math. Scand. **56** (1985), no. 2, 249–275. MR0813640 (87b:46074)

[7] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach*, London Mathematical Society Monographs. New Series, 30, Oxford Univ. Press, Oxford, 2004. MR2111973 (2006a:46070)

[8] D. P. Blecher, P. S. Muhly, and V. I. Paulsen, Categories of operator modules (Morita equivalence and projective modules), Mem. Amer. Math. Soc. **143** (2000), no. 681, viii+94 pp. MR1645699 (2000j:46132)

[9] J. Cuntz, Simple C*-algebras generated by isometries, Comm. Math. Phys. **57** (1977), no. 2, 173–185. MR0467330 (57 #7189)

[10] J. Cuntz, *K*-theory for certain C*-algebras. II, J. Operator Theory **5** (1981), no. 1, 101–108. MR0613050 (84k:46053)

[11] D. Courtney, P. S. Muhly and S. W. Schmidt, Composition operators and endomorphisms, Complex Analysis and Operator Theory **6** (2012), no. 1, 163–188.

[12] K. R. Davidson and E. G. Katsoulis, Operator algebras for multivariable dynamics, Mem. Amer. Math. Soc. **209** (2011), no. 982, viii+53 pp. MR2752983 (2012c:47194)

[13] M. A. Dritschel and S. A. McCullough, Boundary representations for families of representations of operator algebras and spaces, J. Operator Theory **53** (2005), no. 1, 159–167. MR2132691 (2006a:47095)

[14] B. L. Duncan, C*-envelopes of universal free products and semicrossed products for multivariable dynamics, Indiana Univ. Math. J. **57** (2008), no. 4, 1781–1788. MR2440881 (2011d:46121)

[15] B. L. Duncan and J. R. Peters, Operator algebras and representations from commuting semigroup actions. arXiv:1008.2244 [math.OA] (2011).

[16] R. Exel, A new look at the crossed-product of a C*-algebra by an endomorphism, Ergodic Theory Dynam. Systems **23** (2003), no. 6, 1733–1750. MR2032486 (2004k:46119)

[17] N. J. Fowler and I. Raeburn, The Toeplitz algebra of a Hilbert bimodule, Indiana Univ. Math. J. **48** (1999), no. 1, 155–181. MR1722197 (2001b:46093)

[18] A. H. Fuller, Nonself-adjoint Semicrossed Products by Abelian Semigroups, Canad. J. Math. **65** (2013), no. 4, 768–782. MR3071079

[19] M. Hamana, Injective envelopes of operator systems, Publ. Res. Inst. Math. Sci. **15** (1979), no. 3, 773–785. MR0566081 (81h:46071)

[20] A. an Huef and I. Raeburn, Stacey crossed products associated to Exel systems, Integral Equations Operator Theory **72** (2012), no. 4, 537–561. MR2904610

[21] T. Kajiwara, C. Pinzari, and Y. Watatani, Ideal structure and simplicity of the C*-algebras generated by Hilbert bimodules, J. Funct. Anal. **159** (1998), no. 2, 295–322. MR1658088 (2000a:46094)

[22] E. T. A. Kakariadis, Semicrossed products of C*-algebras and their C*-envelopes. arXiv:1102.2252 [math.OA] (2012). To appear in Journal d'Analyse Mathématique.

[23] E. T. A. Kakariadis, The Šilov boundary for operator spaces, Integral Equations Operator Theory **76** (2013), no. 1, 25–38. MR3041719

[24] E. T. A. Kakariadis and E. G. Katsoulis, Contributions to the theory of C*-correspondences with applications to multivariable dynamics, Trans. Amer. Math. Soc. **364** (2012), no. 12, 6605–6630. MR2958949

[25] E. G. Katsoulis and D. W. Kribs, Tensor algebras of C*-correspondences and their C*-envelopes, J. Funct. Anal. **234** (2006), no. 1, 226–233. MR2214146 (2007a:46061)

[26] T. Katsura, A construction of C*-algebras from C*-correspondences, Contemp. Math. **335** (2003), 173–182. MR2029622 (2005k:46131)

[27] T. Katsura, On C*-algebras associated with C*-correspondences, J. Funct. Anal. **217** (2004), no. 2, 366–401. MR2102572 (2005e:46099)

[28] A. Kavruk, V. I. Paulsen, I. G. Todorov, and M. Tomforde, Tensor products of operator systems, J. Funct. Anal. **261** (2011), no. 2, 267–299. MR2793115 (2012h:46094)

[29] C. Kleski, Boundary representations and pure completely positive maps. To appear in J. Operator Theory. arXiv:1110.4149 [math.OA] (2011).

[30] M. Laca, Endomorphisms of $\mathcal{B}(\mathcal{H})$ and Cuntz algebras, J. Operator Theory **30** (1993), no. 1, 85–108. MR1302609 (95m:46100)

[31] M. Laca, From endomorphisms to automorphisms and back: dilations and full corners, J. London Math. Soc. (2) **61** (2000), no. 3, 893–904. MR1766113 (2002a:46094)

[32] E. C. Lance, *Hilbert C*-modules*, London Mathematical Society Lecture Note Series, 210, Cambridge Univ. Press, Cambridge, 1995. MR1325694 (96k:46100)

[33] N. S. Larsen, Crossed products by abelian semigroups via transfer operators, Ergodic Theory Dynam. Systems **30** (2010), no. 4, 1147–1164. MR2669415 (2011j:46114)

[34] X. Li, Semigroup C*-algebras and amenability of semigroups, J. Funct. Anal. **262** (2012), no. 10, 4302–4340. MR2900468

[35] P. S. Muhly and B. Solel, Tensor algebras over C*-correspondences: representations, dilations, and C*-envelopes, J. Funct. Anal. **158** (1998), no. 2, 389–457. MR1648483 (99j:46066)

[36] G. J. Murphy, Simplicity of crossed products by endomorphisms, Integral Equations Operator Theory **42** (2002), no. 1, 90–98. MR1866877 (2003b:46083)

[37] T. Oikhberg and G. Pisier, The "maximal" tensor product of operator spaces, Proc. Edinburgh Math. Soc. (2) **42** (1999), no. 2, 267–284. MR1697398 (2000f:46076)

[38] J. R. Peters, Semicrossed products of C*-algebras, J. Funct. Anal. **59** (1984), no. 3, 498–534. MR0769379 (86e:46063)

[39] P. J. Stacey, Crossed products of C*-algebras by ∗-endomorphisms, J. Austral. Math. Soc. Ser. A **54** (1993), no. 2, 204–212. MR1200792 (94a:46077)

Evgenios T.A. Kakariadis
Pure Mathematics Department, University of Waterloo,
Ontario N2L3G1, Canada
E-mail: ekakaria@uwaterloo.ca
URL: http://users.uoa.gr/~mavro

Justin R. Peters
Department of Mathematics, Iowa State University,
Ames, Iowa, USA
E-mail: peters@iastate.edu