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Optimal Consumption Using a Continuous Income Model

Diplomarbeit

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Contents

1 Introduction

We face many optimization problems every day. Examples are finding the fastest way to get to work, often in combination with the transportation costs, determining the optimal time/price to buy or sell an option or any other derivative, or finding the optimal amount of consumption to gain an otpimal lifetime utility. All this problems can be well modeled by various systems of PDEs. In doing so at some problems arise an uncertainty like variable fuel costs or volatilities in option values. These terms are modeled by a stochastic component leading to a system of SDEs for the problem. But wether the PDE or the SDE case it is common to derive a so-called Hamilton-Jacobi-Bellman equation (HJB) in order to solve the optimization problem.
The first approach is to split up the problem into many little problems. This is done by the dynamic programming principle, that states that every optimal control on a time interval already is an optimal control on every part of the interval. With this principle and a mathematical tool for stochastic calculus, Ito's formula, we can derive a HJB for the optimization problems in order to solve them.
A problem that arises with this approach lies in the nature of the solutions of the HJB. A solution of a HJB is not a priori unique, but we want to find a solution that tells us the (unique) optimal control for our problem. This is handled with a special kind of solution, the viscosity solution. Viscosity solutions of Hamilton-Jacobi-Bellman equations exist and are unique.

In this thesis we face an additional problem. We try to optimize the consumption to gain an optimal lifetime utility. We do not only face a stochastic uncertainty in the income, but also make the (stochastic) target constraint that the financial wealth at the end of the life has to be non-negative. So after explaining the details of optimal control problems and their HJB in chapter 2, we need an adapted version of the dynamic programming principle to derive a HJB for the control problem, the geometric dynamic programming principle. This was introduced by Soner and Touzi [ST02]. To state the problem we first present a continuous (lifetime) income model with a stochastic uncertainty developed by [ST11] in chapter 3. In chapter 4 we develope the optimal control problem based on this income model and derive its associated HJB formulation. Then we use a finite differencing scheme in chapter 5 to solve the HJB numerically. We show convergence to the value function of our control problem and solve a second HJB arising in the problem formulation, telling us the minimal asset value to find an admissible control in dependence of the current income and time. Finally we present some results of the numerical simulation using the parameters of [ST11] in chapter 6.

2 Preliminaries

2.1 Notations

We start with some notations used throughout this thesis. Let $n \in \mathbb{N}$, then a vector $a \in \mathbb{R}^n$ is supposed to be a column vector. We are using the Euclidean norm $\| \cdot \|$ (if not stated otherwise) and $*$ is denoting the scalar product. The transposition of a is denoted by a^t and S^n denotes the set of all symmetric $n \times n$ matrices with real values.

For any smooth function $\phi : [t,T] \times \mathbb{R}^n \to \mathbb{R}$ (for $t \leq T$), we denote the derivative with respect to the first variable by $\partial_t \phi$, while the Jacobian resp. Hessian matrix (with respect to the spatial variables) are denoted by $D\phi$ resp. $D^2\phi$.

For a set $U \subset \mathbb{R}^n$, $int(U), cl(U), \partial(U)$ denote the interior, closure and boundary of U. For a domain $D = [t,T] \times U$, $\partial_p D$ means the parabolic boundary of D, with $\partial_p D := ([t,T) \times \partial U) \cup (\{T\} \times cl(U))$, whereas $int_p D$ denotes the parabolic interior of D, $int_p D := [t,T) \times int(U)$.

2.2 Stochastic Optimal Control

Before we start with modelling our problem we will define the basic structures of a stochastic optimal control problem, based on the remarks of [Pha09] and [Neu09].

A stochastic optimal control problem consits of 3 main components:

1. **State**
2. **Control**
3. **Performance functional**

As we will use a model with stochastic uncertainty in income, we consider a dynamic system in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The state of the system at time t is a set of random variables $X_t(\omega) : \Omega \to \mathbb{R}^n$ used to specify the problem. The current state of the system depends on time t and the event ω and is ought to follow some dynamics. So we have to describe the mapping $X(\omega)(t) : t \mapsto X_t(\omega)$, which is done by a stochastic differential equation.

To get influence on the dynamics we need a so called control, a process $\nu = (\nu_t)_{t \in \mathbb{T}}, \nu_t \in U$ (typically $U \subset \mathbb{R}^d$ and $\mathbb{T} = [t_0, T]$ for some $0 \leq t_0 \leq T$ is the used time interval for our problem) connected with the state variables through the stochastic differential equations. So our system depends on a so called state equation

(we neglect the dependency on ω)

$$dX(t) = \mu(X(t), \nu(t))dt + \sigma(X(t), \nu(t))dW_t$$
$$X(t_0) = x_0$$

with $X(t) = X_t$, $\nu(t) = \nu_t$, $\mu : \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times U \to \mathbb{R}^{n+d}$ and W being a d-dimensional Brownian motion with independent components W_i, W_j for $i \neq j$. x_0 is the initial condition at starting time $t_0 \in \mathbb{T}$.
We choose our control out of a set of admissible controls $\mathcal{U}$ containing all permitted U-valued controls, which will be constrained in our problem.

$$\mathcal{U} := \{\nu : \mathbb{T} \times \Omega \to U | \nu \text{ is measurable, adapted to } W \text{ and essentially bounded}\}$$

To know how to influence the system we need the performance functional, providing us information on the value of the state of our system. The main task in our problem is to find a control ν^* that maximizes (resp. minimizes) a performance (resp. cost) criterion. We therefore maximize (or minimze) a functional $J(X, \nu)$ over all $\nu \in \mathcal{U}$ in the time interval $\mathbb{T}$. Since the state is stochastic, we can only maximize over an expected value, so our funtional is supposed to take the form (in a finite time horizon case):

$$J(t_0, X, \nu) = \mathbb{E}[\int_{t_0}^{T} f(X_t, \omega, \nu_t)dt + g(X_T, \omega)]$$

where the right hand side of the term is our perfomance criterion. Here $t_0 \in \mathbb{T}$ is the starting time of the system $T \in \mathbb{T}$ the stopping time, $X = (X_t)_{t \in \mathbb{T}}$, f denotes a performance function in time t and g denotes the terminal performance. For the infinite horizon case g disappears and f is typically discounted by a factor $e^{-\beta t}$, to ensure that the integral converges. Discounting is also often used in finite time settings, but we concentrate on the case where $\beta = 0$. For other situations, like variable stopping time or combined ones, see e.g. [Pha09].
Now our main task is to find the maximizing (minimizing) control $\nu^* \in \mathcal{U}$, so that our performance functional $J(t_0, X, \nu^*)$ fits the corresponding value function

$$V(t_0, X) = \sup_{\nu \in \mathcal{U}} J(t_0, X, \nu)$$

resp. (minimizing case)

$$V(t_0, X) = \inf_{\nu \in \mathcal{U}} J(t_0, X, \nu)$$

2.3 Viscosity solution

To provide a (numerical) solution for our later developed value function, a Hamilton-Jacobi-Bellman equation (HJB) it is required to have a generalized solution terminology for such a set of equations. It can be shown that the value function is a solution of a corresponding HJB. Since our solution may not be differentiable in the classical sense we use the approach of viscosity solutions, introduced in [CL83], which also has the advantage that viscosity solutions of Hamilton-Jacobi-Bellmann equations exist and are unique. First we define the HJB equation, then we will show how a viscosity solution of that equation is defined.

A heuristical derivation of the HJB will be given for the second order equation. The derivation of the first order case is quite similar, but easier with using a Taylor expansion instead of handling expected values and using Ito's formula. Let us just define the first order HJB and remain the derivation for the stochastic case.

Definition 2.1. ***(first order) Hamilton-Jacobi-Bellman equation****.*
For a finite $T > 0$ and $0 \leq t \leq T$ let us assume a optimal control problem

$$V(t, x_0) = \max_{c \in U} \{ \int_t^T h(x(s), c(s)) ds + g(x(T)) \} \tag{1}$$

with the (multidimensional) state equation $\frac{dx(s)}{ds} = f(x(s), c(s))$ for $t < s < T$ and $x(t) = x_0$ as starting vector. Then the associated Hamilton-Jacobi-Bellman equation is

$$u_t(t, x_0) - H(x_0, u, Du(t, x_0)) = 0 \tag{2}$$

for $t < T$ with the final time condition $u(T, x_0) = g(x(T))$ and the Hamiltonian

$$H(x, u, p) = \inf_{c \in U} \{ -f(x, c) * p - h(x, c) \} \tag{3}$$

and boundary condition $u(T, x_0) = g(x(T))$.

This nonlinear partial differential equation is solved by $V(t, x_0)$ (at least in the weak solution terminology of the viscosity solution). So we try to solve the Hamilton-Jacobi-Bellman equation associated to the control problem to find a solution for (1). Since it is not clear that the HJB has a solution (maybe not in $C^1([0, T] \times \mathbb{R}^n)$) and possible solutions are not necessarily unique, we somehow have to assure to find the right solution for our problem, solving the Hamilton-Jacobi-Bellman equation. As mentioned above this is assured by the so called *viscosity solution*. So now we define the terminology *viscosity solution* as developed in [CL83]. First we define a weak sense of differentiation and with this we can construct the new solution terminology.

Definition 2.2. ***Subdifferential***.
Be $t, T, n \in \mathbb{R}$ *with* $t \leq T$. *For a function* $u(t,x) : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}, D^- u(t,x)$ *denotes the subdifferential of* u *in* (t,x), *i.e.*

$$D^- u(t, \hat{x}) = \{p \in \mathbb{R}^n : \liminf_{x \to \hat{x}} \frac{u(t,x) - u(t,\hat{x}) - p * (x - \hat{x})}{\|x - \hat{x}\|} \geq 0\}$$

Definition 2.3. ***Superdifferential***.
Be t, T, n, u *like above. Then* $D^+ u(t,x)$ *denotes the superdifferential of* u *in* (t,x), *i.e.*

$$D^+ u(t, \hat{x}) = \{p \in \mathbb{R}^n : \limsup_{x \to \hat{x}} \frac{u(t,x) - u(t,\hat{x}) - p * (x - \hat{x})}{\|x - \hat{x}\|} \leq 0\}$$

Now we are able to define first order viscosity (super-/sup-)solutions:

Definition 2.4. ***(first order) Viscosity supersolution***.
A (lower semi-)continuous function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is called a viscosity supersolution of* (2) *if*

$$u_t(t,x) - H(x, u, p) \geq 0 \text{ for every } (t,x) \in [0,T] \times \mathbb{R}^n \text{ and } p \in D^- u(t,x) \tag{4}$$

Definition 2.5. ***(first order) Viscosity subsolution***.
An (upper semi-)continuous function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is called a viscosity subsolution of* (2) *if*

$$u_t(t,x) - H(x, u, p) \leq 0 \text{ for every } (t,x) \in [0,T] \times \mathbb{R}^n \text{ and } p \in D^+ u(t,x) \tag{5}$$

Definition 2.6. ***(first order) Viscosity solution***.
A continuous function $u \in C([0,T] \times \mathbb{R}^n)$ *is called a viscosity solution of* (2) *if* u *is a viscosity subsolution and viscosity supersolution of* (2).

Note that a function $u \in C^1([0,T] \times \mathbb{R}^n, \mathbb{R})$ that is differentiable in the classical sense and satisfies $u_t(t,x) - H(x, Du(t,x)) = 0$ for every $(t,x) \in [0,T] \times \mathbb{R}^n$ also provides a viscosity solution, while on the other hand, a viscosity solution doesn't have to be differentiable (that is why we introduced this kind of solution terminology). But if the viscosity solution u is differentiable in a point $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$, then it satisfies

$$u_t(t_0, x_0) - H(x_0, u, Du(t_0, x_0)) = 0$$

in the classical sense (see [Gru04], Theorem 4.6 for the time-invariant case).

2.4 Second order viscosity solution

Since we will use a model based on a income equation with a stochastic uncertainty, we need to introduce the upper terminology for second order PDEs. In the stochastic case we will derive a second order Hamilton-Jacobi-Bellman equation. Since we face the same problems (a solution is not necessarily twice differentiable, or even not differentiable in the classical sense in all points, possible solutions are not necessary unique) as in the deterministic case, we want to use the idea of the viscositiy solution as well. So we define a second order analogon for the upper terminology (following the definitions of [Ish93] and [CIL92]).

Definition 2.7. ***(second order) Hamilton-Jacobi-Bellman equation***.
For a finite $T > 0$ let us assume a stochastic optimal control problem

$$V(t, x_0) = \max_{c \in U} \{\mathbb{E}[\int_t^T f(x(s), c(s))ds + g(x(T))]\} \tag{6}$$

with the (multidimensional) stochastic state equation $dx(s) = \alpha(x(s), c(s))ds + \beta(x(s), c(s))dW_s$ for some d-dimensional Brownian motion W and $x(t) = x_0$ as starting vector. Then the associated Hamilton-Jacobi-Bellman equation for $t < T$ is

$$-u_t(t, x_0) + H(x_0, Du(t, x_0), D^2u(t, x_0)) = 0 \tag{7}$$

with final time condition $u(T, x_0) = g(x(T))$ and the Hamiltonian

$$H(x, q, A) = \inf_{c \in U} \{-f(x, c) - \alpha(x, c) * q - \frac{1}{2} Trace[(\beta\beta^t)(x, c)A]\}. \tag{8}$$

As stated above we will now give a heuristic derivation of this equation. We therefore use the so following dynamic programming principle, stating that a control problem solved by an optimal control c at starting time t will be solved by the control $\hat{c}$ at starting time $t' > t$, where $\hat{c} = c_{|[t',T]}$. We make a formal definition following [Koh03]:

Definition 2.8. ***Dynamic programming principle***.
Suppose the optimal control problem defined above. Then it follows
$V(t, x_0) = \max_{c \in U} \{\mathbb{E}[\int_t^{t'} f(x(s), c(s))ds + V(t', x(t'))]\}$

The second tool we need for the derivation of the HJB is Ito's formula, a formula helping us to determine the derivation of a stochastic function.

Definition 2.9. ***Ito's formula***.
Be the stochatic variable $x(s)$ described by the formula $dx(s) = \alpha(x(s), c(s))ds + \beta(x(s), c(s))dW_s$ for some Brownian motion W as above. Be $\phi(x(s), s) \in C^{1,2}$. Then

$$d(\phi(x(s), s)) = [\phi_s + \phi_x\alpha + \frac{\beta^2}{2}\phi_{xx}]ds + \beta\phi_x dW.$$

We will now finally give a heuristical derivation of the HJB following [Koh03] and [Neu09]. Suppose the optimal control problem (6). Then the dynamic programming principle gives us:

$$V(t, x_0) = \max_{c\in U}\{\mathbb{E}[\int_t^{t'} f(x(s), c(s))ds + V(t', x(t'))]\}$$

Now using $t' = t + \Delta t$ and letting $\Delta t \to 0$ we can approximate the integral by $f(x_0, \overline{c})\Delta t$, where $\overline{c}$ is the optimal value for $c \in U$. So we get

$$V(t, x_0) \approx \max_{c\in U}\{f(x_0, \overline{c})\Delta t + \mathbb{E}[V(t + \Delta t, x(t + \Delta t))]\} \tag{9}$$

To determine the expected value, we use Ito's formula to derive

$$V(t + \Delta t, x(t + \Delta t)) - V(t, x(t)) = \int_t^{t+\Delta t} V_t + \alpha\nabla V + \frac{1}{2}\beta^2\Delta V d\tau + \int_t^{t+\Delta t} \beta\nabla V dW$$

Taking expected values we get (since the expected value of the dW integral vanishes)

$$\begin{aligned} & \mathbb{E}[V(x(t + \Delta t), t + \Delta t) - V(t, x(t))] \\ = \; & \mathbb{E}[V(x(t + \Delta t), t + \Delta t)] - V(x_0, t) \\ = \; & \mathbb{E}[\int_t^{t+\Delta t} V_t + \alpha\nabla V + \frac{1}{2}\beta^2\Delta V d\tau] \\ \approx \; & \Delta t[V_t + \alpha\nabla V + \frac{1}{2}\beta^2\Delta V] \end{aligned}$$

Together with (9) this leads to

$$V(t, x_0) = \max_{c\in U}\{f(x_0, \overline{c})\Delta t + \Delta t[V_t + \alpha\nabla V + \frac{1}{2}\beta^2\Delta V] + V(t, x_0)\}$$

Since the maximum is independent of V_t and $V(t, x_0)$, taking the limit $\Delta\tau \to 0$ and dividing the equation by $\Delta\tau$ we get the Hamilton-Jacobi-Belmann equation we stated above.

Now we define the analogon of sub- and superdifferential we use to state the second order viscosity solution terminology, call sub- and superjet.

Definition 2.10. ***Subjet.***
Be $t, T, n \in \mathbb{R}$ with $t \leq T$. For a function $u(t,x) : [t,T] \times \mathbb{R}^n \to \mathbb{R}$, $J^{2,-}u(t,x)$ denotes the 2nd order subjet of u in (t,x), i.e.

$$J^{2,-}u(t,\hat{x}) = \{(p,Q)| \liminf_{x \to \hat{x}} \frac{u(t,x) - u(t,\hat{x}) - p * (x-\hat{x}) - \frac{1}{2}(x-\hat{x})^t Q (x-\hat{x})}{\|x-\hat{x}\|^2} \geq 0\}$$

where $(p,Q) \in \mathbb{R}^n \times S^n$.

Definition 2.11. ***Superjet.***
Be t, T, n, u like above. Then $J^{2,+}u(t,x)$ denotes the 2nd order superjet of u in (t,x), i.e.

$$J^{2,+}u(t,\hat{x}) = \{(p,Q)| \limsup_{x \to \hat{x}} \frac{u(t,x) - u(t,\hat{x}) - p * (x-\hat{x}) - \frac{1}{2}(x-\hat{x})^t Q (x-\hat{x})}{\|x-\hat{x}\|^2} \leq 0\}$$

where $(p,Q) \in \mathbb{R}^n \times S^n$.

Finally we can state our main goal of this section, a weak solution terminology for the second order HJB, that provides an unique solution for our associated stochastic optimal control problem.

Definition 2.12. ***(second order) Viscosity supersolution.***
A (lower semi-)continuous function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is called a viscosity supersolution of (7) if

$$-u_t(t,x) + H(x,u,p,Q) \geq 0 \text{ for every } x \in \mathbb{R}^n \text{ and } (p,Q) \in J^{2,-}u(t,x) \tag{10}$$

Definition 2.13. ***(second order) Viscosity subsolution.***
An (upper semi-)continuous function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is called a viscosity subsolution of (7) if

$$-u_t(t,x) + H(x,u,p,Q) \leq 0 \text{ for every } x \in \mathbb{R}^n \text{ and } (p,Q) \in J^{2,+}u(t,x) \tag{11}$$

Definition 2.14. ***(second order) Viscosity solution.***
A continuous function $u \in C([0,T] \times \mathbb{R}^n)$ is called a viscosity solution of (7) if u is a viscosity subsolution and viscosity supersolution of (7).

The weak solution terminology now provides us a unique solution for our optimal control problem. Like e.g. [BCD97] and [Gru04] showed (for the first order case, this can be applied to the second order case like [Neu09] did) the value function as solution of our optimal control problem is also a viscosity solution of the associated HJB.

From [CIL92] we get an uniqueness result using the following comparison principle:

Theorem 2.15. ***Comparison principle.***
Let Ω be an open and bounded subset $\subseteq \mathbb{R}^n$. Let u and v be upper resp. lower semi-continuous functions in $\overline{\Omega}$) with u be a viscosity subsolution and v be a lower semi-continuous viscosity supersolution of (7) *in Ω. Be $u \leq v$ on $\partial\Omega$. Then it follows*

$$u \leq v \text{ in } \overline{\Omega}$$

Now we can state the final result of this section.

Theorem 2.16. *The value function V is a unique viscosity solution of the HJB (7) on any bounded subset of $[0,T] \times \mathbb{R}^n$.*

Proof. We know the value function V is a viscosity solution of the HJB (7).

Let w be a second (not necessarily different) viscosity solution of (7). Then w is a viscosity subsolution and by definition V is a viscosity supersolution. So it follows from the comparison theorem $w \leq V$. With the same argument we know $V \leq w$ and so $w = V$. Hence we have a unique viscosity solution for (7). □

This can be appealed to unbounded solutions on unbounded domains following [CIL92].

2.5 About consumption and its utility

Before we will develope our model to determine optimal consumption of a consumer we take a closer look on the optimal consumption problem. It is a standard problem in financial mathematics to determine the optimal consumption. But to do this we first need a measure that tells us what is preferable to consume. This is done by the utility of the consumption. This utility is defined by a utility function $u : \mathbb{R}_+ \to \mathbb{R}$ attaching a (non-negative) utility to every (non-negative) amount of consumption. In this thesis we want a feasible utility function $u(c)$ to satisfy the following conditions (see [Ina63])

Definition 2.17. ***Inada conditions***

$$\begin{aligned} u(c) &\in C^1(\mathbb{R}_+), \\ u(0) &= 0, \\ u^{'}(c) &> 0, \\ u^{''}(c) &< 0, \\ u^{'}(0) &= \infty, \\ u^{'}(\infty) &= 0. \end{aligned}$$

There are several functions fulfilling these conditions (at least all but $u(0) = 0$) that are generally used. Another condition often applied is that the utility function should have a constant relative risk aversion. Such a function is called a CRRA utility function. The definition of relative risk aversion goes back to Arrow and Pratt in [Arr71] and [Pra64]

Definition 2.18. ***Relative risk aversion***
The relative risk aversion of a utility function $u(c)$ is defined by:

$$-\frac{cu^{''}(c)}{u^{'}(c)}$$

In section 4 we will present a CRRA utility function we will work with in this thesis.

3 Continuous Income Model

In this section we introduce a model evolving the income and asset dynamics of a single consumer, developed by Christian Schluter and Mark Trede [ST11]. Other than former discrete income models, this one is a continuous model, which not only allows us to compute a continuous consumption strategy, but also has the advantage to redefine our optimal consumption strategy if any unexpected change of the income or asset takes place. For economists the income path is a continuous flow with discrete times of payment. So it should be intuitive to use a continuous time income model in an optimal consumption problem, but it is rather little used as discrete models are much easier to solve.
At first we have to define some basic variables:

- $y(t)$: income at time t
- $a(t)$: asset at time t
- $c(t)$: consumption at time t
- r: interest rate for the risk free asset. r is assumed to be constant.

The model starts at a given date t_0, with known asset $a(t_0)$ and income $y(t_0)$, and ends at T, the known date of death.
The first step is now to describe the development of the financial wealth. It follows an ODE that describes the rate of change of the asset as the sum of the asset, paid interest with rate r, and the difference between income and consumption (we concentrate on the case where the consumer only can decide to put his wealth into a risk free asset or to consume):

$$da(t) = r \cdot a(t)dt + (y(t) - c(t))dt$$

As restriction for the final time T we choose $a(T) \geq 0$, which means we do not allow debts at the end of the consumers life (other models make similar but even more restrictive assumptions like $a(t) \geq 0$ for all t or introducing an exit time $\tau = \min\{T, \text{time of bankruptcy}\}$, e.g. see [Set97]). This is necessary to get a limit for the lifetime consumption. Without this restriction, the only solution to our maximizing problem would be to consume as much as you can regardless of the asset and income.

The income follows a stochastic process:

$$y(t) = exp(\tilde{y}(t) + u(t))$$

where $\tilde{y}(t)$ is a deterministic component and $u(t)$ is a stochastic deviation term.
As usual in literature for the deterministic component we use a sum of a constant, a vector of timeinvariant variables and a vector of timevariable influences (multiplied by corresponding estimated parameter vectors):

$$\tilde{y}(t) = \mu + Z_1\beta + Z_2(t)\gamma$$

Here μ is a real parameter, its estimated value is -2.3296. Z_1 is a row vector with timeinvariant variables. It consists of the gender (1 for men, 2 for women) and the time of education in years. The associated vector β has the form $\beta = (\beta_1, \beta_2)^t$ and has an estimated value $(-0.3038, 0.1254)^t$. The row vector Z_2 consists of 3 parameters: the age, the squared age and the time t. The associated vector γ has the following estimated values: $\gamma = (0.0928, -0.001, 0.0407)^t$.
The last step to complete the model is to determine the stochastic component by means of an Ornstein-Uhlenbeck process with mean 0:

$$du(t) = -\eta u(t)dt + \sigma dW(t)$$

where the estimated values are $\eta = 2.324$ and $\sigma^2 = 0.5033$, so the possible values $\eta > 0$ and $\sigma \in \mathbb{R}$ are adhered by the estimations. Here $dW(t)$ is a standard Wiener process.
All estimations of the model were made using US Panel studies of income dynamics (PSID) data with the generalized method of moments (GMM). To yield better results only data sets on an annual basis from 1989 to 1996 were used, as in former estimations a structural interruption were recognized using data from former periods. Further [ST11] used only data of people reporting their income in every period used for the estimations to get more representative values.

4 Problem formulation

4.1 Stochastic optimal control problem

Based on the model we developed in the former section, we now want to maximize the total utility of (lifetime) consumption. First of all we need a function that describes the utility of consumption at any time t. This utility should only depend on the amount of the consumption at the time in dependence of a risk aversion α. If we loosen the Inada condition and do not demand $u(0) = 0$, as two utility functions u and v are equivalent if a positive affine transformation between u and v exists (see [Pra64]), a possible utility function for us would be the CRRA utility function:

$$\tilde{u}(c(t)) = \begin{cases} ln(c(t)) & \text{for } \alpha = 1, \\ (c(t)^{1-\alpha} - 1)/(1-\alpha) & \text{otherwise} \end{cases}$$

The expected value is taken for the case that $c(s)$ is stochastic. Through this definition $\tilde{u}$ is continuous and well-defined in its limit $\alpha \to 1$, because of $ln(c(t)) = \lim_{\beta \to 0} \frac{c(t)^{\beta}-1}{\beta}$. With such an utility function, we can compute the expected total utility in the period $[t, T]$ as the sum of all instantaneous utilities. We therefore use an integral over the utility function restricted to the period $[t, T]$:

$$\mathbb{E}\left(\int_t^T \tilde{u}(c(s))ds\right) \tag{12}$$

Now we want to maximize this functional regarding the asset and expected income as introduced in the former section. To do so, we have to state the stochastic optimal control problem as presented in the preliminaries.
In our case the (stochastic) state variables are the income $y(t)$ and the asset $a(t)$. These are the parameters that tell us how much money we can invest in the risk free asset and in consumption. So they are describing our current and future (financial) wealth. By consuming more or less in a period, the asset and therefore our wealth changes. So our control variable is the consumption $c(t)$. For the controls $c(t)$ we have the constraint $c(t) \geq c_a$, with c_a the autonomous consumption. The end state of the system is $y(T) \in \mathbb{R}, a(T) \geq 0$, which means we don't want to have any debts at the end of our life. This of course does not mean we cannot have debts in lifetime. The perfomance functional is the expected total utility (12) and we do not impose any final time utility.

So our stochastic optimal control problem (SOCP1) states as follows:

$$\begin{aligned} \max_{c \in U} \{ \mathbb{E} \left(\int_{t_0}^{T} \tilde{u}(c(s))ds \right) \} \\ da(t) = r \cdot a(t)dt + (y(t) - c(t))dt \\ y(t) = exp(\tilde{y}(t) + u(t)) \\ \tilde{y}(t) = \mu + Z_1\beta + Z_2(t)\gamma \\ du(t) = -\eta u(t)dt + \sigma dW(t) \\ a(T) \geq 0 \\ y(T) \in \mathbb{R} \text{ arbitrary} \end{aligned}$$

for given $t_0, T, r, a(t_0), y(t_0), c(t_0), \mu, Z_1, Z_2(t_0), \beta, \gamma, \eta, \sigma, u(0)$ and $W(0)$. For further tranformations and from the (exponential) equation for $y(t)$, we choose in fact $y(T) > 0$.

In the next section we will derive a Hamilton-Jacobi-Bellman equation to solve this system numerically. But first we need to transform (SOCP1) a little bit to use the ideas of [BEI10].
Since we have a stochastic target constraint with our condition $a(T) \geq 0$, we cannot derive the associated HJB and show the value function to be a viscosity solution just straightforward. We need a slight different dynamic programming principle, the geometric dynamic programming principle, to get a system we can solve numerically with standard schemes. This all provides us te approach of [BEI10].
The main point of this approach is to split up the system into a constant term (start conditions), a Riemann integral and a stochastic integral. Since the equation of $y(t)$ depends on an exponential function, we need to logarithmize it to get the needed system. We set $\bar{y}(t) = \ln y(t) = \tilde{y}(t) + u(t)$. For simplification we further write $y(t)$ instead of $\bar{y}(t)$ keeping in mind that we use the logarithmized income from now on. With this new notation we can write

$$y(s) = \tilde{y}(s) + u(s), \tag{13}$$

so we get

$$y(s) - y(t) = (\tilde{y}(s) - \tilde{y}(t)) + (u(s) - u(t)) \tag{14}$$

If we now use the stochastic integral equation for $du(t)$, i.e. $u(s) = u(t) - \int_t^s \eta u(x)dx + \int_t^s \sigma dW(x)$, and the integral equation for $\tilde{y}(s) - \tilde{y}(t)$, we finally get:

$$y(s) = y(t) + \int_t^s (\frac{d}{dx}\tilde{y}(x) - \eta u(x)dx) + \int_t^s \sigma dW(x) \tag{15}$$

respectively as stochastic differential equation $dy(t) = (\frac{d}{dt}\tilde{y}(t) - \eta u(t))dt + \sigma dW(t)$. With $u(t) = y(t) - \tilde{y}(t)$ this leads to our SDE for $y(t)$:

$$dy(t) = (\frac{d}{dt}\tilde{y}(t) + \eta\tilde{y}(t) - \eta y(t))dt + \sigma dW(t) \tag{16}$$

And so we get (SOCP2), a new formulation of our stochastic otpimal control problem (notify that we write $y(t)$ for $\bar{y}(t)$, therefore we have a new state variable):

$$\begin{aligned} \max_{c \in U}\{\mathbb{E}\left(\int_{t_0}^{T} \tilde{u}(c(s))ds\right)\} \\ da(t) = r \cdot a(t)dt + (exp(y(t)) - c(t))dt \\ dy(t) = (\frac{d}{dt}\tilde{y}(t) + \eta\tilde{y}(t) - \eta y(t))dt + \sigma dW(t) \\ a(T) \geq 0 \\ y(T) \in \mathbb{R} \text{ arbitrary} \end{aligned}$$

for given $t_0, T, r, a(t_0), y(t_0), \tilde{y}(t_0), c(t_0), \eta, \sigma$ and $W(0)$. Note that there is no hidden constraint for $y(T)$ this time, since it is just the natural logarithm of a positive real number.

4.2 Stochastic target constraint problem

Now we can follow the ideas of [BEI10] that will lead us to a value function V describing our system. We have to declare some notations first and then we prove that all assumption made in that paper hold for our system. For simplification we further use most of the notations of the paper.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a canonical filtered probability space under the usual assumptions, see [BEI10, p. 3505]. We define $\mathcal{U}$ as the collection of progressively measurable processes $\in L^2([0,T] \times \Omega)$. Out of these processes we will choose our consumption path c with values in a given closed subset U of $\mathbb{R}$. U is supposed to be bounded. This should be clear, since the amount of consumption has some natural borders: The autonomous consumption (> 0) and an upper limit, defined by the sum of all consumable goods ($< \infty$).

Set $z := (a(t_0), y(t_0)) \in \mathbb{R} \times \mathbb{R}$ the starting value of the controlled process $Z^c_{t_0,z} := (Y^c_{t_0,y(t_0)}, A^c_{t_0,z})$ with our state variables $(y(s), a(s))$, that are the solutions of the following stochastic differential equations and only take values in $\mathbb{R} \times \mathbb{R}$:

$$\begin{aligned}
y(s) &=: Y^c_{t_0,y(t_0)}(s) \\
&= y(t_0) + \int_{t_0}^{s} \mu_X(Y^c_{t_0,y(t_0)}(x), c(x), x)dx \\
&\quad + \int_{t_0}^{s} \sigma_X(Y^c_{t_0,y(t_0)}(x), c(x))dW(x) \\
a(s) &=: A^c_{t_0,z}(s) \\
&= a(t_0) + \int_{t_0}^{s} \mu_Y(Z^c_{t_0,z}(x), c(x))dx + \int_{t_0}^{s} \sigma_Y(Z^c_{t_0,z}(x), c(x))dW(x)
\end{aligned}$$

To determine μ_X, μ_Y, σ_X and σ_Y, we take a closer look at (SOCP2). There we have $da(t) = r \cdot a(t)dt + (exp(y(t)) - c(t))dt$, which leads us to

$$a(s) = a(t) + \int_{t}^{s} \left[r \cdot a(x) + exp(y(x)) - c(x)\right] dx \tag{17}$$

In our case we easily get from this, (13) and (15):

$$\begin{aligned}
\mu_X(Y^c_{t_0,y(t_0)}(x), c(x), x) &= \frac{d}{dx}\tilde{y}(x) + \eta\tilde{y}(x) - \eta y(x) && (18) \\
\mu_Y(Z^c_{t_0,z}(x), c(x)) &= r \cdot a(x) + exp(y(x)) - c(x) && (19) \\
\sigma_X(Y^c_{t_0,y(t_0)}(x), c(x)) &= \sigma && (20) \\
\sigma_Y(Z^c_{t_0,z}(x), c(x)) &= 0 && (21)
\end{aligned}$$

In (18) we got the term $\frac{d}{dx}\tilde{y}(x) + \eta\tilde{y}(x) = \frac{d}{dx}Z_2(x)\gamma + \eta(\mu + Z_1\beta + Z_2(x)\gamma)$. As stated above $Z_2(t)$ is a vector, that consists of the time t, the age and the squared age. If we define the age by $t + const_{age}$ with $const_{age}$ the age at $t_0 = 0$, we have:

$$Z_2(t) = (t + const_{age}, (t + const_{age})^2, t) \tag{22}$$

and therefore

$$\frac{d}{dt}Z_2(t) = (1, 2 \cdot (t + const_{age}), 1) \tag{23}$$

resp.

$$\frac{d}{dx}Z_2(x) * \gamma = \gamma_1 + 2 \cdot x \cdot \gamma_2 + 2 \cdot const_{age} \cdot \gamma_2 + \gamma_3 = const + 2 \cdot x \cdot \gamma_2 \tag{24}$$

and

$$Z_2(t) * \gamma = \gamma_1 \cdot (t + const_{age}) + \gamma_2 \cdot (t + const_{age})^2 + \gamma_3 \cdot t \tag{25}$$

So we get a dependence on the integration variable in (18), which does not occur in [BEI10]. We have to keep this in mind while we proceed.

Now we define two auxiliary functions that will allow us to state our stochastic optimal control problem in the way [BEI10] did:

$$\begin{aligned} g(Y^c_{t_0,y(t_0)}(t), A^c_{t_0,z}(t)) = g(Z^c_{t_0,z}(t)) &:= A^c_{t_0,z}(t) \\ f(c,t_0) &:= \int_{t_0}^{T} \tilde{u}(c(s))ds \end{aligned}$$

Here f defines the utility gained by the consumption of our system from the start up to the time T. It is therefore a perfomance functional as this is the factor we want to maximize. The function g tells us which asset is left for future consumption (after time t). It is also a performance functional since the higher the asset value the more money we have to consume in the future periods. But note that this functional can be less than zero, as our assumption $a^c(T) \geq 0$ $\mathbb{P}-$a.s. (directly) tells nothing about the asset in the period $[t,T[$.

With this functions we can state a value function with a stochastic target constraint

$$V(t_0,z) := \sup\{\mathbb{E}[f(c,t_0)],\ \mathrm{c} \in \mathcal{U} \text{ s.t. } g(Z^c_{t_0,z}(T)) \geq 0 \ \mathbb{P}\text{-a.s.}\} \tag{26}$$

As shown in [BEI10, p. 3502] we can state this problem in a classical sense

$$V(t_0,z) = \sup\{\mathbb{E}[f(c,t_0)],\ \mathrm{c} \in \mathcal{U} \text{ s.t. } Z^c_{t_0,z}(s) \in D \ \mathbb{P}\text{-a.s. } \forall\, t_0 \leq s \leq T\}$$

by using a viability domain D (see [Aub91] for the definition of a viability domain), that tells us for which combinations of starting time, income and asset we will find a consumption path, that $\mathbb{P}$-a.s. ensures our terminal constraint $a(T) \geq 0$:

$$D := \{(t_0,z) \in [0,T] \times \mathbb{R}^2 : g(Z^c_{t_0,z}(T)) \geq 0 \ \mathbb{P}\text{-a.s. for some } c \in \mathcal{U}\}.$$

To give the value function a mathematical sense, we need to verify some conditions on f and g.

Lemma 4.1. $\mathbb{E}[f(c,t_0)]$ *is well defined for any* $c \in \mathcal{U}$.

Proof. Since the values of c lie in the bounded subset $U \subseteq \mathbb{R}$, with $c_a > 0$ as lower limit, the values of $\tilde{u}(c(t))$ are $\tilde{u}(c(t)) < \infty$ and $\tilde{u}(c(t)) \geq 0$ $\forall t \in [t_0,T]$.
So $-\infty < \int_{t_0}^{T} \tilde{u}(c(s))ds < \infty$.

Lemma 4.2. *g is a locally bounded Borel-measurable map.*

Proof. Borel-measurable: Be $y \in \mathbb{R}$ arbitrary. $\{x = (x_1,x_2) \in \mathbb{R}^2 : g(x) \leq y\} = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \leq y\} = \mathbb{R} \times]-\infty, y] \in \mathcal{B}(\mathbb{R}^2)$.
locally bounded: Let $\epsilon > 0$ arbitrary, $x = (x_1,x_2) \in \mathbb{R}^2$ arbitrary, $a = x_2 + \epsilon$. Be

$U(x)$ an open neighborhood of x in $\mathbb{R}^2$ such that for all $u = (u_1, u_2) \in U(x)$ the inequality $|x_2 - u_2| < \epsilon$ holds. Then

$$|g(u)| < x_2 + \epsilon = a \forall u \in U(x). \tag{27}$$

□

[BEI10] further use some assumptions on the former defined functions, which we will prove now using the Euclidean norms of $\mathbb{R}$ resp. $\mathbb{R}^n$:

Lemma 4.3. (μ_X, σ_X) *as defined by (18) and (20) is Lipschitz continuous.*

Proof. σ_X as a constant function is Lipschitz continuous.
For ease of notation we write a_i, y_i, c_i, x_i for $a_i(x_i), y(x_i), c_i(x_i), x_i$. We further use $\gamma_3 > 0$ and $\eta > 0$.

$$\begin{aligned}
& \|\mu_X(a_1, y_1, c_1, x_1) - \mu_X(a_2, y_2, c_2, x_2)\|^2 \\
= & \|\frac{d}{dx}\tilde{y}(x_1) + \eta\tilde{y}(x_1) - \eta y(x_1) - (\frac{d}{dx}\tilde{y}(x_2) + \eta\tilde{y}(x_2) - \eta y(x_2))\|^2 \\
= & \|\frac{d}{dx}Z_2(x_1)\gamma + \eta(\mu + Z_1\beta + Z_2(x_1)\gamma) - \eta y_1 \\
& -(\frac{d}{dx}Z_2(x_2)\gamma + \eta(\mu + Z_1\beta + Z_2(x_2)\gamma) - \eta y_2)\|^2 \\
= & \|(\gamma_1 + \gamma_3 + 2x_1\gamma_2) + \eta Z_2(x_1)\gamma - \eta y_1 - ((\gamma_1 + \gamma_3 + 2x_2\gamma_2) + \eta Z_2(x_2)\gamma - \eta y_2)\|^2 \\
= & \|2\gamma_2(x_1 - x_2) - \eta(y_1 - y_2) + \eta(Z_2(x_1) - Z_2(x_2))\gamma\|^2 \\
= & \|2\gamma_2(x_1 - x_2) - \eta(y_1 - y_2) \\
& +\eta(x_1 - x_2, (x_1 + const_{age})^2 - (x_2 + const_{age})^2, x_1 - x_2)\gamma\|^2 \\
= & \|(2\gamma_2 + \eta(\gamma_1 + \gamma_3))(x_1 - x_2) - \eta(y_1 - y_2) \\
& +\eta((x_1 + const_{age})^2 - (x_2 + const_{age})^2)\gamma_2\|^2 \\
= & [(2\gamma_2 + \eta(\gamma_1 + \gamma_3))(x_1 - x_2) - \eta(y_1 - y_2) \\
& +\eta((x_1 + const_{age})^2 - (x_2 + const_{age})^2)\gamma_2]^2
\end{aligned}$$

With $(2\gamma_2 + \eta(\gamma_1 + \gamma_3)) =: \alpha > 0$, $const_{age} =: ca$ and $b_i := const_{age} + x_i$ we have

$$\begin{aligned}
& \|\mu_X(a_1, y_1, c_1, x_1) - \mu_X(a_2, y_2, c_2, x_2)\|^2 \\
= \; & \alpha^2(x_1 - x_2)^2 + \eta^2(y_1 - y_2)^2 + \eta^2\gamma_3^2(b_1^2 - b_2^2)^2 - 2\alpha\eta(x_1 - x_2)(y_1 - y_2) \\
& +2\alpha\eta\gamma_2(x_1 - x_2)(b_1^2 - b_2^2) - 2\eta^2\gamma_2(y_1 - y_2)(b_1^2 - b_2^2) \\
\leq \; & \alpha^2(x_1 - x_2)^2 + \eta^2(y_1 - y_2)^2 + 2\alpha\eta(x_1 - x_2)^2 + 2\alpha\eta(y_1 - y_2)^2 \\
& +2\alpha\eta\gamma_2(x_1 - x_2)(b_1^2 - b_2^2) - 2\eta^2\gamma_2(y_1 - y_2)(b_1^2 - b_2^2) + \eta^2\gamma_3^2(b_1^2 - b_2^2)^2 \\
= \; & (\alpha^2 + 2\alpha\eta)(x_1 - x_2)^2 + (\eta^2 + 2\alpha\eta)(y_1 - y_2)^2 + \eta^2\gamma_3^2((x_1 + x_2 + 2ca)(x_1 - x_2))^2 \\
& +(2\alpha\eta\gamma_2(x_1 - x_2) - 2\eta^2\gamma_2(y_1 - y_2))(x_1 + x_2 + 2ca)(x_1 - x_2) \\
= \; & (\alpha^2 + 2\alpha\eta)(x_1 - x_2)^2 + (\eta^2 + 2\alpha\eta)(y_1 - y_2)^2 + 2\alpha\eta\gamma_2(x_1 - x_2)^2(x_1 + x_2 + 2ca) \\
& -2\eta^2\gamma_2(y_1 - y_2)(x_1 + x_2 + 2ca)(x_1 - x_2) + \eta^2\gamma_3^2(x_1 + x_2 + 2ca)^2(x_1 - x_2)^2
\end{aligned}$$

With $x_1, x_2 \in [t, T]$ it follows $x_1 + x_2 \leq 2T$, $x_1 + x_2 + 2ca > 0$ and therefore

$$\begin{aligned}
& \|\mu_X(a_1, y_1, c_1, x_1) - \mu_X(a_2, y_2, c_2, x_2)\|^2 \\
\leq \; & (\alpha^2 + 2\alpha\eta)(x_1 - x_2)^2 + (\eta^2 + 2\alpha\eta)(y_1 - y_2)^2 + 2\alpha\eta\gamma_2(2T + ca)(x_1 - x_2)^2 \\
& -2\eta^2\gamma_2(y_1 - y_2)(x_1 + x_2 + 2ca)(x_1 - x_2) + \eta^2\gamma_3^2(2T + 2ca)^2(x_1 - x_2)^2 \\
\leq \; & (\alpha^2 + 2\alpha\eta + (2\alpha\eta\gamma_2 + \eta^2\gamma_3^2)(2T + ca))(x_1 - x_2)^2 + (\eta^2 + 2\alpha\eta)(y_1 - y_2)^2 \\
& +2\eta^2\gamma_2(2T + ca)(y_1 - y_2)^2 + 2\eta^2\gamma_2(2T + ca)(x_1 - x_2)^2 \\
= \; & \delta_1(x_1 - x_2)^2 + \delta_2(y_1 - y_2)^2 \\
\leq \; & \max\{\delta_1, \delta_2\}((a_1 - a_2)^2 + (y_1 - y_2)^2 + (c_1 - c_2)^2 + (x_1 - x_2)^2) \\
= \; & L \cdot \|(a_1, y_1, c_1, x_1) - (a_2, y_2, c_2, x_2)\|^2
\end{aligned}$$

for $L = \max\{\delta_1, \delta_2\}$, with $\delta_1 = (\alpha^2 + 2\alpha\eta + (2\alpha\eta\gamma_2 + \eta^2\gamma_3^2 + 2\eta^2\gamma_2)(2T + ca))$ and $\delta_2 = (\eta^2 + 2\alpha\eta) + 2\eta^2\gamma_2(2T + ca)$.

□

For μ_Y we can only show local Lipschitz continuity:

Lemma 4.4. (μ_Y, σ_Y) *as defined by (19) and (21) is locally Lipschitz continuous.*

Proof. σ_Y as a constant function is Lipschitz continuous.

For ease of notation we write a_i, y_i, c_i for $a_i(x_i), y(x_i), c_i(x_i)$. We further use $r > 0$.

$$\begin{aligned}
& \|\mu_Y(a_1,y_1,c_1) - \mu_Y(a_2,y_2,c_2)\|^2 \\
= \; & \|r \cdot a_1 + \exp(y_1) - c_1 - (r \cdot a_2 + \exp(y_2) - c_2)\|^2 \\
= \; & \|r(a_1 - a_2) + (\exp(y_1) - \exp(y_2)) - (c_1 - c_2)\|^2 \\
= \; & [r(a_1 - a_2) + (\exp(y_1) - \exp(y_2)) - (c_1 - c_2)]^2 \\
= \; & r^2(a_1 - a_2)^2 + (\exp(y_1) - \exp(y_2))^2 + (c_1 - c_2)^2 - 2r(a_1 - a_2)(c_1 - c_2) \\
& +2r(a_1 - a_2)(\exp(y_1) - \exp(y_2)) - 2(c_1 - c_2)(\exp(y_1) - \exp(y_2)) \\
\leq \; & r^2(a_1 - a_2)^2 + (\exp(y_1) - \exp(y_2))^2 + (c_1 - c_2)^2 + 2r(a_1 - a_2)^2 + 2r(a_1 - a_2)^2 \\
& +2r(\exp(y_1) - \exp(y_2))^2 + 2r(c_1 - c_2)^2 + 2(c_1 - c_2)^2 + 2(\exp(y_1) - \exp(y_2))^2 \\
= \; & (r^2 + 4r)(a_1 - a_2)^2 + (3 + 2r)(\exp(y_1) - \exp(y_2))^2 + (3 + 2r)(c_1 - c_2)^2 \\
\overset{(*)}{\leq} \; & (r^2 + 4r)(a_1 - a_2)^2 + \max\{\exp(y_1)^2, \exp(y_2)^2\} \cdot (3 + 2r)(y_1 - y_2)^2 \\
+ \; & (3 + 2r)(c_1 - c_2)^2 \\
\leq \; & \max\{r^2 + 4r, 3 + 2r, (3 + 2r) \cdot \max\{\exp(y_1)^2, \exp(y_2)^2\}\} \\
\cdot \; & ((a_1 - a_2)^2 + (y_1 - y_2)^2 + (c_1 - c_2)^2) \\
= \; & L \cdot \|(a_1, y_1, c_1) - (a_2, y_2, c_2)\|^2
\end{aligned}$$

with

$$0 < L = \max\{r^2 + 4r, 3 + 2r, (3 + 2r) \cdot \max\{\exp(y_1)^2, \exp(y_2)^2\}\}$$

(*) where we have used $\exp(y_1) - \exp(y_2) \leq \exp(y_1)(y_1 - y_2)$ for $y_1 \geq y_2$.

So μ_Y is Lipschitz continuous in a neighborhood of (a_1, y_1, c_1) for any $(a_1, y_1, c_1) \in \mathbb{R} \times \mathbb{R} \times \mathcal{U}$ and therefore locally Lipschitz continuous in every point. □

As [BEI10] showed, for technical reasons in proving a dynamic programming principle for the value function, we need to restrain the set of admissible controls to a subset

$$\mathcal{U}^t := \{c \in \mathcal{U} : c \text{ independent of } \mathcal{F}_t\}$$

So we finally get our stochastic control problem formulated as

$$\left.\begin{aligned}
V(t_0, z) := & \quad \sup_{c \in \mathcal{U}_{t_0,z}} \mathbb{E}[\int_{t_0}^{T} \tilde{u}(c(s))ds] \\
\mathcal{U}_{t_0,z} := & \quad \{c \in \mathcal{U}^{t_0} : A^c_{t_0,z}(T) \geq 0 \ \mathbb{P} - \text{a.s.}\}
\end{aligned}\right\} \tag{28}$$

in which our value function V describes the maximal achieveable (expected) utility we can gain with the consumption in the period $[t_0, T]$ using all allowed consumption paths, i.e. the consumption paths c, that will $\mathbb{P}$-a.s guarantee us a terminal asset $a(T) \geq 0$ with starting asset and income $(a(t_0), y(t_0)) = z$.

4.3 Derivation of the Hamilton-Jacobi-Bellman equation for the value function

Our last step now is to provide a viscosity characterization for the value function in order to solve the problem numerically. We follow [BEI10] in their main arguments. Essential for the proofs of the PDE characterization of the value function is a geometric dynamic programming principle, developed in [ST02] which can be found in the Appendix.

A direct conclusion of the geometric dynamic programming principle can be formulated by an auxiliary value function $w : [0,T) \times \mathbb{R} \to \mathbb{R}$, with:

$$w(t, y) := \inf \{a \in \mathbb{R} : (t, y, a) \in D\} \tag{29}$$

This function tells us the minimal asset a we are supposed to hold at time t to find a consumption path such that we will $\mathbb{P} - a.s.$ reach our terminal condition $a(T) \geq 0$, while earning an income that developes through our SDE for the income starting with y. So it characterizes the boundary of our viability domain D.
It is now intentional to think if we can find a consumption path at time t for given asset $a(t)$ and some fixed income y, then we can find at least the same path for a greater asset $a^+(t)$ and the same fixed income.
This can be formulated mathematically and we will see it will hold in our system:

Lemma 4.5. *$h(a(t)) := a(t) \mapsto g(\overline{y}, a(t))$ is non-decreasing and right-continuous for constant $\overline{y}$.*

Proof. non-decreasing: Be $a(t) \geq a(s)$. Then it follows: $h(a(t)) = g(\overline{y}, a(t)) = a(t) \geq a(s) = g(\overline{y}, a(s)) = h(a(s))$.
right-continuous: $\lim_{\epsilon \searrow 0} h(a(t) + \epsilon) = \lim_{\epsilon \searrow 0} g(\overline{y}, a(t) + \epsilon) = \lim_{\epsilon \searrow 0} (a(t) + \epsilon) = a(t) = g(\overline{y}, a(t)) = h(a(t))$. □

A consequence of this lemma is our former assumption that we will find at least the same paths for $a^+(t)$ as for $a(t)$:

$$\text{Be } a^+(t) \geq a(t). \text{ Then it follows } \mathcal{U}_{t,y,a^+} \supset \mathcal{U}_{t,y,a} \tag{30}$$

For a further use of the auxiliary value function we will make the following assumption:

Assumption 4.6. *w is continuous and admits a continuous extension $\hat{w}$ on $[0,T]\times\mathbb{R}$ such that $\hat{w}(T,\cdot)\geq 0$.*

If we write $w(T,\cdot)$ we will use this extension to give the expression a sense.

We now take a closer look on the viability domain D. It is determined by the auxiliary value function w, as this function tells us the minimum asset $a(t)$ for a given t and given income $y(t)$, so we can find a comsumption path, satisfying the terminal condition $a(T)\geq 0$ $\mathbb{P}$-a.s. Since we know the minimal consumption path c_{sub} (consuming c_a at any time t) with 30 we can write D as

$$\begin{aligned} D &= \{(t_0,z)\in[0,T]\times\mathbb{R}^2 : \mathcal{U}_{t_0,z}\neq\emptyset\} \\ &= \{(t_0,z)\in[0,T]\times\mathbb{R}^2 : A^{c_{sub}}_{t_0,z}(T)\geq 0 \ \mathbb{P}\text{-a.s.}\} \\ &= \{(t,y,a)\in[0,T]\times\mathbb{R}\times\mathbb{R} : a\geq w(t,y)\} \end{aligned}$$

For our final result we need to split our domain $cl(D)$ into an inner part and the spatial and time boundary: $cl(D) = int_p(D)\cup\partial_Z(D)\cup\partial_T(D)$, whereas

$$\begin{aligned} int_p(D) &= \{(t,y,a)\in[0,T)\times\mathbb{R}\times\mathbb{R} : a > w(t,y)\} \\ \partial_Z(D) = \partial D\cap([0,T)\times\mathbb{R}\times\mathbb{R}) &= \{(t,y,a)\in[0,T)\times\mathbb{R}\times\mathbb{R} : a = w(t,y)\} \\ \partial_T(D) = \partial D\cap(\{T\}\times\mathbb{R}\times\mathbb{R}) &= \{(t,y,a)\in\{T\}\times\mathbb{R}\times\mathbb{R} : a \geq w(t,y)\} \end{aligned}$$

From the geometric dynamic programming principle and Theorem A.2 it follows that our controlled process $Z^c_{t,z}$ will stay in the domain D at any time, if the control c lies in $\mathcal{U}_{t,z}$. Since the auxiliary value function w determines the spatial boundary of D, we can say that in the limits $a\to w(t,y)$ the spatial and time derivations of $A^c_{t,z} - w(\cdot, Y^c_{t,y})$ should be non-negative. Appealing to Ito's lemma (for $w(t, Y^c_{t,y})$) we will get:

$$\begin{aligned} dw(t,y) &= (\frac{\partial w}{\partial y}\cdot\mu_X + \frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}\sigma_X^2)dt + (\frac{\partial w}{\partial y}\cdot\sigma_X)dW_t \\ &= \mathcal{L}^c_Y w(t,y)dt + \sigma_X Dw\, dW_t \end{aligned}$$

with the $\mathcal{L}$ the Dynkin operator associated to the random variable Y, i.e.

$$\mathcal{L}^c_Y w := \frac{\partial w}{\partial y}\cdot\mu_X + \frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}\sigma_X^2.$$

Using $da(t) = r\cdot a(t)dt + (exp(y(t)) - c(t))dt$ we get for $dA^c_{t,z} - dw(\cdot, Y^c_{t,y})\geq 0$:

$$\begin{aligned} \mu_Y(y,a,c(t)) - \mathcal{L}^c_Y w(t,y) &\geq 0, \\ 0 - \sigma_X(y,c)Dw(t,y)\geq 0 \Rightarrow Dw(t,y) &\leq 0 \end{aligned}$$

since $\sigma = \sigma_X(y,c) \geq 0$.

The second equation is clear in a financial context: We fix the time t and have a minimal asset $a(t)$ for give income $y(t)$ that will $\mathbb{P}$-a.s. fulfill the terminal condition. If we now have an (infinitesimal) increase in our income, we would expect to need an (infinitesimal) lower starting asset to consume the minimal consumption path c_{sub} (we will consume as $a \to w(t,y)$) and thus expect a non-positive (in fact even negative) derivative in the direction of the income. So in this context it is clear that σ should be positive as a negative value would tell us a lower asset value would allow us more consumption.

However, this equation only makes sense if w is smooth enough ($w \in C^{1,2}$). Since we only assumed w to be continuous, we need to use test functions to give the expressions a valid meaning. We introduce two sets of test functions as following

$$\begin{aligned}
\mathcal{W}^*(t,y) &= \{\varphi \in C^{1,2}([0,T] \times \mathbb{R}) \\
&\quad \text{s.t. } (w-\varphi) < (w-\varphi)(t,y) = 0 \text{ in a neigborhood of } (t,y)\} \\
\mathcal{W}_*(t,y) &= \{\varphi \in C^{1,2}([0,T] \times \mathbb{R}) \\
&\quad \text{s.t. } (w-\varphi) > (w-\varphi)(t,y) = 0 \text{ in a neigborhood of } (t,y)\}
\end{aligned}$$

In order to define the viscosity solution V on the spatial boundary we need a relaxation of our operators and constraints for this sets of test functions. As we will see, on ∂D V should satisfy the constraint HJB (see [BEI10] but keep in mind that their value function did not depend on the control, so they get a Hamiltonian without the utility term)

$$-\partial_t \varphi(t_0,z) + H_{int}(z, D\varphi(t_0,z), D^2\varphi(t_0,z)) = 0, \tag{31}$$

with

$$H_{int}(t_0,y,a,q,A) = \inf_{c \in U_{int}(t_0,y,a,w)} H^c(y,a,q,A),$$

$$H^c(y,a,q,A) := -u(c) - \mu_Z(y,a,c) * q - \frac{\sigma^2}{2} Trace[A]$$

and

$$U_{int}(t_0,y,a,w) = \{c \in U : Dw(t_0,y) \leq 0, \mu_Y(y,a,c) - \mathcal{L}_Y^c w(t_0,y) \geq 0\}.$$

Since $Dw(t_0,y) \leq 0$ is no constraint for any $c \in U$ we can neglect this term in the definition of U_{int} and write instead

$$U_{int}(t_0,y,a,w) = \{c \in U : \mu_Y(y,a,c) \geq \mathcal{L}_Y^c w(t_0,y)\}.$$

Here we use the function w in the definition. Since we want to use test functions instead, we define a relaxed version of U_{int} and H_{int} for a $\gamma \in \mathbb{R}$ and a smooth test function ϕ

$$U_\gamma(t_0, y, a, \phi) := \{c \in U : \mu_Y(y, a, c) - \mathcal{L}_Y^c \phi(t_0, y) \geq \gamma\}$$

and

$$F_\gamma^\phi(t_0, z, q, A) := \inf_{c \in U_\gamma(t_0, z, \phi)} \{-u(c) - \mu_Z(z, c) * q - \frac{\sigma^2}{2} Trace[A]\}$$

where $\inf \emptyset = \infty$.

Now we finally can define the upper resp. lower semi-relaxed limits of the family of functions $(F_\gamma^\phi)_\gamma$ with smooth $\phi \in \mathcal{W}^*(t, y)$ resp. $\phi \in \mathcal{W}_*(t, y)$:

$$F^{\phi *}(t_0, z, q, A) := \limsup_{\substack{(t', z', q', A') \to (t_0, z, q, A), \\ \gamma' \to 0}} F_{\gamma'}^\phi(t', z', q', A') \tag{32}$$

and

$$F_*^\phi(t_0, z, q, A) := \liminf_{\substack{(t', z', q', A') \to (t_0, z, q, A), \\ \gamma' \to 0}} F_{\gamma'}^\phi(t', z', q', A'). \tag{33}$$

For technical reasons we have to introduce a subset of U, which we neglected in the relaxation, as it is no real subset in our case. For $\delta > 0$ set

$$\mathcal{N}_\delta(y, a, q) := \{c \in U : |\sigma q| \leq \delta\}.$$

As we see $\mathcal{N}_\delta(y, a, q)$ is either $\emptyset$ or U and in fact only depends on q.
To get to our main result we need to confirm that the continuity assumption on $\mathcal{N}_0$ as stated in [BEI10] as Assumption 3.1 and taken from Assumption 2.1 in [BET09] holds for our system.

Assumption. *Let Ψ be a Lipschitz continuous function on $[0, T] \times \mathbb{R}$ and let $\mathcal{O}$ be some open subset of $[0, T] \times \mathbb{R}^2$ such that $\mathcal{N}_0(x, y, \Psi(t, x)) \neq \emptyset$ for all $(t, x, y) \in \mathcal{O}$. Then for every $\epsilon > 0, (t_0, x_0, y_0) \in \mathcal{O}$ and $u_0 \in \mathcal{N}_0(x_0, y_0, \Psi(t_0, x_0))$, there exists an open neighborhood $\mathcal{O}^‘$ of (t_0, x_0, y_0) and a locally Lipschitz continuous map $\hat{u}$ defined on $\mathcal{O}^‘$ such that $|\hat{u}(t_0, x_0, y_0) - u_0| \leq \epsilon$ and $\hat{u}(t, x, y) \in \mathcal{N}_0(x, y, \Psi(t, x))$ on $\mathcal{O}^‘$.*

Lemma 4.7. *We can find a function Ψ that is Lipschitz continuous on $[0,T] \times\mathbb{R}$ and an open subset of $[0, T] \times \mathbb{R}^2$, $\mathcal{O}$, such that $\mathcal{N}_0(y, a, \Psi(t, y)) \neq \emptyset$ for all $(t, y, a) \in \mathcal{O}$.*

Proof. With $\mathcal{N}_0(y,a,q) = \{c \in U : |N^c(y,a,q)| = 0\}$ we get $\mathcal{N}_0(y,a,q) = \{c \in U : 0 = \sigma q\}$.
Be Ψ a function on $[0,T] \times \mathbb{R} \to \mathbb{R}$ with $\Psi \equiv 0$ and therefore Lipschitz continuous. Then $\mathcal{N}_0(y,a,\Psi(t,y)) = U \; \forall (t,y,a) \in \mathcal{O}$, for any open subset $\mathcal{O} \neq \emptyset$ of $[0,T] \times \mathbb{R}^2$. Since the function c_{sub} (i.e. $c_{sub} \equiv c_a$ for all t), lies in U, $U \neq \emptyset$ and therefore $\mathcal{N}_0(y,a,\Psi(t,y)) \neq \emptyset$ for all $(t,y,a) \in \mathcal{O}$. □

Now we can state the final PDE characterization of the value function, assuming U to be bounded and using upper- resp. lower-semicontinuous envelopes.

Theorem 4.8. *The value function can be characterized by the following equations: V_* is a viscosity supersolution on $cl(D)$ of*

$$\left.\begin{aligned} (-\partial_t\varphi + H^*\varphi)(t,y,a) \geq 0 \quad &if \quad (t,y,a) \in int_p(D) \\ \forall \phi \in \mathcal{W}^*(t,y),\; (-\partial_t\varphi + F^{\phi *}\varphi)(t,y,a) \geq 0 \quad &if \qquad (t,y,a) \in \partial_Z D \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} \varphi(T,y,a) \geq f_*(c,T) \text{ for all } c \in U, \text{ if } (t,y,a) \in \partial_T D, \\ a > w(T,y), H^*\varphi(T,y,a) < \infty \end{aligned}\right\} \tag{35}$$

V^* is a viscosity subsolution on $cl(D)$ of

$$\left.\begin{aligned} (-\partial_t\varphi + H_*\varphi)(t,y,a) \leq 0 \quad &if \quad (t,y,a) \in int_p(D) \\ \forall \phi \in \mathcal{W}_*(t,y),\; (-\partial_t\varphi + F^{\phi}_*\varphi)(t,y,a) \leq 0 \quad &if \qquad (t,y,a) \in \partial_Z D \end{aligned}\right\} \tag{36}$$

$$\varphi(T,y,a) \leq f^*(c,T) \text{ for all } c \in U, \text{ if } (t,y,a) \in \partial_T D \text{ and } H_*\varphi(T,y,a) > -\infty \tag{37}$$

where $\forall (t,z) \in cl(D)$

$$V_*(t,z) := \liminf_{(t',z') \in int_p(D) \to (t,z)} V(t',z'),\; V^*(t,z) := \limsup_{(t',z') \in int_p(D) \to (t,z)} V(t',z')$$

and H^* resp. H_* are the upper resp. lower semi-continuous envelopes of $H = \inf_{c \in U} H^c$.

Proof. See [BEI10] Theorem 3.1 and Corollary 3.1. Note that they use only a continuity of μ_X, μ_Y, σ_X and σ_Y and not the Lipschitz continuity. So we have no problem with μ_Y only being locally Lipschitz continuous and not globally. The local Lipschitz continuity in every point guarantees a continuity in every point, and so a global continuity.
For Corollary 3.1 we need either U be bounded or $|\sigma_Y(y,a,c)| \to \infty$ as $|c| \to \infty$ and

$$\limsup_{|c| \to \infty, c \in U} \frac{|\mu_Y(y,a,c)| + |\mu_X(y,c)| + |\sigma_X(y,c)|^2}{|\sigma_Y(y,a,c)|^2} = 0 \; \forall (y,a) \in \mathbb{R}^2$$

and a local uniform convergence with respect to (y,a).
Since $\sigma_Y \equiv \sigma$, the second condition cannot be fulfilled. But as we stated above, we assume U to be bounded. □

It is somehow clear to assume U to be bounded, since we want to consume at least the autonomuos consumption c_a at all times. For the upper limit of consumption we have not an exactly amount of maximum consumption, but it is natural to have an upper bound $c_{max} < \infty$ here as well, since we use a logarithmic-like and therefore strictly concave utility function. So if we already spent a lot of our asset for consumption in a period t_1, it costs us a lot of our asset to gain a little more utility out of our additional consumption. If we would use this money in a future period t_2, we would have a better utility out of this. So it's rational not to spent an infinity amount of money in a period for consumption (and we assume our asset not to be infinity).
If we now take functions $c_1, c_2 : \mathbb{R} \to \mathbb{R}$, with $c_{sub} \equiv c_a$ and $c_{sup} \equiv c_{max}$ at any time t and use the $L^2 - Norm$ on $[t_0, T]$ it is easy to see that c_{sub} and c_{sup} are (except for null sets) lower resp. upper bounds (using $c(t) > 0\ \forall t$).

Since we use a compact set of admissible controls and the auxiliary value function f is therefore a bounded and continuous function on $[t_0, T]$, we can replace the lower resp. upper semi-continuous envelopes in the theorem with the actual functions and receive the following corollary:

Corollary 4.9. *The value function can be characterized by the following equations: V is a viscosity solution on $cl(D)$ of*

$$\left.\begin{aligned} (-\partial_t\varphi + H\varphi)(t,y,a) = 0 \quad &if \quad (t,y,a) \in int_p(D) \\ \forall \phi \in \mathcal{W}^*(t,y),\ (-\partial_t\varphi + F^{\phi *}\varphi)(t,y,a) \geq 0 \quad &if \quad (t,y,a) \in \partial_Z D \\ \forall \phi \in \mathcal{W}_*(t,y),\ (-\partial_t\varphi + F^{\phi}_*\varphi)(t,y,a) \leq 0 \quad &if \quad (t,y,a) \in \partial_Z D \end{aligned}\right\} \tag{38}$$

$$\varphi(T,y,a) = f(c,T) = 0 \text{ for all } c \in U,\ \text{if } (t,y,a) \in \partial_T D \tag{39}$$

5 Numerical Scheme

Now we want to give a numerical solution to our stochastic optimal control problem, formulated by the HJB (38) and (39). Since we compute a continuous approximation of V and w we ignore the fact that we use testfunctions on the spatial boundary and the problem can be reformulated to:
Our value function defined by equation (28) is a viscosity solution of the following HJB:

$$\begin{aligned}(-\partial_t V + HV)(t,y,a) = 0 \quad &\text{if} \quad (t,y,a) \in int_p(D)\\ (-\partial_t V + H_{int}V)(t,y,a) = 0 \quad &\text{if} \quad (t,y,a) \in \partial_Z D\\ V(T,y,a) = 0 \quad &\text{if} \quad (t,y,a) \in \partial_T D\end{aligned}$$

For this problem we now will derive a (in the fully implicit time stepping case) unconditionally stable finite difference scheme, based on [FL07] and [CF08].

5.1 Finite difference scheme

5.1.1 Discretization

Before we start to define the finite difference scheme in detail, we have to introduce some variables, we use to determine our computational domain.
As usual for a (financial) control problem, we solve our problem backwards in time, introducing a new time variable $\tau := T - t_0$. So for $\tau = 0$ we have $t_0 = T$ and at $\tau = T$ it follows $t_0 = 0$. With this notation we solve the problem for $V(\tau, y, a)$ from $\tau = 0$ to $\tau = T$. For computational reasons we have to define a spatial area in which we will solve our problem. Therefore we will set some variables $a_{min}, a_{max}, y_{min}, y_{max}$ as boundaries in a resp. y-direction. Note that from our viability domain D we get $a_{min} = a_{min}(y_{max}, \tau)$. So if we try to choose a_{min} in such a way that $\partial D \cap [y_{min}, y_{max}]$ lies in our computational domain for all τ, we would use an approximation of $w(\tau = T, y_{max})$ as this is the lowest value that w can take in our computational domain. As approximation we use $a_{min} = [c_a - \exp(y_{max})] \cdot (T - t_0)$, which will becomes clearer in the following. For a_{max} and y_{max} we can assume there will be a limit of consumption c_{max} you will never cross, even if you gain an extra asset or income. So it's rather intuitive to have such limits. For the limit y_{min} we can say the income can never have a negative value. Since we use the natural logarithm of the income for y we still face the problem to have no lower limit here. But if we assume to have a minimal (not logarithmized) income $>> 0$, we get an arbitrary lower boundary $y_{min} >> -\infty$.

We now define a grid on which we will discretize our equation. Let $\Delta\tau$ be the time step, with τ^n denoting $n \cdot \Delta\tau$. Set $p_i \in \mathbb{R}^2$ with $i = 1, \dots P$ the 2-dimensional points on a grid with y and a as axes for a firm time τ^n. Here P is the number of nodes, $p_1 = (y_{min}, a_{min}), p_P = (y_{max}, a_{max})$. Set $v \in \mathbb{R}^P$, $v_i^n = V^n(p_i)$, where $V^n(p_i) = V(\tau^n, p_i)$ and $v^n = [v_1^n, \dots, v_P^n]^t$. For the spatial directions set Δy resp. Δa as spacing between 2 nodes in y resp. a-direction.

Now with

$$\mathcal{L}^c V = \mu_Z(y, a, c) * DV + \frac{\sigma^2}{2} \text{Trace}[D^2 V] \tag{40}$$

we can state our problem (in the inner domain) as

$$V_\tau = \sup_{c \in U} \{\mathcal{L}^c V + u(c)\} \tag{41}$$

since we have the equations

$$\begin{aligned}
- \quad & V_t(t, z) + H(t, z, DV, D^2 V) = 0 \\
\Leftrightarrow \quad & V_\tau(\tau, z) + H(\tau, z, DV, D^2 V) = 0 \\
\Leftrightarrow \quad & V_\tau(\tau, z) = -H(\tau, z, DV, D^2 V) \\
\Leftrightarrow \quad & V_\tau(\tau, z) = -\inf_{c \in U} \{-u(c) - \mu_Z(y, a, c) * DV - \frac{\sigma^2}{2} \text{Trace}[D^2 V]\} \\
\Leftrightarrow \quad & V_\tau(\tau, z) = \sup_{c \in U} \{u(c) + \mu_Z(y, a, c) * DV + \frac{\sigma^2}{2} \text{Trace}[D^2 V]\} = \sup_{c \in U} \{\mathcal{L}^c V + u(c)\}
\end{aligned}$$

This operator $\mathcal{L}^c V$ will be discretized by using forward, backward, or central differencing in the spatial directions. Therefore we introduce the forward, backward, and central differencing operators as following (neglecting the dependence on the time τ) :

For the first derivatives in one spatial dimension we use

$$\begin{aligned}
\textit{(forward)} \ \frac{\partial V}{\partial y_j}(p_i) \quad &\approx \quad \frac{1}{\Delta y_j}[V(p_i + e_j \Delta y_j) - V(p_i)], & (42) \\
\textit{(backward)} \ \frac{\partial V}{\partial y_j}(p_i) \quad &\approx \quad \frac{1}{\Delta y_j}[V(p_i) - V(p_i - e_j \Delta y_j)] \text{ and} & (43) \\
\textit{(central)} \ \frac{\partial V}{\partial y_j}(p_i) \quad &\approx \quad \frac{1}{2\Delta y_j}[V(p_i + e_j \Delta y_j) - V(p_i - e_j \Delta y_j)], & (44)
\end{aligned}$$

where $j \in \{1, 2\}; y_1 = y, y_2 = a; e_j$ is the normal unit vector in y_j direction.

For the second derivatives we use the usual differencing operator

$$\frac{\partial^2 V}{\partial y_j^2}(p_i) \approx \frac{1}{\Delta y_j^2}[V(p_i + e_j \Delta y_j) + V(p_i - e_j \Delta y_j) - 2V(p_i)] \tag{45}$$

With this differencing operators we can discretize (40) at the node (τ^n, v_i) as

$$(\mathcal{L}^c_{\Delta y,\Delta a} v^n)_i = \alpha_i^n(c) v^n_{i-x} + \beta_i^n(c) v^n_{i-1} + \gamma_i^n(c) v^n_{i+1} + \delta_i^n(c) v^n_{i+x} - \zeta_i^n(c) v^n_i, \tag{46}$$

where $\zeta_i^n(c) = \alpha_i^n(c) + \beta_i^n(c) + \gamma_i^n(c) + \delta_i^n(c)$. v^n_{i-x} and v^n_{i+x} denote the values of V at time τ^n and node $p_i - e_2\Delta a$ resp. $p_i + e_2\Delta a$, while $v^n_{i\pm1} = V^n(p_i \pm e_1\Delta y)$. We will declare the coefficients in the following. For now we only have to know that we will have a *positive coefficient condition*, which is specified by

Assumption 5.1. *Positive coefficient condition*

$$\alpha_i^n(c), \beta_i^n(c), \gamma_i^n(c), \delta_i^n(c) \geq 0 \ \forall c \in U. \tag{47}$$

Now we are able to discretize equation (41) by either using Crank-Nicolson or fully implicit time stepping with

$$\frac{v_i^{n+1} - v_i^n}{\Delta\tau} = (1-\theta) \sup_{c^{n+1}\in U} \{(\mathcal{L}^{c^{n+1}}_{\Delta y,\Delta a} v^{n+1})_i + u_i^{n+1}\} + \theta \sup_{c^n \in U} \{(\mathcal{L}^{c^n}_{\Delta y,\Delta a} v^n)_i + u_i^n\} \tag{48}$$

using $\theta = 0$ for fully implicit and $\theta = \frac{1}{2}$ for Crank-Nicolson time stepping. Here u_i^m denotes the discretization of $u(c)$ at time step m and node p_i.

5.1.2 Boundary handling

To solve the problem numerically we still need to apply boundary conditions to our discretized HJB. In the time direction we have an explicit boundary $\tau = 0$ with boundary condition $V(0, y, a) = 0$.
In the spatial direction we have the viability domain D as solving area. Hence we have to compute D first by solving a HJB for $w(\tau^n, y)$, which represents ∂D on a time step τ^n and $y \in [y_{min}, y_{max}]$. As stated in [BEI10], [BET09] and [ST02] w is a (discontinuous) viscosity solution of

$$\sup_{c\in U, 0\geq D\varphi} \{\mu_Y(\cdot, \varphi, c) - \mathcal{L}^c_X \varphi - \partial_t \varphi\} = 0, \ \varphi(T, \cdot) = 0. \tag{49}$$

$\mathcal{L}^c_X \varphi := \mu_X * D\varphi + \frac{1}{2}[\sigma_X \sigma_X^t D^2\varphi]$. Since $\partial_t\varphi$ is independent of the control, we get:

$$\sup_{c\in U, 0\geq D\varphi} \{\mu_Y(\cdot, \varphi, c) - \mathcal{L}^c_X \varphi\} = \partial_t \varphi, \ \varphi(T, \cdot) = 0. \tag{50}$$

We solve this equation backwards in time with τ as above. So we get

$$\begin{aligned}
& \sup_{c\in U, 0\geq D\varphi} \{\mu_Y(\cdot, \varphi, c) - \mathcal{L}^c_X \varphi\} = -\partial_\tau \varphi, \ \varphi(0, \cdot) = 0 \\
\Leftrightarrow \ & -\sup_{c\in U, 0\geq D\varphi} \{-(\mathcal{L}^c_X \varphi - \mu_Y(\cdot, \varphi, c))\} = \partial_\tau \varphi, \ \varphi(0, \cdot) = 0 \\
\Leftrightarrow \ & \inf_{c\in U, 0\geq D\varphi} \{\mathcal{L}^c_X \varphi - \mu_Y(\cdot, \varphi, c)\} = \partial_\tau \varphi, \ \varphi(0, \cdot) = 0 \\
\Leftrightarrow \ & \inf_{c\in U, 0\geq D\varphi} \{\mathcal{L}^c_X \varphi - r\varphi - (exp(y) - c)\} = \partial_\tau \varphi, \ \varphi(0, \cdot) = 0.
\end{aligned}$$

So we solve this HJB with the same discretization as above, but with an extra term $-\mu_Y(\cdot,\varphi,c)$ instead of the utility function and the sup replaced by an inf (minimization instead of maximization). Hence we get the discretization:

$$\frac{w_j^{n+1}-w_j^n}{\Delta\tau} = (1-\theta)\inf_{c^{n+1}\in U}\{(\mathcal{L}_{\Delta y}^{c^{n+1}}w^{n+1})_j + d_j^{n+1}\} + \theta \inf_{c^n\in U}\{(\mathcal{L}_{\Delta y}^{c^n}w^n)_j + d_j^n\} \quad (51)$$

with $w_j^n = w(\tau^n, p_j)$, $j = 1,\ldots,x$, where x denotes the amount of nodes in the y-direction (and will denote it further). $w^n = [w_1^n,\ldots,w_x^n]^t$, $\mathcal{L}_{\Delta y}^{c^n}w^n$ defined similar to above, but with only two coefficients (ψ, ν) on the off-diagonals and a slight different coefficient on the diagonal, -($\psi+\nu+r$). This coefficients have to fulfill the positive coefficient condition. In fact the coefficients are well known: They are β and γ as we will see. At last $d_j^n = -exp(y) + c_j^n$, where y is the y-component of p_j. This discretization can be computed quite easily, since we know that the consumption used to determine the inf terms is the autonomous consumption, so $c^{n+1} = c^n = c_a$ and we do not have to determine the inf terms with optimization loops.

∂D is the lower boundary in the a-direction. So after computing w, we can apply boundary conditions for points p_j, where the a-coordinate lies on ∂D.

Since ∂D is the limit, where we have to consume c_a in every period to fulfill the terminal condition $a(T) \geq 0$ $\mathbb{P}$-a.s., the Dirichlet boundary condition here is $V(\tau, y, w(\tau, y)) = \tau\cdot\tilde{u}(c_a)$. For points (t, y, a) with $a < w(t, y)$ we set $V = \tau\cdot\tilde{u}(c_a)$ as well.

If we choose a_{max} large enough, i.e. $a_{max} = (T-t_0)\cdot c_{max}$, we got such a large asset, that the maximal consumption c_{max} can be consumed every period regardless the (non-negative) income, which will be clearer soon. So our boundary condition on the spatial boundary $a = a_{max}$ is $V(\tau,\cdot,a_{max}) = \tau\cdot\tilde{u}(c_{max})$.

While getting closer to y_{min} consumption becomes more and more independent of the income. As we use y_{min} as approximation for zero income, we assume the consumption on the limit y_{min} to be the asset consumed over the left periods, but at least to be the autonomuos consumption. Since we neglect the income and get a non-stochastic optimization problem on the boundary, it is natural to have such a linearity assumption, while using a concave utility function.

To determine the right boundary value on y_{min} we will now solve the optimization problem

$$\max_c \int_{t_0}^{T} u(c(s))ds, \tag{52}$$

$$\frac{da(t)}{dt} - ra(t) + c(t) = 0, \tag{53}$$

$$a(T) = 0. \tag{54}$$

This is a normal optimization problem with side conditions, so we use its Lagrangian to solve it

$$L(c, a, \lambda, \mu) = \int_{t_0}^{T} u(c(s))ds + \lambda \int_{t_0}^{T} \frac{da(t)}{dt} - ra(t) + c(t)dt + \mu \int_{t_0}^{T} a(T)dt \tag{55}$$

In order to solve the problem we get the following equations

$$\begin{aligned} 0 &= \frac{\partial L}{\partial c} \Rightarrow 0 = u^{'}(c) + \lambda \Rightarrow c = (-\lambda)^{-\frac{1}{\alpha}}, \\ 0 &= \frac{\partial L}{\partial a} \Rightarrow \frac{d}{dt}\lambda - r\lambda = 0 \Rightarrow \lambda = e^{rt} \cdot \kappa, \\ 0 &= \frac{\partial L}{\partial \lambda} \Rightarrow \frac{da(t)}{dt} - ra(t) + c(t) = 0, \\ 0 &= \frac{\partial L}{\partial \mu} \Rightarrow a(T) = 0. \end{aligned}$$

Inserting these equations into each other we obtain

$$\left.\frac{da(t)}{dt}\right|_{t=T} = ra(T) - (-e^{rT} \cdot \kappa)^{-\frac{1}{\alpha}}$$

using $a(T) = 0$ and approximating $\left.\frac{da(t)}{dt}\right|_{t=T}$ with $\frac{a(T)-a(t)}{T-t}$ we gain

$$-\frac{a(t)}{T-t} = -(-e^{rT} \cdot \kappa)^{-\frac{1}{\alpha}} \Rightarrow \kappa = -\left[\frac{a(t)}{T-t}\right]^{-\alpha} \cdot e^{-rT}$$

putting these parts together we can determine the optimal consumption as

$$c = (-\lambda)^{-\frac{1}{\alpha}} = (-e^{rt} \cdot \kappa)^{-\frac{1}{\alpha}} = (-e^{rt} \cdot -e^{-rT} \cdot \left[\frac{a(t)}{T-t}\right]^{-\alpha})^{-\frac{1}{\alpha}}$$

and finally get the result

$$c(t) = \frac{a(t)}{T-t} \cdot e^{\frac{r(T-t)}{\alpha}} \tag{56}$$

Hence the boundary condition on the boundary $y = y_{min}$ is $V(\tau^n, y_{min}, a) = \tau^n \tilde{u}(c_{y_{min}})$, where $p_i \in D \cap (y_{min}, \cdot)$. Here $c_{y_{min}} = \max(c_a, \min(c_{max}, \frac{a(t)}{T-t} \cdot e^{\frac{r(T-t)}{\alpha}}))$. To avoid technical difficulties, we set $V(\tau^n, y_{min}, a) = \tau^n \tilde{u}(c_a)$, where $a < w(\tau, y_{min})$. This will be explained in section (5.1.4).

Note that in the points $p_i = (y_{min}, a_{max})$ and $p_i = (y_{min}, w(\tau, y_{min}))$ this boundary condition fits with the above ones, which will be useful for stating the convergence of the discretization. This is easily seen by

$$\begin{aligned}
& c(t) \text{ at } (y_{min}, a_{max}) \\
= & \frac{a_{max}}{T-t} \cdot e^{\frac{r(T-t)}{\alpha}} \\
= & \frac{(T-t_0)\cdot c_{max}}{T-t} \cdot e^{\frac{r(T-t)}{\alpha}} \\
\geq & c_{max}
\end{aligned}$$

and

$$\begin{aligned}
& c(t) \text{ at } (y_{min}, w(t, y_{min})) \\
= & \frac{w(t, y_{min})}{T-t} \cdot e^{\frac{r(T-t)}{\alpha}} \\
\leq & c_a \text{ for } w(t, y_{min}) \leq \frac{(T-t)\cdot c_a}{e^{\frac{r(T-t)}{\alpha}}}
\end{aligned}$$

With this second inequation we get the boundary value for w at y_{min}. Together we see, we have a continuous boundary condition, fitting the boundary conditions of the points, where former boundary conditions were applied to.

In the boundary $y = y_{max}$ we make a typical assumption. We that guess the second order derivative in the outside direction vanishes. This guess will cause some errors in the boundary region, but if we choose y_{max} large enough the errors in regions of our interest will be small (see [FL07] and [BDR95]). If we take a closer look at the critical points (t,y_{max}, a_{max}) and (t,$y_{max}, w(t, y_{max})$) we see that the former given boudary values on these points are solutions of the HJB with vanishing second order derivative. So we have continuous boundary conditions on this side as well and all in all continuous boundary conditions on the whole computational domain.

5.1.3 The discretization matrix and its coefficients

For computational reasons it is useful to get a matrix form of discretizations (48) resp. (51).
First, we will derive a matrix G with $(\mathcal{L}^c_{\Delta y,\Delta a} v^n)_i = [G(c)v^n]_i$. Since we know

$$(\mathcal{L}^c_{\Delta y,\Delta a} v^n)_i = \alpha_i^n(c)v_{i-x}^n + \beta_i^n(c)v_{i-1}^n + \gamma_i^n(c)v_{i+1}^n + \delta_i^n(c)v_{i+x}^n - \zeta_i^n(c)v_i^n$$

it follows that G is a sparse matrix with only entries $\alpha_i, \beta_i, \gamma_i, \delta_i, -\zeta_i$. From the above equation we see that G has the following form (before appealing the boundary conditions and domain D and neglecting dependence of n)

$$G^n = \begin{pmatrix} -\zeta_1 \; \gamma_1 \; 0 \; \dots \; 0 \; \delta_1 \; 0 \; \dots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \\ \beta_2 \; -\zeta_2 \; \gamma_2 \; 0 \; \dots \; 0 \; \delta_2 \; 0 \; \dots \; 0 \; 0 \; 0 \; 0 \; 0 \\ 0 \; \beta_3 \; -\zeta_3 \; \gamma_3 \; 0 \; \dots \; 0 \; \delta_3 \; 0 \; \dots \; 0 \; 0 \; 0 \; 0 \\ \dots \\ \alpha_k \; 0 \; \dots \; 0 \; \beta_k \; -\zeta_k \; \gamma_k \; 0 \; \dots \; 0 \; \delta_k \; 0 \; \dots \; 0 \\ \dots \\ 0 \; 0 \; \dots \; 0 \; \alpha_l \; 0 \; \dots \; \beta_l \; -\zeta_l \; \gamma_l \; 0 \; \dots \; 0 \; \delta_l \\ \dots \\ 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; \dots \; 0 \; \alpha_P \; 0 \; \dots \; 0 \; \beta_P \; -\zeta_P \end{pmatrix}$$

It is useful to define the coefficients now and make sure assumption (5.1) holds for them. From (40) and the usage of forward/backward/central differencing for the discretization we obtain:

$$\alpha_i^n = \frac{\sigma^2}{2(\Delta a)^2} + \frac{\max(0, -\mu_Y)}{\Delta a} \tag{57}$$

$$\beta_i^n = \frac{\sigma^2}{2(\Delta y)^2} + \frac{\max(0, -\mu_X)}{\Delta y} \tag{58}$$

$$\gamma_i^n = \frac{\sigma^2}{2(\Delta y)^2} + \frac{\max(0, \mu_X)}{\Delta y} \tag{59}$$

$$\delta_i^n = \frac{\sigma^2}{2(\Delta a)^2} + \frac{\max(0, \mu_Y)}{\Delta a} \tag{60}$$

for forward/backward differencing and

$$\alpha_i^n = \frac{\sigma^2}{2(\Delta a)^2} - \frac{\mu_Y}{2\Delta a} \tag{61}$$

$$\beta_i^n = \frac{\sigma^2}{2(\Delta y)^2} - \frac{\mu_X}{2\Delta y} \tag{62}$$

$$\gamma_i^n = \frac{\sigma^2}{2(\Delta y)^2} + \frac{\mu_X}{2\Delta y} \tag{63}$$

$$\delta_i^n = \frac{\sigma^2}{2(\Delta a)^2} + \frac{\mu_Y}{2\Delta a} \tag{64}$$

for central differencing.

To get a discretization of higher order we try to use central differencing as much as possible (see [WF08]). However, we can only use central differencing if

$$|\mu_X(p_i, c, \tau)| \leq \frac{\sigma^2}{\Delta y} \text{ and}$$

$$|\mu_Y(p_i, c)| \leq \frac{\sigma^2}{\Delta a}$$

for all $c \in U$, to ensure the positive coefficient condition holds (see [OS05] and references therein). Otherwise we use forward or backward differencing, so the positive coefficient condition holds for all of our coefficients.

The above version of G does not contain any boundary conditions, so we have to manipulate the matrix.
We therefore introduce an additional vector $F^n \in \mathbb{R}^P$, containing the boundary values of V in the points p_i, where $a = a_{max}$, $y \in \{y_{min}\} \cap D$ or $p_i \in \partial D$ and for technical reasons for all points $p_i \notin D$,

$$F_i^n = \begin{cases} \tau^n \cdot \tilde{u}(c_{max}) & \text{for } i = P - x + 1, \dots, P, \\ \tau^n \cdot \tilde{u}(c_a) & \text{for } p_i \in \partial D \vee p_i + mx \in \partial D \text{ for } m > 1, \\ \tau^n \cdot \tilde{u}(c_{y_{min}}) & \text{for } i = mx + 1, m \geq 0, i \leq P - x + 1, \\ & p_i \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Next we set the rows of G to zero where $F = \tau^n \cdot \tilde{u}(c_{max})$, $F = \tau^n \cdot \tilde{u}(c_a)$ or $F = \tau^n \cdot \tilde{u}(c_{ymin})$.
For the term of the utility function we introduce a vector U^n with

$$U_j^n = \begin{cases} u(c_j^n) & \text{for } (n, p_j) \in D \cap ([t_0, T) \times (y_{min}, y_{max}] \times (a_{min}, a_{max})) \\ 0 & \text{otherwise} \end{cases}$$

and $U^T = 0_P$ since we do not have any final time utility.
Now with the matrices F and G and vector U, we can write the discretization (48) as

$$\begin{aligned} [\mathrm{I} - (1-\theta)\Delta\tau G^{n+1}(c^{n+1})]v^{n+1} = {} & [\mathrm{I} + \theta\Delta\tau G^n(c^n)]v^n + [F^{n+1} - F^n] \\ & + (1-\theta)\Delta\tau U^{n+1}(c^{n+1}) + \theta\Delta\tau U^n(c^n). \end{aligned}$$

Here I denotes the $P \times P$ identity matrix and c^n is detemined by

$$c^n \in \arg\sup_{c \in U} \{G^n(c)v^n + U^n(c)\}. \tag{65}$$

Since G consists only of the coefficients $\alpha, \beta, \gamma, \delta$ on the off-diagonals, where the entries are different from zero and of a combination $-(\alpha+\beta+\gamma+\delta)$ on the diagonal, where it is different from zero, the matrix $[\mathrm{I} - (1-\theta)\Delta\tau G^{n+1}(c^{n+1})]$ is a M-matrix.

Definition 5.2. ***M-matrix.***
A real $n \times n$ matrix M is called M-matrix if the following conditions hold for M (see e.g. [Bur06])*:*

- $m_{ij} \leq 0 \; \forall i \neq j$,
- $M^{-1} > 0$.

This is equivalent to (see [Var00])*:*

- $m_{ij} \leq 0 \; \forall i \neq j$,
- $m_{ii} > 0 \; \forall 1 \leq i \leq n$,
- *M is diagonally dominant.*

For the discretization of w we use the same techniques. We first derive a matrix H with $(\mathcal{L}_{\Delta y}^{c_i^n} w^n)_i = [H(c)w^n]_i$. Since we got only one spatial variable for w, we get only coefficients ψ_i^n and ν_i^n we have to discretize analogous to above, using central differencing as much as possible, and forward/backward differencing to ensure the positive coefficient condition for ψ_i^n and ν_i^n.
The coefficients are:

$$\psi_i^n = \frac{\sigma^2}{2(\Delta y)^2} + \frac{\max(0, -\mu_X)}{\Delta y} \tag{66}$$

$$\nu_i^n = \frac{\sigma^2}{2(\Delta y)^2} + \frac{\max(0, \mu_X)}{\Delta y} \tag{67}$$

$$\tag{68}$$

for forward/backward differencing and

$$\psi_i^n = \frac{\sigma^2}{2(\Delta y)^2} - \frac{\mu_X}{2\Delta y} \tag{69}$$

$$\nu_i^n = \frac{\sigma^2}{2(\Delta y)^2} + \frac{\mu_X}{2\Delta y} \tag{70}$$

$$\tag{71}$$

for central differencing, thus using central differencing if

$$|\frac{\mu_X}{2\Delta y}| \leq \frac{\sigma^2}{2(\Delta y)^2}. \tag{72}$$

As boundary conditions we use the approximations

$$\begin{aligned} w(0, \cdot) &= 0, \\ w(\tau, y_{min}) &= \frac{\tau c_a}{\exp(\frac{r\tau}{\alpha})}, \\ w(\tau, y_{max}) &= [\frac{c_a}{\exp(\frac{r\tau}{\alpha})} - \exp(y_{max})]\tau. \end{aligned}$$

The approximation for the upper boundary value is the solution of the control problem with a permanent (non-stochastic) income $\exp(y_{max})$. This is computed

analogous to the boundary value on the lower value.
To introduce them into the discretization, we set the first and the last row of H to zero and add a vector J^n, including the boundary values,

$$J^n = [\tau^n c_a \cdot \exp(-\frac{r\tau^n}{\alpha}),\ 0,\ \ldots,\ 0,\ (\frac{c_a}{\exp(\frac{r\tau^n}{\alpha})} - \exp(y_{max}))\tau^n]^t.$$

Now with the vector D^n

$$D_j^n = \begin{cases} d_j^n & \text{for } 1 < j < x, \\ 0 & j = 1, j = x \end{cases}$$

the discretization (51) of w can be written as

$$\begin{aligned} [\mathrm{I} - (1-\theta)\Delta\tau H^{n+1}(c^{n+1})]w^{n+1} &= [\mathrm{I} + \theta\Delta\tau H^n(c^n)]w^n + J^{n+1} - J^n \\ &+ (1-\theta)\Delta\tau D^{n+1}(c^{n+1}) + \theta\Delta\tau D^n(c^n) \end{aligned}$$

with $c^n = c_a$ for all n.

5.1.4 Convergence of the discretization

The last step before solving our problem numerically is now to show that the discretizations defined above are convergent to the viscosity solutions we are looking for. To do so, we first have to define what the convergence of the discretizations means and then prove that our discretizations fulfill this defintition. Let us begin with the basic definitions of stability, consistency and monotonicity:

Definition 5.3. ***Stability.***
The discretizations (48) *and* (51) *are stable if*

$$\|v^n\| \leq c_1,\ \|w^n\| \leq c_2,$$

for $0 \leq n \leq P$, $\Delta\tau \to 0$, $\Delta y \to 0$, $\Delta a \to 0$ *and* c_1 *resp.* c_2 *are independent of* $\Delta\tau$, Δy *and* Δa *resp.* $\Delta\tau$ *and* Δy.

For an easier notation of the consistency it is useful to define

$$\begin{aligned} K_i^{n+1}(v^{n+1}, v^n, \Delta y, \Delta a, \Delta\tau) := \frac{v_i^{n+1} - v_i^n}{\Delta\tau} &- (1-\theta) \sup_{c^{n+1}\in U} \{(\mathcal{L}_{\Delta y,\Delta a}^{c^{n+1}} v^{n+1})_i + u_i^{n+1}\} \\ &- \theta \sup_{c^n \in U} \{(\mathcal{L}_{\Delta y,\Delta a}^{c^n} v^n)_i + u_i^n\} \end{aligned}$$

and

$$\begin{aligned} L_j^{n+1}(w^{n+1}, w^n, \Delta y, \Delta\tau) := \frac{w_j^{n+1} - w_j^n}{\Delta\tau} &- (1-\theta) \inf_{c^{n+1}\in U} \{(\mathcal{L}_{\Delta y}^{c^{n+1}} w^{n+1})_j + d_j^{n+1}\} \\ &- \theta \inf_{c^n \in U} \{(\mathcal{L}_{\Delta y}^{c^n} w^n)_j + d_j^n\} \end{aligned}$$

Now we are able to define the consistency condition as follows:

Definition 5.4. ***Consistency.***
The discretizations (48) *and* (51) *are consistent if*

$$\lim_{\substack{\Delta\tau\to 0\\ \Delta y\to 0\\ \Delta a\to 0}} |(\phi_\tau - \sup_{c\in U}\{\mathcal{L}^c\phi + u\})_i^{n+1} - K_i^{n+1}(\phi^{n+1}, \phi^n, \Delta y, \Delta a, \Delta\tau)| = 0$$

and

$$\lim_{\substack{\Delta\tau\to 0\\ \Delta y\to 0}} |(\phi_\tau - \inf_{c\in U}\{\mathcal{L}_X^c\phi - r\phi + d\})_i^{n+1} - L_i^{n+1}(\phi^{n+1}, \phi^n, \Delta y, \Delta\tau)| = 0,$$

for any smooth $\phi \in C^{1,2}$, *with* $\phi_i^n = \phi(p_i, \tau^n)$, $\phi^n = [\phi_1, \ldots, \phi_P]^t$, *with* $d = -exp(y) + c$, *where* y *is the* y*-component of* p_i.

Note that for the consistency we defined above it is necessary to have continuous boundary conditions. Otherwise we had to define consistency for a discontinuous case, which would have created some technical difficulties, we avoided by the choice of the values for V on the boundary. In the same way we set the value of V in the regions $\notin D$ to the same value as on the boundary to avoid a discontinuity here. By multiplication of the utility with τ^n, which is done, since we consume the same amount in every period, we avoid a discontinuity of V in the limit $\tau^n \to 0$. The same takes place on the critical points of w. So we just need this continuous consistency definition.

The last condition we need to define in order to prove the convergence is the monotonicity property:

Definition 5.5. ***Monotonicity.***
The discretizations (48) *and* (51) *are monotone if for all* i *and* $\epsilon^l = [\epsilon_1^l, \ldots, \epsilon_P^l]^t \in \mathbb{R}^P$, $\epsilon_j^l \geq 0$, $\epsilon_i^{n+1} = 0$

$$K_i^{n+1}(v^{n+1} + \epsilon^{n+1}, v^n + \epsilon^n, \Delta y, \Delta a, \Delta\tau) \leq K_i^{n+1}(v^{n+1}, v^n, \Delta y, \Delta a, \Delta\tau),$$

resp.

$$L_i^{n+1}(w^{n+1} + \epsilon^{n+1}, w^n + \epsilon^n, \Delta y, \Delta\tau) \leq L_i^{n+1}(w^{n+1}, w^n, \Delta y, \Delta\tau).$$

After defining the basic conditions for convergence, we now will verify them for our system. This proofs follow the ideas of [FL07].

Theorem 5.6. *The discretizations* (48) *and* (51) *are stable according to Definition* (5.3), *if* $\alpha_i^n, \beta_i^n, \gamma_i^n, \delta_i^n, \psi_i^n, \nu_i^n$ *and* r *fulfill the positive coefficent condition and* $\theta\Delta\tau(\alpha_i^n + \beta_i^n + \gamma_i^n + \delta_i^n) \leq 1$ *resp.* $\theta\Delta\tau(\psi_i^n + \nu_i^n + r) \leq 1$ *for all* i.

Proof. We begin with the stability of (48).

In the inner domain (i.e. $int(D) \cap]y_{min}, y_{max}[\times]a_{min}, a_{max}[$) we get from (48):

$$\begin{aligned}
\frac{v_i^{n+1} - v_i^n}{\Delta\tau} &= (1-\theta) \sup_{c^{n+1} \in U} \{(\mathcal{L}^{c^{n+1}}_{\Delta y, \Delta a} v^{n+1})_i + u_i^{n+1}\} + \theta \sup_{c^n \in U} \{(\mathcal{L}^{c^n}_{\Delta y, \Delta a} v^n)_i + u_i^n\} \\
\Leftrightarrow v_i^{n+1} &= v_i^n + \theta\Delta\tau[\alpha_i^n v_{i-x}^n + \beta_i^n v_{i-1}^n + \gamma_i^n v_{i+1}^n + \delta_i^n v_{i+x}^n - \zeta_i^n v_i^n + u_i^n] \\
&+ \Delta\tau(1-\theta)[\alpha_i^{n+1} v_{i-x}^{n+1} + \beta_i^{n+1} v_{i-1}^{n+1} + \gamma_i^{n+1} v_{i+1}^{n+1} + \delta_i^{n+1} v_{i+x}^{n+1} - \zeta_i^{n+1} v_i^{n+1} \\
&\quad + u_i^{n+1}] \\
\Leftrightarrow v_i^{n+1} &= v_i^n + \theta\Delta\tau[\alpha_i^n v_{i-x}^n + \beta_i^n v_{i-1}^n + \gamma_i^n v_{i+1}^n + \delta_i^n v_{i+x}^n + u_i^n] \\
&\quad + \Delta\tau(1-\theta)[\alpha_i^{n+1} v_{i-x}^{n+1} + \beta_i^{n+1} v_{i-1}^{n+1} + \gamma_i^{n+1} v_{i+1}^{n+1} + \delta_i^{n+1} v_{i+x}^{n+1} + u_i^{n+1}] \\
&\quad - \theta\Delta\tau[\alpha_i^n + \beta_i^n + \gamma_i^n + \delta_i^n] v_i^n \\
&\quad - \Delta\tau(1-\theta)[\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1} + \delta_i^{n+1}] v_i^{n+1},
\end{aligned}$$

where the coefficients are the ones for the maximizing control. So we get

$$\begin{aligned}
&(1 + \Delta\tau(1-\theta)[\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1} + \delta_i^{n+1}])|v_i^{n+1}| \\
\leq\ & \Delta\tau(1-\theta)[\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1} + \delta_i^{n+1}]\|v^{n+1}\|_\infty + \Delta\tau((1-\theta)u_i^{n+1} + \theta u_i^n) \\
&+ \theta\Delta\tau[\alpha_i^n + \beta_i^n + \gamma_i^n + \delta_i^n]\|v^n\|_\infty + (1 - \theta\Delta\tau[\alpha_i^n + \beta_i^n + \gamma_i^n + \delta_i^n])\|v^n\|_\infty \\
=\ & \|v^n\|_\infty + \Delta\tau(1-\theta)[\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1} + \delta_i^{n+1}]\|v^{n+1}\|_\infty \\
&+ \Delta\tau((1-\theta)u_i^{n+1} + \theta u_i^n).
\end{aligned}$$

Now if $\|v^{n+1}\|_\infty = |v_j^{n+1}|$ with p_j in the inner domain then it follows that

$$\begin{aligned}
&(1 + \Delta\tau(1-\theta)[\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1} + \delta_i^{n+1}])|v_i^{n+1}| \\
\leq\ & \|v^n\|_\infty + \Delta\tau(1-\theta)[\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1} + \delta_i^{n+1}]|v_j^{n+1}| + \Delta\tau((1-\theta)u_i^{n+1} + \theta u_i^n) \\
&\text{for all inner points } p_i \\
\Rightarrow\ & (1 + \Delta\tau(1-\theta)[\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1} + \delta_i^{n+1}])|v_j^{n+1}| \\
\leq\ & \|v^n\|_\infty + \Delta\tau(1-\theta)[\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1} + \delta_i^{n+1}]|v_j^{n+1}| + \Delta\tau((1-\theta)u_i^{n+1} + \theta u_i^n) \\
\Rightarrow\ & \|v^{n+1}\|_\infty = |v_j^{n+1}| \leq \|v^n\|_\infty + \Delta\tau((1-\theta)\|u^{n+1}\|_\infty + \theta\|u^n\|_\infty) \\
\leq\ & \|v^n\|_\infty + \Delta\tau(1-\theta)\max_n(\|u^{n+1}\|_\infty) + \theta\max_n(\|u^n\|_\infty) \\
=\ & \|v^n\|_\infty + \Delta\tau\max_{n,i}(|u_i^n|)
\end{aligned}$$

If $\|v^{n+1}\|_\infty = |v_j^{n+1}|$ with p_j on the boundary of the domain then we obtain

$$\|v^{n+1}\|_\infty = \|F^{n+1}\|_\infty$$

Together we obtain the inequality

$$\begin{aligned}
&\|v^{n+1}\|_\infty \leq \max\{\|v^n\|_\infty + \Delta\tau\max_{n,i}(|u_i^n|), \|F^{n+1}\|_\infty\} \\
\Rightarrow\ & \|v^n\|_\infty \leq \max\{\|v^{n-1}\|_\infty + \Delta\tau\max_{n,i}(|u_i^n|), \|F^n\|_\infty\}
\end{aligned}$$

and so with $\|F\|_\infty := \max_n \|F^n\|_\infty$

$$\|v^n\|_\infty \le \max\{\|v^0\|_\infty + T\max_{n,i}(|u_i^n|), \|F\|_\infty\}$$

Since $v^0 = V(T, p_i)$ and we easily see $\|F^n\|_\infty = |\tau^n \cdot \tilde{u}(c_{max})|$ it follows $\|F\|_\infty = T \cdot \tilde{u}(c_{max})$ and since $\max_{n,i}(|u_i^n|)$ is independent of $\Delta\tau, \Delta y$ and Δa we have an estimate for $\|v^n\|_\infty$ independent of $\Delta\tau, \Delta y$ and Δa.

For the stability of w we use the same steps.
From (51) we get in the inner domain $]y_{min}, y_{max}[$

$$\begin{aligned}
\frac{w_j^{n+1} - w_j^n}{\Delta\tau} &= (1-\theta)\inf_{c^{n+1}\in U}\{(\mathcal{L}_{\Delta y}^{c^{n+1}} w^{n+1})_j + d_j^{n+1}\} + \theta \inf_{c^n\in U}\{(\mathcal{L}_{\Delta y}^{c^n} w^n)_j + d_j^n\} \\
\Leftrightarrow w_i^{n+1} &= w_i^n + \theta\Delta\tau[\psi_i^n w_{i-1}^n + \nu_i^n w_{i+1}^n - (\psi_i^n + \nu_i^n + r)w_i^n + d_i^n] \\
&\quad + \Delta\tau(1-\theta)[\psi_i^{n+1} w_{i-1}^{n+1} + \nu_i^{n+1} w_{i+1}^{n+1} - (\psi_i^{n+1} + \nu_i^{n+1} + r)w_i^{n+1} \\
&\quad + d_i^{n+1}] \\
\Leftrightarrow w_i^{n+1} &= w_i^n + \theta\Delta\tau[\psi_i^n w_{i-1}^n + \nu_i^n w_{i+1}^n + d_i^n] \\
&\quad + \Delta\tau(1-\theta)[\psi_i^{n+1} w_{i-1}^{n+1} + \nu_i^{n+1} w_{i+1}^{n+1} + d_i^{n+1}] \\
&\quad - \theta\Delta\tau[\psi_i^n + \nu_i^n + r]w_i^n \\
&\quad - \Delta\tau(1-\theta)[\psi_i^{n+1} + \nu_i^{n+1} + r]w_i^{n+1},
\end{aligned}$$

where the coefficients are the ones for the minimizing control. So we get as above

$$\begin{aligned}
&(1 + \Delta\tau(1-\theta)[\psi_i^{n+1} + \nu_i^{n+1} + r])|w_i^{n+1}| \\
\le\ & \|w^n\|_\infty + \Delta\tau(1-\theta)[\psi_i^{n+1} + \nu_i^{n+1} + r]\|w^{n+1}\|_\infty + |[\Delta\tau(1-\theta)d_i^{n+1} + \Delta\tau\theta d_i^n]|.
\end{aligned}$$

Now if $\|w^{n+1}\|_\infty = |w_j^{n+1}|$ with $j \ne \{1, x\}$, so p_j lies in the interior of the domain, then it follows like above

$$\begin{aligned}
\|w^{n+1}\|_\infty &\le \|w^n\|_\infty + \Delta\tau((1-\theta)d_i^{n+1} + \theta d_i^n) \\
&\le \|w^n\|_\infty + \Delta\tau((1-\theta)\|d^{n+1}\|_\infty + \theta\|d^n\|_\infty) \\
&\le \|w^n\|_\infty + \Delta\tau((1-\theta)\max_n\{\|d^{n+1}\|_\infty\} + \theta\max_n\{\|d^n\|_\infty\}) \\
&= \|w^n\|_\infty + \Delta\tau\max_{n,i}\{|d_i^n|\}.
\end{aligned}$$

If $\|w^{n+1}\|_\infty = |w_j^{n+1}|$ with p_j on the boundary of the domain we obtain

$$\|w^{n+1}\|_\infty = \|J^{n+1}\|_\infty.$$

Finally we obtain the inequality

$$\begin{aligned}
&\|w^{n+1}\|_\infty \le \max\{\|w^n\|_\infty + \Delta\tau\max_{n,i}\{|d_i^n|\}, \|J^{n+1}\|_\infty\} \\
\Rightarrow\ &\|w^n\|_\infty \le \max\{\|w^{n-1}\|_\infty + \Delta\tau\max_{n,i}\{|d_i^n|\}, \|J^n\|_\infty\}
\end{aligned}$$

and hence, with $\|J\|_\infty := \max_n \|J^n\|_\infty$

$$\|w^n\|_\infty \leq \max\{\|w^0\|_\infty + T \max_{n,i}\{|d_i^n|\}, \|J\|_\infty\}$$

Since $w^0 = w(T, \cdot)$ and we easily see $\|J^n\|_\infty = |\tau^n \cdot (c_a - \exp(y_{max}))|$ for y_{max} large enough, it follows $\|J\|_\infty = T \cdot |(c_a - \exp(y_{max}))|$ and since $\max_{n,i}\{|d_i^n|\}$ is independent of $\Delta\tau$ and Δy, we have an estimate for $\|w^n\|_\infty$ independent of $\Delta\tau$ and Δy. □

Theorem 5.7. *The discretizations* (48) *and* (51) *are monotone according to Definition (5.5), if $\alpha_i^n, \beta_i^n, \gamma_i^n, \delta_i^n, \psi_i^n, \eta_i^n$ and r fulfill the positive coefficent condition and $\theta\Delta\tau(\alpha_i^n + \beta_i^n + \gamma_i^n + \delta_i^n) \leq 1$ resp. $\theta\Delta\tau(\psi_i^n + \eta_i^n + r) \leq 1$ for all i.*

Proof. First we show the monotonicity for (48), by using the matrix notations K as introduced above. We split up the proof and first show that $K_i^{n+1}(v^{n+1} + \epsilon^{n+1}, v^n, \Delta y, \Delta a, \Delta\tau) \leq K_i^{n+1}(v^{n+1}, v^n, \Delta y, \Delta a, \Delta\tau)$ and then for the second component $K_i^{n+1}(v^{n+1}, v^n + \epsilon^n, \Delta y, \Delta a, \Delta\tau) \leq K_i^{n+1}(v^{n+1}, v^n, \Delta y, \Delta a, \Delta\tau)$, with ϵ^{n+1} and ϵ^n as described in the definition (5.5), which in addition shows the monotonicity of (48).

Let $\epsilon \in \mathbb{R}^P$ and K_i^{n+1} as above. Then we deduce (surpressing the dependence of c for the coefficients)

$$
\begin{aligned}
& K_i^{n+1}(v^{n+1}+\epsilon^{n+1}, v^n, \Delta y, \Delta a, \Delta\tau) - K_i^{n+1}(v^{n+1}, v^n, \Delta y, \Delta a, \Delta\tau) \\
= \quad & -(1-\theta) \sup_{c^{n+1}\in U}\{\alpha_i^{n+1} v_{i-x}^{n+1} + \beta_i^{n+1} v_{i-1}^{n+1} + \gamma_i^{n+1} v_{i+1}^{n+1} + \delta_i^{n+1} v_{i+x}^{n+1} - \zeta_i^{n+1} v_i^{n+1} \\
+ \quad & \alpha_i^{n+1}\epsilon_{i-x}^{n+1} + \beta_i^{n+1}\epsilon_{i-1}^{n+1} + \gamma_i^{n+1}\epsilon_{i+1}^{n+1} + \delta_i^{n+1}\epsilon_{i+x}^{n+1} + u_i^{n+1}\} \\
- \quad & \theta \sup_{c^n\in U}\{\alpha_i^n v_{i-x}^n + \beta_i^n v_{i-1}^n + \gamma_i^n v_{i+1}^n + \delta_i^n v_{i+x}^n - \zeta_i^n v_i^n u_i^n\} \\
+ \quad & (1-\theta) \sup_{c^{n+1}\in U}\{\alpha_i^{n+1} v_{i-x}^{n+1} + \beta_i^{n+1} v_{i-1}^{n+1} + \gamma_i^{n+1} v_{i+1}^{n+1} + \delta_i^{n+1} v_{i+x}^{n+1} - \zeta_i^{n+1} v_i^{n+1} \\
& +u_i^{n+1}\} + \theta \sup_{c^n\in U}\{\alpha_i^n v_{i-x}^n + \beta_i^n v_{i-1}^n + \gamma_i^n v_{i+1}^n + \delta_i^n v_{i+x}^n - \zeta_i^n v_i^n + u_i^n\} \\
= \quad & (1-\theta) \inf_{c^{n+1}\in U}\{-(\alpha_i^{n+1}\epsilon_{i-x}^{n+1} + \beta_i^{n+1}\epsilon_{i-1}^{n+1} + \gamma_i^{n+1}\epsilon_{i+1}^{n+1} + \delta_i^{n+1}\epsilon_{i+x}^{n+1}) \\
- \quad & (\alpha_i^{n+1} v_{i-x}^{n+1} + \beta_i^{n+1} v_{i-1}^{n+1} + \gamma_i^{n+1} v_{i+1}^{n+1} + \delta_i^{n+1} v_{i+x}^{n+1} - \zeta_i^{n+1} v_i^{n+1} + u_i^{n+1})\} \\
+ \quad & (1-\theta) \sup_{c^{n+1}\in U}\{\alpha_i^{n+1} v_{i-x}^{n+1} + \beta_i^{n+1} v_{i-1}^{n+1} + \gamma_i^{n+1} v_{i+1}^{n+1} + \delta_i^{n+1} v_{i+x}^{n+1} - \zeta_i^{n+1} v_i^{n+1} \\
& +u_i^{n+1}\} \\
\leq \quad & (1-\theta)[\sup_{c^{n+1}\in U}\{-(\alpha_i^{n+1}\epsilon_{i-x}^{n+1} + \beta_i^{n+1}\epsilon_{i-1}^{n+1} + \gamma_i^{n+1}\epsilon_{i+1}^{n+1} + \delta_i^{n+1}\epsilon_{i+x}^{n+1}) \\
- \quad & \sup_{c^{n+1}\in U}\{\alpha_i^{n+1} v_{i-x}^{n+1} + \beta_i^{n+1} v_{i-1}^{n+1} + \gamma_i^{n+1} v_{i+1}^{n+1} + \delta_i^{n+1} v_{i+x}^{n+1} - \zeta_i^{n+1} v_i^{n+1} + u_i^{n+1}\}] \\
+ \quad & (1-\theta) \sup_{c^{n+1}\in U}\{\alpha_i^{n+1} v_{i-x}^{n+1} + \beta_i^{n+1} v_{i-1}^{n+1} + \gamma_i^{n+1} v_{i+1}^{n+1} + \delta_i^{n+1} v_{i+x}^{n+1} - \zeta_i^{n+1} v_i^{n+1} \\
& +u_i^{n+1}\} \\
= \quad & (1-\theta) \sup_{c^{n+1}\in U}\{-(\alpha_i^{n+1}\epsilon_{i-x}^{n+1} + \beta_i^{n+1}\epsilon_{i-1}^{n+1} + \gamma_i^{n+1}\epsilon_{i+1}^{n+1} + \delta_i^{n+1}\epsilon_{i+x}^{n+1})\} \\
= \quad & -(1-\theta) \inf_{c^{n+1}\in U}\{\alpha_i^{n+1}\epsilon_{i-x}^{n+1} + \beta_i^{n+1}\epsilon_{i-1}^{n+1} + \gamma_i^{n+1}\epsilon_{i+1}^{n+1} + \delta_i^{n+1}\epsilon_{i+x}^{n+1}\} \\
\leq \quad & -(1-\theta) \inf_{c^{n+1}\in U}\{\min\{\epsilon_{i-x}^{n+1}, \epsilon_{i-1}^{n+1}, \epsilon_{i+1}^{n+1}, \epsilon_{i+x}^{n+1}\}(\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1} + \delta_i^{n+1})\} \\
= \quad & -(1-\theta)\min\{\epsilon_{i-x}^{n+1}, \epsilon_{i-1}^{n+1}, \epsilon_{i+1}^{n+1}, \epsilon_{i+x}^{n+1}\} \inf_{c^{n+1}\in U}\{\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1} + \delta_i^{n+1})\} \\
\leq \quad & 0
\end{aligned}
$$

Now let $\epsilon \in \mathbb{R}^P$ and K_i^n be defined as above. Then we conclude

$$\begin{aligned}
& K_i^{n+1}(v^{n+1}, v^n + \epsilon^n, \Delta y, \Delta a, \Delta\tau) - K_i^{n+1}(v^{n+1}, v^n, \Delta y, \Delta a, \Delta\tau) \\
= & - \frac{\epsilon_i^n}{\Delta\tau} \\
& - (1-\theta) \sup_{c^{n+1}\in U} \{\alpha_i^{n+1} v_{i-x}^{n+1} + \beta_i^{n+1} v_{i-1}^{n+1} + \gamma_i^{n+1} v_{i+1}^{n+1} + \delta_i^{n+1} v_{i+x}^{n+1} - \zeta_i^{n+1} v_i^{n+1} \\
& + u_i^{n+1}\} \\
& - \theta \sup_{c^n\in U} \{\alpha_i^n v_{i-x}^n + \beta_i^n v_{i-1}^n + \gamma_i^n v_{i+1}^n + \delta_i^n v_{i+x}^n - \zeta_i^n v_i^n + u_i^n \\
& + \alpha_i^n \epsilon_{i-x}^n + \beta_i^n \epsilon_{i-1}^n + \gamma_i^n \epsilon_{i+1}^n + \delta_i^n \epsilon_{i+x}^n - \zeta_i^n \epsilon_i^n\} \\
& + (1-\theta) \sup_{c^{n+1}\in U} \{\alpha_i^{n+1} v_{i-x}^{n+1} + \beta_i^{n+1} v_{i-1}^{n+1} + \gamma_i^{n+1} v_{i+1}^{n+1} + \delta_i^{n+1} v_{i+x}^{n+1} - \zeta_i^{n+1} v_i^{n+1} \\
& + u_i^{n+1}\} \\
& + \theta \sup_{c^n\in U} \{\alpha_i^n v_{i-x}^n + \beta_i^n v_{i-1}^n + \gamma_i^n v_{i+1}^n + \delta_i^n v_{i+x}^n - \zeta_i^n v_i^n + u_i^n\} \\
= & - \frac{\epsilon_i^n}{\Delta\tau} + \theta \inf_{c^{n+1}\in U} \{-(\alpha_i^n \epsilon_{i-x}^n + \beta_i^n \epsilon_{i-1}^n + \gamma_i^n \epsilon_{i+1}^n + \delta_i^n \epsilon_{i+x}^n - \zeta_i^n \epsilon_i^n) \\
& - (\alpha_i^n v_{i-x}^n + \beta_i^n v_{i-1}^n + \gamma_i^n v_{i+1}^n + \delta_i^n v_{i+x}^n - \zeta_i^n v_i^n + u_i^{n+1})\} \\
& + \theta \sup_{c^n\in U} \{\alpha_i^n v_{i-x}^n + \beta_i^n v_{i-1}^n + \gamma_i^n v_{i+1}^n + \delta_i^n v_{i+x}^n - \zeta_i^n v_i^n + u_i^{n+1}\} \\
\leq & - \frac{\epsilon_i^n}{\Delta\tau} + \theta[\sup_{c^n\in U} \{-(\alpha_i^n \epsilon_{i-x}^n + \beta_i^n \epsilon_{i-1}^n + \gamma_i^n \epsilon_{i+1}^n + \delta_i^n \epsilon_{i+x}^n - \zeta_i^n \epsilon_i^n) \\
& - \sup_{c^n\in U} \{\alpha_i^n v_{i-x}^n + \beta_i^n v_{i-1}^n + \gamma_i^n v_{i+1}^n + \delta_i^n v_{i+x}^n - \zeta_i^n v_i^n\}] \\
& + \theta \sup_{c^n\in U} \{\alpha_i^n v_{i-x}^n + \beta_i^n v_{i-1}^n + \gamma_i^n v_{i+1}^n + \delta_i^n v_{i+x}^n - \zeta_i^n v_i^n\} \\
= & - \frac{\epsilon_i^n}{\Delta\tau} + \theta \sup_{c^n\in U} \{-\alpha_i^n \epsilon_{i-x}^n - \beta_i^n \epsilon_{i-1}^n - \gamma_i^n \epsilon_{i+1}^n - \delta_i^n \epsilon_{i+x}^n + \zeta_i^n \epsilon_i^n\} \\
\leq & - \frac{\epsilon_i^n}{\Delta\tau} + \theta \sup_{c^n\in U} \{\zeta_i^n \epsilon_i^n\} + \theta \sup_{c^n\in U} \{-\alpha_i^n \epsilon_{i-x}^n - \beta_i^n \epsilon_{i-1}^n - \gamma_i^n \epsilon_{i+1}^n - \delta_i^n \epsilon_{i+x}^n\} \\
\leq & - \frac{\epsilon_i^n}{\Delta\tau} + \theta \frac{\epsilon_i^n}{\theta\Delta\tau} - \theta \inf_{c^n\in U} \{\alpha_i^n \epsilon_{i-x}^n + \beta_i^n \epsilon_{i-1}^n + \gamma_i^n \epsilon_{i+1}^n + \delta_i^n \epsilon_{i+x}^n\} \\
\leq & - \theta \min\{\epsilon_{i-x}^n, \epsilon_{i-1}^n, \epsilon_{i+1}^n, \epsilon_{i+x}^n\} \inf_{c^n\in U} \{\alpha_i^n + \beta_i^n + \gamma_i^n + \delta_i^n\} \\
\leq & \quad 0
\end{aligned}$$

and therefore (48) is monotone as defined above.
For the monotonicity of w we have quite similar estimations. Let L^{n+1}, L^n and ϵ be as defined above, with $\epsilon_i^{n+1} = 0$. Be $\theta\Delta\tau(\psi_i^n + \nu_i^n + r) \leq 1$ for all i. We get the

estimate $L_i^{n+1}(w_i^{n+1}+\epsilon_i^{n+1}, w_i^n, \Delta y, \Delta\tau) - L(w_i^{n+1}, w_i^n, \Delta y, \Delta\tau) \leq 0$ by

$$
\begin{aligned}
& L_i^{n+1}(w_i^{n+1}+\epsilon_i^{n+1}, w_i^n, \Delta y, \Delta\tau) - L(w_i^{n+1}, w_i^n, \Delta y, \Delta\tau) \\
= & -\ (1-\theta) \inf_{c^{n+1}\in U}\{\psi_i^{n+1}w_{i-1}^{n+1} + \nu_i^{n+1}w_{i+1}^{n+1} - (\psi_i^{n+1}+\nu_i^{n+1}+r)w_i^{n+1} + d_i^{n+1} \\
& +\ \psi_i^{n+1}\epsilon_{i-1}^{n+1} + \nu_i^{n+1}\epsilon_{i+1}^{n+1}\} \\
& -\ \theta \inf_{c^n\in U}\{\psi_i^n w_{i-1}^n + \nu_i^n w_{i+1}^n - (\psi_i^n+\nu_i^n+r)w_i^n + d_i^n\} \\
& +\ (1-\theta) \inf_{c^{n+1}\in U}\{\psi_i^{n+1}w_{i-1}^{n+1} + \nu_i^{n+1}w_{i+1}^{n+1} - (\psi_i^{n+1}+\nu_i^{n+1}+r)w_i^{n+1} + d_i^{n+1}\} \\
& +\ \theta \inf_{c^n\in U}\{\psi_i^n w_{i-1}^n + \nu_i^n w_{i+1}^n - (\psi_i^n+\nu_i^n+r)w_i^n + d_i^n\} \\
= & \quad (1-\theta) \sup_{c^{n+1}\in U}\{(\psi_i^{n+1}+\nu_i^{n+1}+r)w_i^{n+1} - \nu_i^{n+1}w_{i+1}^{n+1} - \psi_i^{n+1}w_{i-1}^{n+1} - d_i^{n+1} \\
& -\ \psi_i^{n+1}\epsilon_{i-1}^{n+1} - \nu_i^{n+1}\epsilon_{i+1}^{n+1}\} \\
& -\ (1-\theta) \sup_{c^{n+1}\in U}\{-\psi_i^{n+1}w_{i-1}^{n+1} - \nu_i^{n+1}w_{i+1}^{n+1} + (\psi_i^{n+1}+\nu_i^{n+1}+r)w_i^{n+1} - d_i^{n+1}\} \\
\leq & \quad (1-\theta)[\sup_{c^{n+1}\in U}\{(\psi_i^{n+1}+\nu_i^{n+1}+r)w_i^{n+1} - \nu_i^{n+1}w_{i+1}^{n+1} - \psi_i^{n+1}w_{i-1}^{n+1} - d_i^{n+1}\} \\
& -\ \sup_{c^{n+1}\in U}\{\psi_i^{n+1}\epsilon_{i-1}^{n+1} + \nu_i^{n+1}\epsilon_{i+1}^{n+1}\}] \\
& -\ (1-\theta) \sup_{c^{n+1}\in U}\{-\psi_i^{n+1}w_{i-1}^{n+1} - \nu_i^{n+1}w_{i+1}^{n+1} + (\psi_i^{n+1}+\nu_i^{n+1}+r)w_i^{n+1} - d_i^{n+1}\} \\
= & -\ (1-\theta) \sup_{c^{n+1}\in U}\{\psi_i^{n+1}\epsilon_{i-1}^{n+1} + \nu_i^{n+1}\epsilon_{i+1}^{n+1}\} \\
= & \quad (1-\theta) \inf_{c^{n+1}\in U}\{-\psi_i^{n+1}\epsilon_{i-1}^{n+1} - \nu_i^{n+1}\epsilon_{i+1}^{n+1}\} \\
\leq & \quad (1-\theta)\min\{\epsilon_{i-1}^{n+1}, \epsilon_{i+1}^{n+1}\} \inf_{c^{n+1}\in U}\{-(\psi_i^{n+1}+\nu_i^{n+1})\} \\
\leq & \quad 0.
\end{aligned}
$$

In the other component we get the equations

$$
\begin{aligned}
& L_i^{n+1}(w_i^{n+1}, w_i^n + \epsilon_i^n, \Delta y, \Delta\tau) - L(w_i^{n+1}, w_i^n, \Delta y, \Delta\tau) \\
= & - \frac{\epsilon_i^n}{\Delta\tau} \\
& - (1-\theta) \inf_{c^{n+1}\in U} \{\psi_i^{n+1} w_{i-1}^{n+1} + \nu_i^{n+1} w_{i+1}^{n+1} - (\psi_i^{n+1} + \nu_i^{n+1} + r) w_i^{n+1} + d_i^{n+1}\} \\
& - \theta \inf_{c^n\in U} \{\psi_i^n w_{i-1}^n + \nu_i^n w_{i+1}^n - (\psi_i^n + \nu_i^n + r) w_i^n + d_i^n \\
& + \psi_i^n \epsilon_{i-1}^n + \nu_i^n \epsilon_{i+1}^n - (\psi_i^n + \nu_i^n + r)\epsilon_i^n)\} \\
& + (1-\theta) \inf_{c^{n+1}\in U} \{\psi_i^{n+1} w_{i-1}^{n+1} + \nu_i^{n+1} w_{i+1}^{n+1} - (\psi_i^{n+1} + \nu_i^{n+1} + r) w_i^{n+1} + d_i^{n+1}\} \\
& + \theta \inf_{c^n\in U} \{\psi_i^n w_{i-1}^n + \nu_i^n w_{i+1}^n - (\psi_i^n + \nu_i^n + r) w_i^n + d_i^n\} \\
= & - \frac{\epsilon_i^n}{\Delta\tau} + \theta \sup_{c^n\in U} \{-\psi_i^n w_{i-1}^n - \nu_i^n w_{i+1}^n + (\psi_i^n + \nu_i^n + r) w_i^n - d_i^n \\
& - (\psi_i^n \epsilon_{i-1}^n + \nu_i^n \epsilon_{i+1}^n - (\psi_i^n + \nu_i^n + r)\epsilon_i^n)\} \\
& - \theta \sup_{c^n\in U} \{-\psi_i^n w_{i-1}^n - \nu_i^n w_{i+1}^n + (\psi_i^n + \nu_i^n + r) w_i^n - d_i^n\} \\
\leq & - \frac{\epsilon_i^n}{\Delta\tau} + \theta \sup_{c^n\in U} \{-\psi_i^n w_{i-1}^n - \nu_i^n w_{i+1}^n + (\psi_i^n + \nu_i^n + r) w_i^n - d_i^n\} \\
& - \theta \sup_{c^n\in U} \{\psi_i^n \epsilon_{i-1}^n + \nu_i^n \epsilon_{i+1}^n - (\psi_i^n + \nu_i^n + r)\epsilon_i^n)\} \\
& - \theta \sup_{c^n\in U} \{-\psi_i^n w_{i-1}^n - \nu_i^n w_{i+1}^n + (\psi_i^n + \nu_i^n + r) w_i^n - d_i^n\} \\
= & - \frac{\epsilon_i^n}{\Delta\tau} - \theta \sup_{c^n\in U} \{\psi_i^n \epsilon_{i-1}^n + \nu_i^n \epsilon_{i+1}^n - (\psi_i^n + \nu_i^n + r)\epsilon_i^n)\} \\
= & - \frac{\epsilon_i^n}{\Delta\tau} + \theta \inf_{c^n\in U} \{(\psi_i^n + \nu_i^n + r)\epsilon_i^n) - (\psi_i^n \epsilon_{i-1}^n + \nu_i^n \epsilon_{i+1}^n)\} \\
\leq & - \frac{\epsilon_i^n}{\Delta\tau} + \theta \sup_{c^n\in U} \{(\psi_i^n + \nu_i^n + r)\epsilon_i^n)\} - \theta \sup_{c^n\in U} \{\psi_i^n \epsilon_{i-1}^n + \nu_i^n \epsilon_{i+1}^n\} \\
\leq & - \frac{\epsilon_i^n}{\Delta\tau} + \theta \frac{\epsilon_i^n}{\theta\Delta\tau} - \theta \sup_{c^n\in U} \{\psi_i^n \epsilon_{i-1}^n + \nu_i^n \epsilon_{i+1}^n\} \\
= & - \theta \sup_{c^n\in U} \{\psi_i^n \epsilon_{i-1}^n + \nu_i^n \epsilon_{i+1}^n\} \\
= & \ \theta \inf_{c^n\in U} \{-\psi_i^n \epsilon_{i-1}^n - \nu_i^n \epsilon_{i+1}^n\} \\
\leq & \ \theta \min\{\epsilon_{i-1}^n, \epsilon_{i+1}^n\} \inf_{c^n\in U} \{-(\psi_i^n + \nu_i^n)\} \\
\leq & \ 0.
\end{aligned}
$$

So we get the monotonicity of (51). □

The last step before stating the convergence of our discretizations is the consistence. We will prove this following the ideas of [FL07] and [Neu09] using Taylor series expansions around certain points to show the order of the differencing operators in the spatial directions and in time direction and then using them to estimate the error of the discretization operators and showing them to converge to zero as the grid distances converge to zero.

Theorem 5.8. *The discretizations* (48) *and* (51) *are consistent according to Definition* (5.4)*.*

Proof. Be $\phi(\tau; y, a) \in C^{1,2}([0,T] \times \mathbb{R}^2)$ a smooth test function. (48) is consistent, if

$$\lim_{\substack{\Delta\tau\to 0\\ \Delta y\to 0\\ \Delta a\to 0}} |(\phi_\tau - \sup_{c\in U}\{\mathcal{L}^c\phi + u\})_i^{n+1} - K_i^{n+1}(\phi^{n+1}, \phi^n, \Delta y, \Delta a, \Delta\tau)| = 0$$

First we determine the order of the differencing operators we use for the discretizations by Taylor series expansion. As this is equally for every differencing operator and widely discussed in literature, we only give one example of the order of the differencing operator of $D^2\phi$.

Set $\hat{p} = p_i + e_1\Delta y$ and $\overline{p} = p_i - e_1\Delta y$ for some i with $p_i \in int(D) \cap \,]y_{min}, y_{max}[\, \times \,]a_{min}, a_{max}[$. Then we can develop ϕ in a Taylor serious around these points and get

$$\phi(\tau,\hat{p}) = \phi(\tau,p_i) + \frac{\partial\phi}{\partial y}(\tau,p_i)\Delta y + \frac{\partial^2\phi}{(\partial y)^2}(\tau,p_i)\frac{(\Delta y)^2}{2} + \frac{\partial^3\phi}{(\partial y)^3}(\tau,\hat{p})\frac{(\Delta y)^3}{6} \tag{73}$$

and

$$\phi(\tau,\overline{p}) = \phi(\tau,p_i) - \frac{\partial\phi}{\partial y}(\tau,p_i)\Delta y + \frac{\partial^2\phi}{(\partial y)^2}(\tau,p_i)\frac{(\Delta y)^2}{2} - \frac{\partial^3\phi}{(\partial y)^3}(\tau,\overline{p})\frac{(\Delta y)^3}{6} \tag{74}$$

So we get

$$\begin{aligned} & \frac{\phi(\tau,\hat{p}) + \phi(\tau,\overline{p}) - 2\phi(\tau,p_i)}{(\Delta y)^2} \\ = \; & \frac{\partial^2\phi}{(\partial y)^2}(\tau,p_i) + \frac{\partial^3\phi}{(\partial y)^3}(\tau,\hat{p})\frac{\Delta y}{6} - \frac{\partial^3\phi}{(\partial y)^3}(\tau,\overline{p})\frac{\Delta y}{6} \\ = \; & \frac{\partial^2\phi}{(\partial y)^2}(\tau,p_i) + \frac{\Delta y}{6}[\frac{\partial^3\phi}{(\partial y)^3}(\tau,\hat{p}) - \frac{\partial^3\phi}{(\partial y)^3}(\tau,\overline{p})] \\ = \; & \frac{\partial^2\phi}{(\partial y)^2}(\tau,p_i) + O(\Delta y) \end{aligned}$$

Analogous we get the differencing operator for the second derivation in a-direction is of order $O(\Delta a)$ and the first derivation differencing operators are of order $O((\Delta y)^2)$, $O((\Delta a)^2)$ resp. $O(\Delta\tau)$ for the time derivative. So we obtain as first result

$$\|(\mathcal{L}^c_{\Delta y,\Delta a}\phi^n)_i - (\mathcal{L}^c\phi^n)_i\|_\infty = O(\max(\Delta y, \Delta a)) \tag{75}$$

Since μ_Z and σ_Z are time continuous functions we can state as well

$$\|\mathcal{L}^{c^{n+1}}_{\Delta y,\Delta a}\phi^{n+1} - \mathcal{L}^{c^n}\phi^n\|_\infty = O(\max(\Delta y, \Delta a)) + O(\Delta\tau) \tag{76}$$

Finally we obtain the consistency estimate

$$
\begin{aligned}
& |(\phi_\tau - \sup_{c\in U}\{\mathcal{L}^c\phi + u\})_i^{n+1} - \frac{\phi_i^{n+1}-\phi_i^n}{\Delta\tau} \\
& +(1-\theta)\sup_{c^{n+1}\in U}\{(\mathcal{L}^{c^{n+1}}_{\Delta y,\Delta a}\phi^{n+1})_i + u_i^{n+1}\} + \theta\sup_{c^n\in U}\{(\mathcal{L}^{c^n}_{\Delta y,\Delta a}\phi^n)_i + u_i^n\}| \\
\leq\; & |(\phi_\tau)_i^{n+1} - \frac{\phi_i^{n+1}-\phi_i^n}{\Delta\tau}| \\
& + \sup_{c\in U}|\{\mathcal{L}^c\phi + u\}_i^{n+1} - (1-\theta)\{(\mathcal{L}^c_{\Delta y,\Delta a}\phi)_i^{n+1} + u_i^{n+1}\} - \theta\{(\mathcal{L}^c_{\Delta y,\Delta a}\phi)_i^n + u_i^n\}| \\
\leq\; & O(\Delta\tau) + (1-\theta)\sup_{c\in U}|\{\mathcal{L}^c\phi+u\}_i^{n+1} - \{(\mathcal{L}^c_{\Delta y,\Delta a}\phi)_i^{n+1} + u_i^{n+1}\}| \\
& +\theta\sup_{c\in U}|\{\mathcal{L}^c\phi+u\}_i^{n+1} - \{(\mathcal{L}^c_{\Delta y,\Delta a}\phi)_i^n + u_i^n\}| \\
=\; & O(\Delta\tau) + O(\max(\Delta y,\Delta a))
\end{aligned}
$$

If we now take the limits $\tau \to 0, \Delta y \to 0, \Delta a \to 0$, we get the consistency of $\mathcal{L}^c_{\Delta y,\Delta a}$. For the consistency of (51) we use the same Taylor expansion series, where $p_i \in]y_{min}, y_{max}[$ and use additionally the time continuity of d, to get the same estimations with L_i^{n+1} and d instead of K_i^{n+1} and u, with an additionally term $-r\phi$. □

Now we are able to formulate the main theorem of this section.

Theorem 5.9. *The discretizations* (48) *and* (51) *are convergent to the viscosity solution of their underlying HJB.*

Proof. See [FL07], Theorem 5.1 and references therein (e.g. [BDR95]). □

5.2 Scaling

Since we use data from the US-PSID that are typically in 1000 US-$, we have to scale our system before implementing it to obtain a problem, we can solve on a reasonable grid.

5.2.1 Scaling A

The first way to scale our system is to introduce the scaled variables for wealth, income and consumption as follows:

$$\check{a}(t) = \frac{a(t)}{1000}, \tag{77}$$

$$\check{y}(t) = \frac{exp(y(t))}{1000}, \tag{78}$$

$$\check{c}(t) = \frac{c(t)}{1000}. \tag{79}$$

Now we have to update our differential equations with this scaled variables. As we easily see we obtain:

$$
\begin{aligned}
& d\breve{a}(t) \\
= & \frac{da(t)}{1000} \\
= & \frac{ra(t) + exp(y(t)) - c(t)}{1000} dt \\
= & (r\breve{a}(t) + \breve{y}(t) - \breve{c}(t))dt
\end{aligned}
$$

So $\mu_Y(\breve{y}(t), \breve{a}(t), \breve{c}(t), t) = r\breve{a}(t) + \breve{y}(t) - \breve{c}(t)$, σ_Y remains zero.
Next we update μ_X and σ_X with

$$
\begin{aligned}
& d\breve{y}(t) \\
= & d\frac{exp(y(t))}{1000} \\
= & \frac{exp(y(t))}{1000} \cdot dy(t) \\
= & \breve{y}(t) \cdot dy(t) \\
= & \breve{y}(t)(\frac{d}{dt}\tilde{y}(t) + \eta\tilde{y}(t) - \eta y(t))dt + \breve{y}(t)\sigma dW(t) \\
= & \breve{y}(t)(\frac{d}{dt}\tilde{y}(t) + \eta\tilde{y}(t) - \eta \cdot log(\breve{y}(t) \cdot 1000))dt + \breve{y}(t)\sigma dW(t)
\end{aligned}
$$

So we obtain

$$
\mu_X(\breve{y}(t), \breve{a}(t), \breve{c}(t), t) = \breve{y}(t)(\frac{d}{dt}\tilde{y}(t) + \eta\tilde{y}(t) - \eta log(\breve{y}(t) \cdot 1000)), \tag{80}
$$

$$
\sigma_X(\breve{y}(t), \breve{a}(t), \breve{c}(t)) = \sigma_X(\breve{y}(t)) = \sigma\breve{y}(t). \tag{81}
$$

Note that we can divide each variable by e.g. 10,000 instead of 1,000 and all equations stay the same, except for the multiplier in the *log* term, which has to be equal to the divisor.

At last we will make a choice for the autonomous consumption. We scale our system to have zero utility while consuming c_a (so a negative utility would arise if the consumption lay under this limit). To do so we choose $c_a = 1$. Then we obtain (using that the utility function stays the same for the scaled system)

$$
\tilde{u}(c_a) = \begin{cases} log(c_a) = log(1) = 0 & \text{for } \alpha = 1, \\ \frac{c_a^{1-\alpha}-1}{1-\alpha} = \frac{1^{1-\alpha}-1}{1-\alpha} = 0 & \text{otherwise} \end{cases}
$$

as wanted.

5.2.2 Scaling B

Since we saw that the equations for μ_X and σ_X underlie a big change if we use scaling method A, we give another way to scale the model that will not face this problems. This time we introduce the scaled variables for wealth and consumption as follows:

$$\breve{a}(t) = \frac{a(t)}{1000}, \tag{82}$$

$$\breve{c}(t) = \frac{c(t)}{1000}. \tag{83}$$

We do not scale the variable for the income anymore and use the logarithmized version we introduced in the problem formulation. The update of the differential equations will be as follows:

$$\begin{aligned} & d\breve{a}(t) \\ = & \frac{da(t)}{1000} \\ = & \frac{ra(t) + exp(y(t)) - c(t)}{1000} dt \\ = & (r\breve{a}(t) + \frac{exp(y(t))}{1000} - \breve{c}(t))dt \end{aligned}$$

So

$$\mu_Y(y(t), \breve{a}(t), \breve{c}(t)) = r\breve{a}(t) + \frac{exp(y(t))}{1000} - \breve{c}(t), \tag{84}$$

$$\sigma_Y = 0. \tag{85}$$

Since we did not change $y(t)$ anymore and the wealth and consumption did not appear in the derivation of μ_Y and σ_Y, they are unchanged.

At last we scale the system again to obtain a utility of zero from the autonomous consumption and therefore set $c_a = 1$ as above.

5.3 Iteration

We will now implement the developed discretization. After initializing the variables and start values, we have to compute first w and then V in every time period from T (initial time) to t_0. Therefore we use the matrix equations of the discretizations. This equations depend mainly on the coefficients $\alpha, \beta, \gamma, \delta, \nu$ and ψ and with them on the optimal control c. So in each step we have to compute the optimal control and then the vectors/matrices F, G, H, J and U and therefore w and V. The optimal control for computing w is always the autonomous consumption, so we dont have to do any optimization loop in computing it. To get the optimal control for V we have to do so. As initial guess we use the optimal control we computed in the last time step. Then we compute the matrices and vectors needed to solve the discretization and with them the new vector for V. While we have no convergence we compute with this V a new optimal control and repeat computing F, G and U and with them V. So our iteration based on the policy iteration in [FL07] has the pseudo-code:

```
Initialize parameters
Intitialize w(0), V(0)
Initialize auxiliary matrices and boundary vectors
for n = 1 to T/Δτ do
   compute_new_DHJ(w(n), D(n), H(n), J(n), c_a)
   w(n+1) = computeW(w(n), D(n), H(n), J(n), D(n+1), H(n+1), J(n+1), c_a)
   for k = 0 to convergence do
      compute_new_FGU(V(n), F(n), G(n), c)
      V(n+1) = computeV(V(n), F(n), G(n), F(n+1), G(n+1), U(n+1), U(n))
      compute_new_c(V(n+1))
   end for
end for
```

Algorithm 1: OptimalConsumption

With the following algorithms used to compute w and V

$$w(n+1) = [I - (1-\theta)\Delta\tau H(n+1)]^{-1}([I + \theta\Delta\tau H(n)]w(n) + J(n+1) - J(n) + (1-\theta)\Delta\tau D(n+1) + \theta\Delta\tau D(n))$$

Algorithm 2: computeW

$$V(n+1) = [I - (1-\theta)\Delta\tau G(n+1)]^{-1}([I + \theta\Delta\tau G(n)]V(n) + F(n+1) - F(n) + (1-\theta)\Delta\tau U(n+1) + \theta\Delta\tau U(n))$$

Algorithm 3: computeV

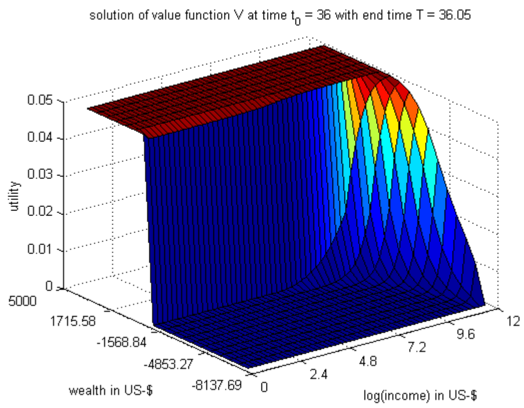
To compute the optimal control c we use the first order condition for a maximum $D_c V = 0$, so we get $u'(c) = V_a$ and therefore $c = V_a^{-\frac{1}{\alpha}}$ for $V_a \neq 0$ resp. $u(c) = const.$ for $V_a = 0$, so c is constant in that case. α denotes the risk aversion of the investor. We discretize V_a with backward differencing.

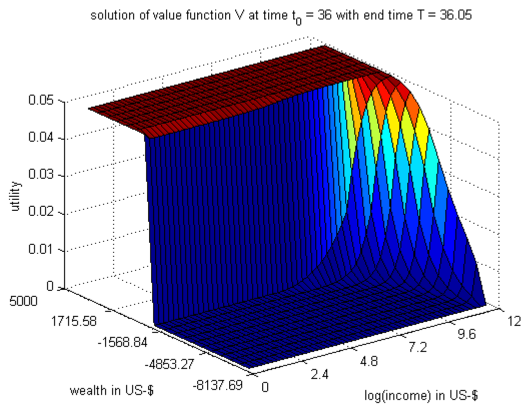
6 Results of the numerical simulation

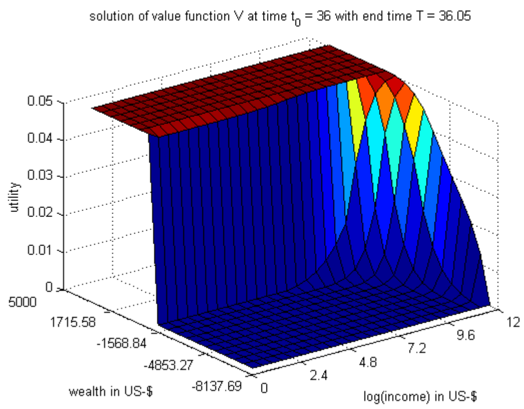
In this chapter we present some results of the implementation. Since the computation domain has to be sufficient large to apply the boundary conditions it is only practical to compute the solution in very small time intervals.

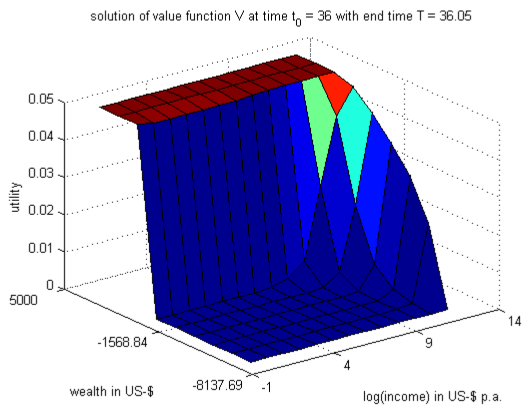
We begin with examples of the unscaled model. The model parameters were the ones estimated by [ST11]. For the parameters of the computations see the appendix.

In the first example we iterated from $t_0 = 36$ to $T = 36.05$. We used fully implicit and Crank-Nicolson time stepping. As we see below we get no difference in the value function with this two methods. This is in fact the case for every iteration we made so in the following we only present the results of the fully implicit time stepping.

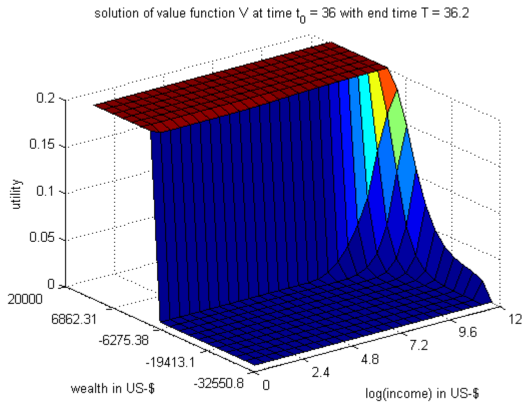


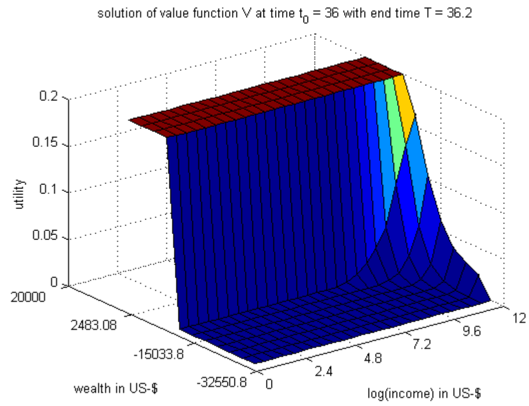




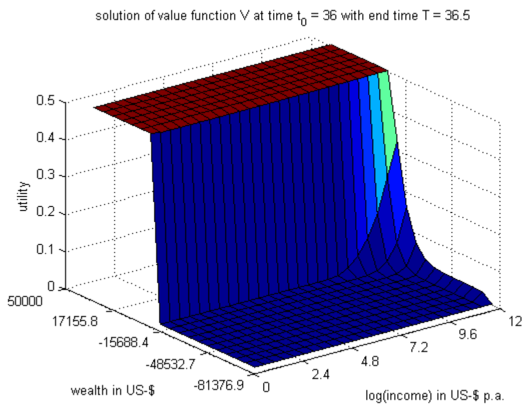


In the first row we see the value function computed on a 882 node grid with 5,000 time steps and Crank-Nicolson resp. fully implicit time stepping. The second row shows computations for a 450 node grid with 5,000 time steps and a 117 node grid with 500 time steps. Comparing all plots we see the slope in the right corner refines as we use a more fine grid. On the left side we see a little dint at the top front of the value function. This shows up in other iterations as well, if the grid is fine enough. This behavior will be discussed when we talk about the results of scaling method b. The next two plots show us computations for the time interval $[36, 36.2]$. The left



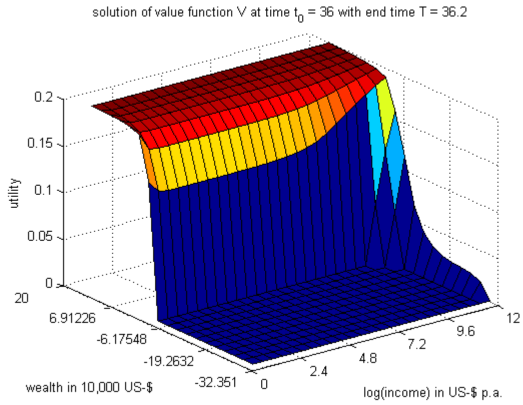


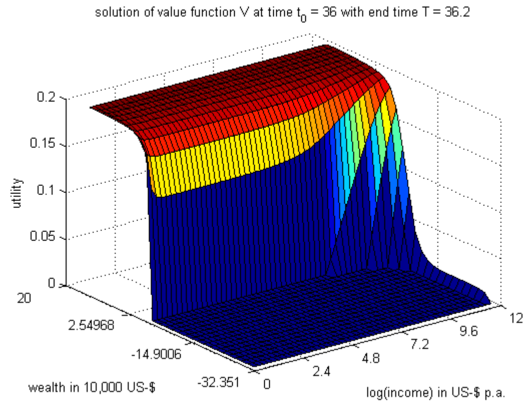
plot was computed using 450 nodes and 20,000 time steps, whereas the right one uses only 275 nodes and 2,000 time steps. Both plots show that the value function takes a similar form as in the first example. We get a little dint in the left plot again. The slope on the right refines as the grid refines like above. If we take a closer look at the right boundary we see the function behaves different from the first example. Here we have a more convex boundary, while in the above plot we got a concave one. This becomes even more clear if we take a look at the computed value function for the interval $[36, 36.5]$. These differences are an effect of the boundary

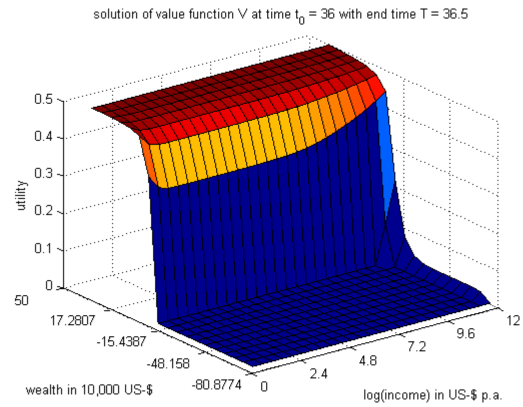


conditions. We guessed the second derivative in the $y-$direction to vanish. No other conditions were imposed. So the values of V on the left of the boundary influence the boundary values a lot. In the first computations this values were a lot higher than in the last ones. This results from the form of the auxiliary value function w, which gets closer to the right boundary as we increase the time interval. This beahvior is again a consequence of the boundary conditions (for w this time) which may cause an inaccuracy on the right boundary.

To discuss more about the form of the value function we compare this results with the results of the scaled system with scaling method b. The first row shows us the



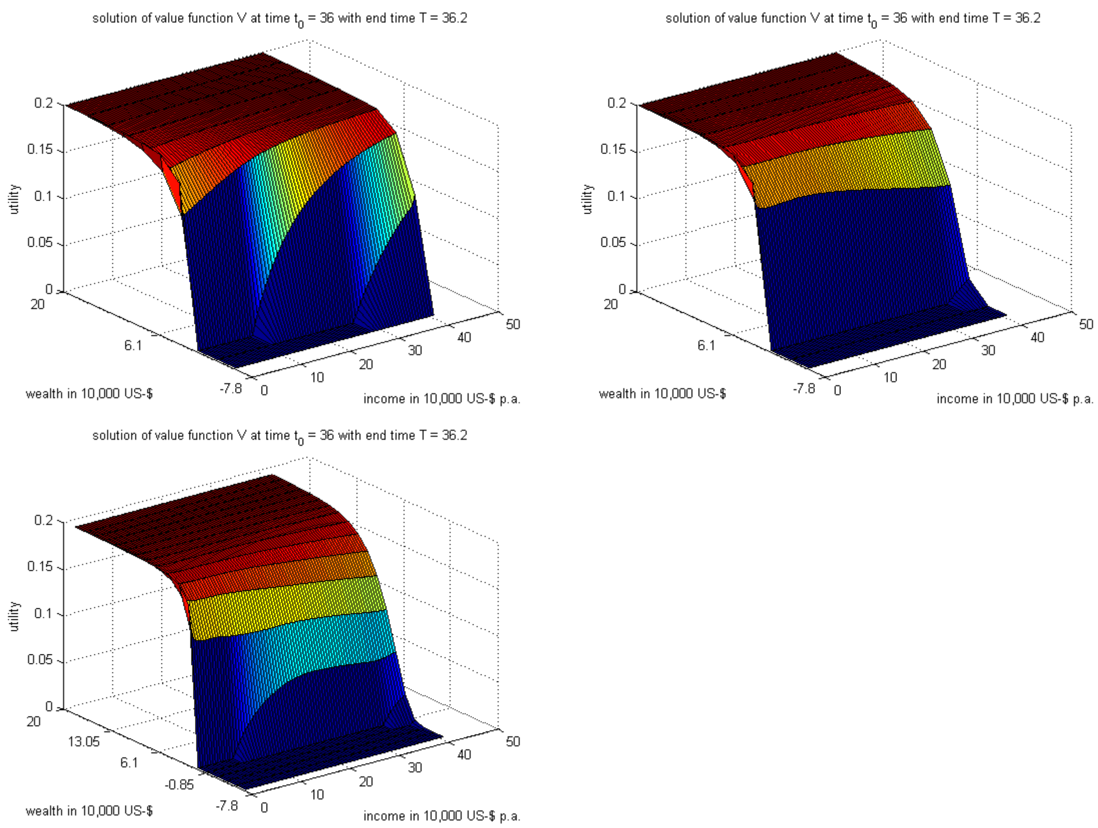




plot of computations for the interval $[36, 36.2]$ using 450 resp. 1323 nodes and 20,000 time steps. The third plot is computed for the interval $[36, 36.5]$ using 450 nodes and 50,000 time steps.

All three plots show the same behavior of the value function. We see a rapidly rising of the value function if we increase the wealth until we reach the maximum consumption and a slightly rising if we increase the income in the left. In the region of y_{max} this rising gets more rapidly as well, but not as much as in the wealth direction. But in difference to the unscaled system the rising in the wealth direction is a little bit softer and we see more shades. This is an effect of the utility function. It is logarithmic like and rises very fast in the beginning. In regions of thousands of $ the increase is very small. So the scaled system shows the form of the value function much better than the unscaled and we expect the little dint in the unscaled model to be a tiny version of this shade. We even expect it to become clearer if we would scale the system by 10,000 instead of 1,000.

The last results that we present are computed with the scaled system using scaling method a. This three plots are all computed for the time interval $[36, 36.2]$. In the



first computation we used a constant income function, i.e. $dy(t) = 0dt$ and therefore $\mu_X = 0$ and $\sigma_X = 0$. The plots in the first row were computed using 810 nodes and 2,000 time steps, the third one using 1539 nodes and 20,000 time steps.

The first plot cannot be compared to the other plots directly as we used a different income path. But it illustrates the structure of gaining more utility in both wealth and income direction very well. The rising becomes more linear in the income direction here. On the left edge we see the inaccuracy caused by the boundary conditions again. Since we used a non-stochastic income this time we expect the boundary conditions on y_{min} to be exact. So the inaccuracy has to be caused by the boundary conditions applied on y_{max} or by the not very fine approximation of w.

In the other two plots we see a strange behavior of the value function in the income direction. Here the utility falls in the most cases if we gain more income. This is in fact a wrong behavior and does not make any sense in economic viewing. This shows that scaling method a is not recommended to use.

7 Discussion and outlook

In the beginning of this thesis we learned about the basics of optimal control theory, the famous Hamilton-Jacobi-Bellman equation and its weak solution, the viscosity solution. In addition we made some conditions for utility functions we wanted to use in this paper.
After presenting the income model and the basic problem based on this, we deduced the associated Hamilton-Jacobi-Bellman equation and faced the problem of a stochastic target constraint which forced us to use a non-standard geometric dynamic programming principle to do so.
In the following we developed a finite differencing scheme to solve this HJB numerically. We applied boundary conditions on a changing computation (viability) domain while solving a second HJB to determine the boundary of this domain. Furthermore we proved the convergence of the finite differencing scheme and verified it to converge against the unique optimal value function.
In a last step we made some example computations and gave a short overview about the most important characteristics of the behavior of the value function. Therefore we used an unscaled and two different scaled systems. We saw that not all scalings are reasonable in a numerical sense (or only if we take the income as non changing parameter). Moreover we recognized that we can only compute solutions in very small time intervals as a bigger interval would lead to a sufficient larger computational domain and make numerical computations more and more impractically. This even becomes worse since the time step depends on the maximal values of the underlying PDEs and this rising maximal values would force us to use smaller time steps. Together this makes the numerical solution on larger time intervals uncomputable.

If we take a close look at the model we used to compute the optimal consumption, we can see there are some simplifications. A possible extension would be a stochastic interest rate. But this would lead to more computations and make a numerical solution even more impractically. An extension that would not face this problem would be to take constant but different interest rates for borrowing and lending. This could make the model more realistic without influencing the numerical solvability. Furthermore we could think about introducing a final time utility giving descents a little utility. If the condition $a(T) \geq 0$ should be influenced by this is another consideration. Additionally we could discuss about different forms of the utility function as we saw the utility function was impractically in the original

unscaled system. But all this would go beyond the scope of this thesis.
From a numerical point of view we could consider to adapt the finite differencing scheme so it uses a minimal computational domain in each time step. The maximal wealth could be adapted in time and we could gain more computability while using less nodes therefore. It would also be considerable to think about complete different numerical schemes that would allow us to compute the value function in adequate big time intervals.

A Appendix

A.1 Geometric dynamic programming principle

While proving the viscosity characterization of our value function we need a dynamic programming principle. This is taken from [BEI10]

Theorem A.1. *Fix* $(t, z) \in int_p(D)$ *and let* $\{\theta^c, c \in \mathcal{U}\}$ *be a family of stopping times with values in* $[t_0, T]$. *Then,*

$$\begin{aligned} V(t_0, z) &\leq \sup_{c \in \mathcal{U}_{t_0,z}} \mathbb{E}[f^*(Z^c_{t_0,z}(\theta^c))\mathbf{1}_{\theta^c = T} + V^*(\theta^c, Z^c_{t_0,z}(\theta^c))\mathbf{1}_{\theta^c < T}], \\ V(t_0, z) &\geq \sup_{c \in \mathcal{U}_{t_0,z}} \mathbb{E}[f_*(Z^c_{t_0,z}(\theta^c))\mathbf{1}_{\theta^c = T} + V_*(\theta^c, Z^c_{t_0,z}(\theta^c))\mathbf{1}_{\theta^c < T}] \end{aligned}$$

where V_* *(resp.* V^**) denotes the lower-semicontinuous (resp. upper-semicontinuous) envelope of* V.

Proof. See [BEI10] Theorem 6.1 with using $J(t_0, z; c) := \mathbb{E}[f(Z^c_{t_0,z}(T))]$ for $c \in \mathcal{U}$ instead of $J(t, z, \nu) := \mathbb{E}[f(Z^\nu_{t,z}(T))]$ for $\nu \in \mathcal{U}$. □

Theorem A.2. *For any* $(t, z) \in [0, T) \times \mathbb{R}^2$ *and* $c \in \mathcal{U}^t$, *we have*

$$\exists \tilde{c} \in \mathcal{U}_{t,z} \ s.t.\ c = \tilde{c} \ on \ [t, \theta) \Longleftrightarrow (\theta, Z^c_{t,z}(\theta)) \in D \mathbb{P} - a.s.$$

Proof. See [BEI10] for the set $\mathcal{U}^t$ and [ST02] for the whole set $\mathcal{U}$. □

A.2 Details on the parameters

In this section we give a short overview about the paramters we used for the implementations. All model setups used the paramters estimated by [ST11], that are in detail:

$\eta = 2.324$
$\sigma = \sqrt{0.5033}$
$\mu = -2.3296$
$\beta_1 = -0.3038$
$\beta_2 = 0.1254$
$\gamma_1 = 0.928$
$\gamma_2 = -0.001$
$\gamma_3 = 0.0407$

In addition we used for all setups:

gender = male ($Z_{11} = 1$)
$Z_{12} = 13.5$
$r = 1.02$
riskaversion $\alpha = 2$
$c_a = 1$
$t_0 = 36$

A.2.1 Unscaled model setups

For the first setup we used

$T = 36.05$
$\Delta_a = 750$
$\Delta_y = 0.25$
$\Delta_\tau = 1.0000e - 005$
$\theta = 0.5$

The second one is the same with $\theta = 0$.

We proceed with

$T = 36.05$
$\Delta_a = 750$
$\Delta_y = 0.5$
$\Delta_\tau = 1.0000e - 005$
$\theta = 0$

and

$T = 36.05$
$\Delta_a = 1500$
$\Delta_y = 1$
$\Delta_\tau = 1.0000e - 004$
$\theta = 0$

For computations with the unscaled model in the intervals $[36, 36.2]$ and $[36, 36.5]$ we used

$T = 36.2$
$\Delta_a = 3000$
$\Delta_y = 0.5$
$\Delta_\tau = 1.0000e - 005$
$\theta = 0$

and

$T = 36.2$
$\Delta_a = 5000$
$\Delta_y = 0.5$
$\Delta_\tau = 1.0000e - 004$
$\theta = 0$

respectively

$T = 36.5$
$\Delta_a = 7500$
$\Delta_y = 0.5$
$\Delta_\tau = 1.0000e - 005$
$\theta = 0$

A.2.2 Scaling type A model setups

In the first computation of the scaled model using type A scaling we used $\sigma = 0$ and $\eta, Z_{11}, Z_{12}, \beta_i = 0$. The other parameters are

$T = 36.2$
$\Delta_a = 3$
$\Delta_y = 0.5$
$\Delta_\tau = 1.0000e - 004$
$\theta = 0.5$

The last two computations use $\sigma, \eta, Z_{1i}, \beta_j$ as estimated. We further use

$T = 36.2$
$\Delta_a = 3$
$\Delta_y = 0.5$
$\Delta_\tau = 1.0000e - 004$
$\theta = 0$

respectively

$T = 36.2$
$\Delta_a = 1.5$
$\Delta_y = 0.5$
$\Delta_\tau = 1.0000e - 005$
$\theta = 0$

A.2.3 Scaling type B model setups

For the computations of the scaled model using scaling type B we used

$T = 36.2$
$\Delta_a = 3000$
$\Delta_y = 0.5$
$\Delta_\tau = 1.0000e - 005$
$\theta = 0$

and

$T = 36.2$
$\Delta_a = 2000$
$\Delta_y = 0.25$
$\Delta_\tau = 1.0000e - 005$
$\theta = 0$

respectively

$T = 36.5$
$\Delta_a = 7500$
$\Delta_y = 0.5$
$\Delta_\tau = 1.0000e - 005$
$\theta = 0$

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EIDESSTATTLICHE ERKLÄRUNG

Ich versichere hiermit, dass ich meine Diplomhausarbeit „*Optimal Consumption Using a Continuous Income Model*“ selbstständig und ohne fremde Hilfe angefertigt habe, und dass ich alle von anderen Autoren wörtlich übernommenen Stellen wie auch die sich an die Gedankengänge anderer Autoren eng anlehnenden Ausführungen meiner Arbeit besonders gekennzeichnet und die Quellen zitiert habe und neben dem Programm Matlab keine weiteren Hilfsmittel als die im Literaturverzeichnis angegebenen verwendet habe.

Münster, den 21.04.2011

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