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Mean-Field and Kinetic Market Models

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Abstract

This thesis was concerned with the illustration and development of kinetic models, related to financial markets. Three models with particular attention to their long time behaviour were presented. The derivation of a novel model with an explicit analysis based on results from particle and statistical physics was the main purpose. In favour of a realistic illustration we considered homogeneous as well as heterogeneous conditions. In conclusion we yielded mean-field partial differential equations for the corresponding models.

Contents

1. Introduction

Agent-based modelling is a widespread tool utilised in several fields of research like opinion formation in sociology [26], [10], ant behaviour in entomology [15] or analysis of financial market properties in economics [21], [17], [27]. The characterisation of agent behaviour is of humongous use considering the investigation of the actions on kinetic markets. Since financial markets have a large impact on an array of other markets they are of special interest. The modelling of agent behaviour on financial markets is a challenging problem due to the identification of the crucial elements that influence the system's dynamics. These models rely on the system size, more precisely the overall number of agents trading on that market has an effect on the long time behaviour of the system.

On the one hand this thesis is concerned with the presentation of kinetic market models with special regard to the evolution of the wealth of traders, their propensity to invest and the group size of specific groups. On the other hand we concentrate on the development of a novel kinetic model concerning the agents' decision to trade.

This thesis is organised as follows. Chapter 2 exposes a detailed introduction of three models based on parameters that are essential regarding transactions on financial markets. In the first section, we take a closer look at the wealth of traders by analysing the evolution. In addition this model pays attention to the stock price development. This is done by deriving a *master equation* which leads to a system of *Fokker-Planck equations* in the asymptotic limit.
The next model deals with the trading tendency of agents whereupon a differentiation into three trading groups is arranged. Furthermore an exchange between the groups is included on the basis of the trading strategies of the groups. The resulting system is analysed with respect to the long tome behaviour of the agents and the long time behaviour of the price, which is coupled with the system.
The final model concentrates on the evolution of the size of groups based on the ideas of an ant model. A case-by-case analysis is organised for the two and three group case. The dynamics are caused by the exchange rates resulting in a *Fokker-*

Planck equation. A concluding mean-field consideration finalises this chapter.
A new model, fractionally based on the previous models, is established in chapter 3. It deals with the trading decision of agents relating to their propensity to trade, the fraction of stocks held and their wealth. A detailed introduction of the basics is presented, followed by an analysis of the model for the homogeneous and heterogeneous case. Additionally, a derivation of the corresponding *mean-field equations* for the two cases is illustrated.
Chapter 4 gives a summary of the obtained results by a characterisation of the exact proceeding in the paper. The outlook provides information about the numerical simulation of the mean-field model introduced in the previous chapter. Furthermore variations of the model basics are mentioned.

2. Kinetic Market Models

Quite a number of literature dealing with the actions taking place on financial markets ([2], [13], [18], [21], [25]) can be found. These models exhibit a kinetic background and they differ in the properties of financial markets they want to analyse. This chapter begins with the introduction of three kinetic models, which display their own model characteristics with different underlying parameters.

2.1. Wealth Model

This section is concerned with a model that deals with the wealth evolution of agents and the evolution of the price for a stock. We take a look at the properties of financial markets and see that agents interact through binary exchanges and speculative trading, see also [3], [8]. Basically we follow the ideas of [7].

2.1.1. Model Principles

We have a set of agents $i = 1, ..., N$ and w_i is the wealth of agent i. The portfolio of an agent can consist of stocks and bonds, so each agent decides which fraction γ_i of his wealth he wants to invest in stocks and which fraction $(1 - \gamma_i)$ he wants to invest in bonds. This choice of fractions can be reset at each time step.
So the essence of the model's dynamic is the composition of the portfolio. For reasons of simplicity we disregard dividends and stochastic elements for the moment.
A new wealth w_i' is of the following form

$$w_i' = (1 - \gamma_i)w_i(1 + b) + \gamma_i w_i(1 + r'), \tag{2.1}$$

where b is the interest rate of bonds, $r' = \frac{S'-S}{S}$ is the rate of return of the stock, S is the price of the stock and S' is the new price of the stock.

We denote by n_i the number of stocks of the agent i. It is obvious that $\gamma_i w_i = n_i S$, because the fraction of invested wealth in stocks and the price for n_i stocks are worth the same.
Hence we can rewrite (2.1) into

$$\begin{aligned} w_i' &= w_i + w_i(1-\gamma_i)b + w_i\gamma_i\left(\frac{S'-S}{S}\right) \\ &= w_i + (w_i - n_i S)b + n_i(S'-S) \end{aligned} \tag{2.2}$$

As mentioned before, the dynamics are based on the division of the wealth at the next time step. In order to find an optimal division, agents estimate the return distribution of the stock for the next period and find an optimal composition of stocks and bonds that maximises their utility function $U(w)$(for details see [18]).
If we have a hypothetical price S^h, the agents will find the optimal fraction γ_i^h that maximises their expected utility $E[U]$. We assume that all agents have the same expected utility and therefore $\gamma_i^h(S^h) = \gamma^h(S^h)$, regardless of their wealth.
With the identity $\gamma_i w_i = n_i S$ and [18], we get a monotonically decreasing demand curve that characterises the desired number of stocks for every agent i depending on the hypothetical stock price S^h and the hypothetical new wealth w_i^h, obtained from (2.1) with $S' = S^h$:

$$n_i^h = n_i^h(S^h) = \frac{\gamma^h(S^h)w_i^h(S^h)}{S^h}.$$

The total number of stocks $n = \sum_{i=1}^N n_i$ is conserved, hence we get the new price S' through the market clearing condition, which implies that the total demand equals the supply

$$n = \sum_{i=1}^{N} n_i^h(S')$$

and a new equilibrium price S' is generated.

The next step is to introduce a source of stochastic noise to the model. This is done by extending the assumptions for the proportion of investments γ_i and the rate of return of the stock r' for a more realistic illustration.
Due to that we define the distribution of wealth w as $f(w,t)$, $w \in \mathbb{R}_+$, $t > 0$, representing the probability for an agent to have a wealth w at time t. Next, we assume

that at time t the percentage of wealth invested is in the form

$$\gamma(\xi) = \mu(S) + \xi,$$

where ξ is a random variable in $[-z, z]$, $z = \min\{-\mu(S), 1 - \mu(S)\}$ and it is distributed according to the probability density function $\Phi(\mu(S), \xi)$ with zero mean and variance ζ^2. Φ represents the individual strategy of an agent around the optimal choice $\mu(S)$ and we assume Φ to be independent of the agent's wealth. The optimal demand curve $\mu(\cdot)$ is assumed to be a given monotonically non increasing function of the price S with $0 < \mu(0) < 1$.
For $f(w,t)$ given and normalised, $\int_0^\infty f(w,t)dw = 1$, the price equals the averaged expectation of invested wealth

$$S = \frac{1}{n} E[\gamma w], \tag{2.3}$$

where $E[X]$ denotes the expected value of the random variable X. Since γ and w are independent, we can formulate (2.3) at each time t and the following identity obtains

$$S(t) = \frac{1}{n} E[\gamma] E[w] \overset{E[\xi]=0}{=} \frac{1}{n} \mu(S(t)) \bar{w}(t), \tag{2.4}$$

with the mean wealth defined as

$$\bar{w}(t) := E[w] = \int_0^\infty f(w,t) w \, dw$$

and the optimal choice as

$$\mu(S) := \int \Phi(\mu(S), \xi) \xi \, d\xi.$$

After trading, the agent's new wealth depends on the future price S' and the pre-trade percentage of wealth invested γ corresponding to

$$w'(S', \gamma, \eta) = (1-\gamma) w (1+b) + \gamma w (1 + r(S', \eta)), \tag{2.5}$$

where the expected return rate of the stock is given by

$$r(S', \eta) = \frac{S' - S + D + \eta}{S}.$$

Here, $D \geq 0$ is a constant dividend payed by the company which issued the stocks. η is a random variable distributed according to $\Theta(\eta)$ with zero mean and variance σ^2 and η includes fluctuations due to price uncertainty and dividends. Furthermore we assume $\eta \in [-d, d]$ with $d \in (0, S' + D]$, this condition ensures that $w' \geq 0$. The dynamic of the wealth model is then determined by the agent's new fraction of wealth invested in stocks $\gamma'(\xi') = \mu(S') + \xi'$, with ξ' being a random variable in $[-z', z']$, $z' = \min\{\mu(S'), 1-\mu(S')\}$, and distributed according to $\Phi(\mu(S'), \xi')$. Then we have the demand-supply relation

$$S' = \frac{1}{n} E[\gamma' w']$$

and because of the independence of γ' and w'

$$S' = \frac{1}{n} E[\gamma'] E[w'] = \frac{1}{n} \mu(S') E[w']. \tag{2.6}$$

By rearranging (2.5) we obtain

$$w'(S', \gamma, \eta) = w(1+b) + \gamma w(r(S', \eta) - b)$$

and then forming an expectation obtains

$$\begin{aligned} E[w'] &= E[w](1+b) + E[\gamma w](E[r(S', \eta)] - b) \\ &= \bar{w}(t)(1+b) + \mu(S)\bar{w}(t)\left(\frac{S' - S + D}{S} - b\right). \end{aligned} \tag{2.7}$$

We attach $E[w']$ into (2.6) to get the identity

$$S' = \frac{1}{n}\mu(S')\bar{w}(t)\left[(1+b) + \mu(S)\left(\frac{S' - S + D}{S} - b\right)\right].$$

Transposing equation (2.4) yields $\bar{w} = \frac{nS}{\mu(S)}$, herewith we can eliminate the dependence on the mean wealth $\bar{w}(t)$ and conclude

$$\begin{aligned} S' &= \frac{\mu(S')}{\mu(S)}\left[S(1+b) + \mu(S)(S' - S + D - Sb)\right] \\ &= \frac{\mu(S')}{\mu(S)}\left[(1-\mu(S))S(1+b) + \mu(S)(S' + D)\right] \\ &= \frac{(1-\mu(S))\mu(S')}{(1-\mu(S'))\mu(S)}(1+b)S + \frac{\mu(S')}{1-\mu(S')}D \end{aligned} \tag{2.8}$$

Annotations for the Future Price and the Stock & Bond Return-Relation

The equation for the future value of the stock price S' and in the same way the difference of the average return of stocks $\bar{r}(S')$ and the constant return of bonds b

require a more detailed analysis concerning the optimal investment choice $\mu(S)$ and $\mu(S')$. We define

$$g(S) = \frac{1-\mu(S)}{\mu(S)} S.$$

Due to $\frac{dg(S)}{dS} = -\frac{d\mu(S)}{dS}\frac{S}{\mu(S)^2} + \frac{1-\mu(S)}{\mu(S)} > 0$, with $\mu(S)$ being a monotonically decreasing function and $\mu(0) \in (0,1)$, $g(S)$ is a monotonically increasing function with respect to S and therefore a unique solution for the future price

$$S' = g^{-1}(g(S)(1+b)+D) > S$$

exists. Note that the equation for the future price is given by $g(S') = g(S)(1+b)+D$. The unique solution $S' = S$ holds in case there are no dividends $D = 0$ and bond rates $b = 0$ payed.

The investment in stocks is usually more risky than the investment in bonds, thus we wonder under which certain conditions the average return of stocks, defined as $\bar{r}(S') = E[r(S',\eta)] = \frac{S'-S+D}{S}$ is above the bonds rate b.
For this reason we consider

$$\bar{r}(S') - b = \frac{(\mu(S')-\mu(S))(1+b)}{(1-\mu(S'))\mu(S)} + \frac{\mu(S')D}{S(1-\mu(S'))}. \tag{2.9}$$

Given the threshold for the variation of the investment choices

$$\frac{\mu(S')-\mu(S)}{\mu(S)\mu(S')} S \geq -\frac{D}{1+b},$$

with (2.9) we yield

$$\begin{aligned}
\bar{r}(S') - b &= \frac{(\mu(S')-\mu(S))(1+b)}{(1-\mu(S'))\mu(S)} - \frac{\mu(S')(1+b)}{S(1-\mu(S'))} \cdot \left(-\frac{D}{1+b}\right) \\
&\geq \frac{(\mu(S')-\mu(S))(1+b)}{(1-\mu(S'))\mu(S)} - \frac{\mu(S')(1+b)}{S(1-\mu(S'))} \frac{(\mu(S')-\mu(S))}{\mu(S)\mu(S')} S \\
&= 0 \\
\Rightarrow \bar{r}(S') &\geq b.
\end{aligned}$$

Hence if the lower limit is not reached, the bonds return lies upon the average return and consequently the earnings of a less risky investment are larger than the earnings of a risky one.

Example

In case of a constant investment $\mu = C \in (0,1)$ the equations of the actual and future price are of the following form

$$g(S) = \frac{(1-C)S}{C}$$
$$g(S') = \frac{(1-C)S(1+b)}{C} + D$$

and they lead to

$$\begin{aligned} S' &= g^{-1}(g(S')) \\ &= (1+b)S + \frac{C}{1-C}D. \end{aligned}$$

Additionally the average stocks return is always larger than the bonds return $\bar{r}(S') - b = \frac{C\,D}{S(1-C)} \geq 0$.

2.1.2. Price & Wealth Evolution

This model characterises a non-stationary financial market where the average wealth is not conserved and this fact induces price changes. In this part of the paragraph we attend to the development of the wealth distribution and in detail we will analyse the evolution of the mean wealth and the price with particular attention on the bounds.

Using [5] we obtain the following linear kinetic equation for the evolution of the wealth distribution in Boltzmann formulation [11] which includes all possibilities of changes in the distribution $f(w,t)$:

$$\frac{\partial f(w,t)}{\partial t} = \int_{-d}^{d}\int_{-z}^{z} \left(\underbrace{\beta('w \to w)\frac{1}{j(\xi,\eta,t)}f('w,t)}_{gain\ of\ wealth} - \underbrace{\beta(w \to w')f(w,t)}_{loss\ of\ wealth} \right) d\xi\, d\eta. \tag{2.10}$$

The first part of the integral takes into account all possible gains realised when coming from a pre-trade wealth $'w$ to the given wealth w with a probability per unit time $\beta('w \to w)$.

Inverse dynamics yield

$$'w = \frac{w}{j(\xi, \eta, t)}, \qquad j(\xi, \eta, t) = (1+b) + \gamma(\xi)(r(S', \eta) - b),$$

where S' is given by (2.6).

Thus $'w$ is obtained by discounting w with the function $j(\xi, \eta, t)$ that consists of stochastic factors that caused the growth of wealth. $j(\xi, \eta, t)$ is also needed to conserve the total number of agents $\frac{d}{dt}\int_0^\infty f(w,t)dw = 0$.
The latter part includes all possible losses of the given wealth w with w' given by (2.5). The process rate takes the form

$$\beta(w \to w') = \Phi(\mu(S), \xi)\Theta(\eta).$$

Annotations for Further Computations
(2.10) is the basic equation of the model. Next, we introduce some notations which will help on further investigation of the evolution.
Let $\mathcal{M}_0$ be the space of all probability measures in $\mathbb{R}_+$ and

$$\mathcal{M}_p = \left\{ \Psi \in \mathcal{M}_0 : \int_{\mathbb{R}_+} |\vartheta|^p \, \Psi(\vartheta)\, d\vartheta < +\infty, p \geq 0 \right\}$$

is the space of all *Borel probability measures* of finite momentum of order p, equipped with the topology of the weak convergence of the measures.
Let $\mathcal{F}_p(\mathbb{R}_+), p > 1$ be the class of all real functions on $\mathbb{R}_+$ with $g(0) = g'(0) = 0$. The m-th derivate of g is defined as $g^{(m)}$ and $g^{(m)}(v)$ is Hölder continuous of order δ such that

$$\left\| g^{(m)} \right\|_\delta = \sup_{v \neq w} \frac{\left| g^{(m)}(v) - g^{(m)}(w) \right|}{|v-w|^\delta} < \infty,$$

where the integer m and $\delta \in (0, 1]$ are chosen in this way $m + \delta = p$.
Obviously $\Theta \in \mathcal{M}_p$ with $\eta \in [-d, d]$ for all $p > 0$ since following inequality holds

$$\int_{-d}^{d} |\eta|^p \, \Theta(\eta)\, d\eta \leq |d|^p .$$

In order to simplify the following computations that shall lead us to the upper boundaries of the wealth evolution and to a solution of our partial differential equation, we assume Θ to be obtained from a given random variable Y with $E[Y] = 0$ and $\sigma^2 = 1$. Therefore Θ is the density of σY with variance σ^2. Taking this into consideration, we get the dependence of the higher moments of Θ on σ for $p > 2$ by

$$\int_{-d}^{d} |\eta|^p \, \Theta(\eta) \, d\eta = E\left[|\sigma Y|^p\right] = \sigma^p E\left[|Y|^p\right].$$

Boundaries

We can write (2.10) in weak form as a *Boltzmann equation*

$$\frac{d}{dt}\int_0^\infty f(w,t)\phi(w)dw = \int_0^\infty \int_{-d}^{d}\int_{-z}^{z} \Phi(\mu(S),\xi)\Theta(\eta)f(w,t)(\phi(w') - \phi(w))\,d\xi\,d\eta\,dw. \tag{2.11}$$

The right side of the equation is also called *Boltzmann's collision integral.* If we think of a weak solution f of the initial problem (2.10), with an initial probability density $f_0(w) \in \mathcal{M}_p, p > 1$, we think of any probability density $f \in C^1(\mathbb{R}_+, \mathcal{M}_p)$ which satisfies (2.11) for all $t > 0$ and $\phi \in \mathcal{F}_p(\mathbb{R}_+)$ and the following is valid for all $\phi \in \mathcal{F}_p(\mathbb{R}_+)$

$$\lim_{t\to 0}\int_0^\infty f(w,t)\phi(w)dw = \int_0^\infty f_0(w,t)\phi(w)dw.$$

By using (2.11) we can analyse the macroscopic quantities, the moments of $f(w,t)$. Hence the next step is to investigate the wealth evolution for $\phi(w) = 1$, $\phi(w) = w$ and $\phi(w) = w^p, p \geq 2$.

- If we choose $\phi = 1$, we preserve the total number of traders with $\frac{d}{dt}\int_0^\infty f(w,t)dw = 0$.

- The choice $\phi(w) = w$ yields the evolution in time of the average wealth and consequently a description for the price evolution with (2.4).
  Since we have

  $$\frac{d}{dt}\int_0^\infty f(w,t)w\,dw = \left(b + \mu(S)\left(\frac{S' - S + D}{S} - b\right)\right)\int_0^\infty f(w,t)w\,dw$$

  from (2.7), the mean wealth is not conserved and in fact non-decreasing due to the right hand side of the identity. Rewriting yields

  $$\frac{d}{dt}\bar{w}(t) = \left((1-\mu(S))b + \mu(S)\bar{r}(S')\right)\bar{w}(t). \tag{2.12}$$

  We use (2.4) again, transform and include the equation above, so we get

  $$\frac{d}{dt}S(t) = \frac{\mu(S(t))}{\mu(S(t)) - \dot{\mu}(S(t))S(t)}\left((1-\mu(S(t)))b + \mu(S(t))\bar{r}(S'(t))\right)S(t),$$

where S' is given by (2.8) and $\dot{\mu}(S) = \frac{d\mu(S)}{dS} \leq 0$. (2.9) and the monotonicity of μ cause

$$\bar{r}(S') \leq M := b + \frac{D}{S(0)(1-\mu(S(0)))}.$$

If we use (2.12) we obtain the boundary

$$\bar{w}(t) \leq \bar{w}(0)\exp(Mt). \tag{2.13}$$

From (2.4) we get immediately

$$\frac{S(t)}{\mu(S(t))} \leq \frac{S(0)}{\mu(S(0))}\exp(Mt),$$

which results in

$$S(t) \leq S(0)\exp(Mt). \tag{2.14}$$

Example

For the constant case $\mu(\cdot) = C, C \in (0,1)$ mentioned above, we have an explicit expression for the growth of the average wealth $\bar{w}(t)$ as well as for the price $S(t)$

$$\bar{w}(t) = \bar{w}(0)\exp(bt) - (1-\exp(bt))\frac{nD}{1-C}$$
$$S(t) = S(0)\exp(bt) - (1-\exp(bt))\frac{nD}{1-C}.$$

- Next, we choose $\phi(w) = w^p$ and therefore

$$\frac{d}{dt}\int_0^\infty f(w,t)w^p\,dw = \int_0^\infty \int_{-d}^{d}\int_{-z}^{z} \Phi(\mu(S),\xi)\Theta(\eta)f(w,t)(w'^p - w^p)\,d\xi\,d\eta\,dw.$$

Furthermore, we have a Taylor series of $\phi(w')$ expanded around w of second order

$$w'^p = w^p + pw^{p-1}(w'-w) + \frac{1}{2}p(p-1)\tilde{w}^{p-2}(w'-w)^2,$$

with $\tilde{w} = \vartheta w' + (1-\vartheta)w$ for some $\vartheta \in [0,1]$.
Setting the expansion into the upper integro-differential equation and a few bounded above estimates yield

$$\frac{d}{dt}\int_0^\infty w^p f(w,t)dw \leq A_p(S)\int_0^\infty w^p f(w,t)dw, \tag{2.15}$$

where

$$A_p(S) = 2C_p\Big[b^p + b^{p-2}\left(1+\frac{c_2}{S^2}\left((S'-S)^2 + D^2 + \sigma^2 E(|Y|^2)\right)\right) \\ + b^2\left(1+\frac{c_{p-2}}{S^{p-2}}\left((S'-S)^{p-2} + D^{p-2} + \sigma^{p-2}E(|Y|^{p-2})\right)\right)\Big],$$

with C_p, c_2, c_{p-2} being suitable chosen constants.

All these results are summarised in

Theorem 1. *Let $f_0 \in \mathcal{M}_p$ be a probability density with $p = 2+\delta$ for some $\delta > 0$. Then the average wealth $\bar{w}(t)$ is increasing exponentially with time following* (2.13). *As a consequence, if $\mu(S)$ is a non increasing function, the price S does not grow more than exponentially as in* (2.14). *Similarly, higher order moments do not increase more than exponentially and we have the bound* (2.15).

2.1.3. Asymptotic Behaviour

In the previous paragraph we analysed the behaviour of the kinetic equation while the test function $\phi(w)$ took various forms in order to investigate the time evolution of the average wealth respectively the bound for the moments of higher order. Now we are going to have a look at the general form of the testfunction in the underlying weak formulation of the Boltzmann equation (2.11).

We use the weak formulated kinetic equation and a second order Taylor expansion of ϕ around w with $\tilde{w} = \vartheta w' + (1-\vartheta)w$ for some $\vartheta \in [0,1]$. Composing these two parts, we get an equation for *Boltzmann's collision integral*

$$\int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S),\xi)\Theta(\eta) f(w,t)(\phi(w') - \phi(w))\, d\xi\, d\eta\, dw$$
$$= \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S),\xi)\Theta(\eta) f(w,t) w(b+\gamma(r(S',\eta)-b))\phi'(w)\, d\xi\, d\eta\, dw$$
$$+ \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S),\xi)\Theta(\eta) f(w,t) w^2(b+\gamma(r(S',\eta)-b))^2\phi''(w)\, d\xi\, d\eta\, dw$$
$$+ R_b(S,S'),$$

where the remainder $R_b(S,S')$ is given through an extension with zero by

$$\frac{1}{2}\int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S),\xi)\Theta(\eta) f(w,t) w^2(b+\gamma(r(S',\eta)-b))^2(\phi''(\tilde{w}) - \phi''(w))\, d\xi\, d\eta\, dw.$$

By using the first and second moments of the random variables ξ and η, precisely $E[\xi] = 0$, $E[\xi^2] = \zeta^2$ and $E[\eta] = 0$, $E[\eta^2] = \sigma^2$, and a scaling for the time which includes $\tau = bt$ and in consequence $\tilde{f}_b(w,\tau) = f(w,t)$, $\tilde{S}(\tau) = S(t)$, $\tilde{\mu}(\tilde{S}) = \mu(S)$, we obtain the scaled weak formulation of the kinetic equation

$$
\begin{aligned}
&\frac{d}{d\tau}\int_0^\infty \tilde{f}_b(w,\tau)\phi(w)dw \\
&= \int_0^\infty \tilde{f}_b(w,\tau)w\left(1+\tilde{\mu}(\tilde{S})\left(\frac{\tilde{S}'+D-\tilde{S}}{b\tilde{S}}-1\right)\right)\phi'(w)dw \\
&\quad + \frac{1}{2}\int_0^\infty \tilde{f}_b(w,\tau)w^2\left(b+(\zeta^2+\tilde{\mu}(\tilde{S})^2)\left(\frac{(\tilde{S}'-\tilde{S})^2}{b\tilde{S}^2}+\frac{\sigma^2+D^2}{b\tilde{S}^2}+2D\frac{\tilde{S}'-\tilde{S}}{b\tilde{S}^2}\right.\right. \\
&\qquad \left.\left. +b-2\frac{\tilde{S}'+D-\tilde{S}}{\tilde{S}}\right)+2\tilde{\mu}(\tilde{S})\left(\frac{\tilde{S}'+D-\tilde{S}}{\tilde{S}}-b\right)\right)\phi''(w)dw \\
&\quad + \frac{1}{b}R_b(\tilde{S},\tilde{S}').
\end{aligned}
\tag{2.16}
$$

Now we consider the limit of (2.16) for $b \to 0$. For that reason we assume that

$$
\lim_{b\to 0}\frac{\sigma^2}{b} = \nu, \qquad \lim_{b\to 0}\frac{D}{b} = \lambda.
$$

Next we know

$$
\lim_{b\to 0} R_b(\tilde{S},\tilde{S}') = 0,
$$

based on estimates of the remainder $R_b(\tilde{S},\tilde{S}')$.
Besides we define

$$
\kappa(\tilde{S}) := \frac{\tilde{\mu}(\tilde{S})(1-\tilde{\mu}(\tilde{S}))}{\tilde{\mu}(\tilde{S})(1-\tilde{\mu}(\tilde{S}))-\tilde{S}\dot{\tilde{\mu}}(\tilde{S})},
$$

lying in $(0,1]$, $\dot{\tilde{\mu}}(\tilde{S}) = \frac{d\tilde{\mu}(\tilde{S})}{d\tilde{S}} \le 0$, with the limit

$$
\lim_{b\to 0}\frac{\tilde{S}'-\tilde{S}}{b} = \kappa(\tilde{S})\left(\tilde{S}+\frac{\tilde{\mu}(\tilde{S})}{1-\tilde{\mu}(\tilde{S})}\lambda\right).
$$

In conclusion we get

$$\frac{d}{d\tau}\int_0^\infty \tilde{f}(w,\tau)\phi(w)dw$$
$$= \left(1+\tilde{\mu}(\tilde{S})\left((\kappa(\tilde{S})-1)+\frac{(\tilde{\mu}(\tilde{S})(\kappa(\tilde{S})-1)+1)}{1-\tilde{\mu}(\tilde{S})}\frac{\lambda}{\tilde{S}}\right)\right)\int_0^\infty \tilde{f}(w,\tau)w\phi'(w)dw$$
$$+\frac{1}{2}\frac{(\tilde{\mu}(\tilde{S})^2+\zeta^2)}{\tilde{S}^2}\nu\int_0^\infty \tilde{f}(w,\tau)w^2\phi''(w)dw.$$

This equation can be seen as a weak form of a *Fokker-Planck equation* (FPE):

$$\frac{\partial}{\partial\tau}\tilde{f}=\frac{\partial}{\partial w}\left(-A(\tau)w\tilde{f}+\frac{1}{2}B(\tau)\frac{\partial}{\partial w}w^2\tilde{f}\right), \tag{2.17}$$

with

$$A(\tau)=1+\tilde{\mu}(\tilde{S})\left((\kappa(\tilde{S})-1)+\frac{(\tilde{\mu}(\tilde{S})(\kappa(\tilde{S})-1)+1)}{1-\tilde{\mu}(\tilde{S})}\frac{\lambda}{\tilde{S}}\right)$$
$$B(\tau)=\frac{(\tilde{\mu}(\tilde{S})^2+\zeta^2)}{\tilde{S}^2}\nu$$

The outcome of this analysis is

Theorem 2. *Let $f_0\in\mathcal{M}_p$ be a probability density with $p=2+\delta$ for some $\delta>0$ and $\sigma^2=\nu b$, $D=\lambda b$ for $b\to 0$, $\sigma\to 0$, $D\to 0$.*
Then the solution to the weak form of the Boltzmann equation (2.11) *for the scaled probability density $\tilde{f}_b(w,\tau)$, with $\tau=rt$, converges, up to extraction of a subsequence, to a probability density $\tilde{f}(w,\tau)$ as $b\to 0$. This density is a solution of the weak form of the FPE* (2.17)

Remark

- For the evolution of the first moment we obtain

$$\dot{\bar{w}}(\tau)=\frac{d}{d\tau}\int_0^\infty \tilde{f}(w,\tau)wdw=A(\tau)\bar{w}(\tau).$$

- The evolution of the second moment is given by

$$\dot{\bar{e}}(\tau)=\frac{d}{d\tau}\int_0^\infty \tilde{f}(w,\tau)w^2dw=(2A(\tau)+B(\tau))\bar{e}(\tau).$$

The next step is to find self-similar solutions. For this reason we scale the original solution $\tilde{f}(w,\tau)$ in the following manner

$$\tilde{f}(w,\tau) = \frac{1}{w}\tilde{g}(\chi,\tau), \qquad \chi = \log(w)$$

If we take

$$\begin{aligned}
\frac{\partial}{\partial\tau}\tilde{f}(w,\tau) &= \frac{1}{w}\frac{\partial}{\partial\tau}\tilde{g}(\chi,\tau),\\
\frac{\partial}{\partial w}w\tilde{f}(w,\tau) &= \frac{1}{w}\frac{\partial}{\partial\chi}\tilde{g}(\chi,\tau),\\
\frac{\partial^2}{\partial w^2}w^2\tilde{f}(w,\tau) &= \frac{1}{w}\left(\frac{\partial}{\partial\chi}\tilde{g}(\chi,\tau) + \frac{\partial^2}{\partial\chi^2}\tilde{g}(\chi,\tau)\right)
\end{aligned}$$

into account, we obtain the linear convection-diffusion equation

$$\frac{\partial}{\partial\tau}\tilde{g}(\chi,\tau) = \left(\frac{B(\tau)}{2} - A(\tau)\right)\frac{\partial}{\partial\chi}\tilde{g}(\chi,\tau) + \frac{B(\tau)}{2}\frac{\partial^2}{\partial\chi^2}\tilde{g}(\chi,\tau).$$

This identity admits the self-similar solution (see [19])

$$\tilde{g}(\chi,\tau) = \frac{1}{\sqrt{2b(\tau)\pi}}\exp\left(-\frac{(\chi + \frac{b(\tau)}{2} - a(\tau))^2}{2b(\tau)}\right),$$

with

$$\begin{aligned}
a(\tau) &= \int_0^\tau A(s)ds + C_1, & b(\tau) &= \int_0^\tau B(s)ds + C_2,\\
C_1 &= a(0), & C_2 &= b(0).
\end{aligned}$$

Reverting to the original notation, we obtain the lognormal asymptotic behaviour of the model

$$\tilde{f}(w,\tau) = \frac{1}{w\sqrt{2b(\tau)\pi}}\exp\left(-\frac{(\log(w) + \frac{b(\tau)}{2} - a(\tau))^2}{2b(\tau)}\right)$$

and a precise description of $a(\tau)$ and $b(\tau)$:

$$\begin{aligned}
a(\tau) &= \int_0^\tau \frac{\dot{\bar{w}}(s)}{\bar{w}(s)}ds + C_1 = \log(\bar{w}(\tau))\\
b(\tau) &= \int_0^\tau \left(\frac{\dot{\bar{e}}(s)}{\bar{w}(e)} - 2\frac{\dot{\bar{w}}(s)}{\bar{w}(s)}\right)ds + C_2 = \log\left(\frac{\bar{e}(\tau)}{(\bar{w}(\tau))^2}\right).
\end{aligned}$$

2.2. Trading Tendency Model

This model's parameters are the market price and the behaviour of two types of traders acting on a financial market. We differentiate between chartists and fundamentalists. The characterisation of the different parameters is followed by a model extension where interaction is included. The outcome of this extended model is a system of well-known equations. In this section we basically follow the illustrations of [22], [23] and [24].

2.2.1. Modelling the Trading Tendency of Chartists

The chartists' trading tendency depends on information with no fundamental background. In this case, we focus on the price trend and the behaviour of other traders as non-fundamental sources, whereupon the latter parameter appears in forms of mimetic contagion and herding.
We call chartists optimistic when they have the propensity to buy and we call them pessimistic if they have the propensity to sell. These traders reevaluate their expectations in the light of the market's development according to $\frac{S'(t)}{S(t)}$, where $S(t)$ is the actual market price and $S'(t)$ represents the price change.

The collective behaviour of a system of trading agents can be described by introducing a state variable $y \in [-1, 1]$ to which we refer as to the trading tendency of an agent. Positive values of y represent potential buyers, while negative values characterise potential sellers, close to $y = 0$ we have undecided agents.
We denote by $f_C(y, t)$ the density function of chartists having propensity y at time t.
The traders compare their strategies and reevaluate them on the basis of interactions. With (y, y_*) representing the pre-interaction tendency and (y', y'_*) representing the post-interaction tendency, the binary interactions $(y, y_*) \to (y', y'_*)$ cause the system's dynamic via

$$y' = (1 - \alpha_1 H(y) - \alpha_2)y + \alpha_1 H(y) y_* + \alpha_2 \Phi\left(\frac{S'(t)}{S(t)}\right) + D_C(y)\eta$$
$$y'_* = (1 - \alpha_1 H(y) - \alpha_2)y_* + \alpha_1 H(y_*) y + \alpha_2 \Phi\left(\frac{S'(t)}{S(t)}\right) + D_C(y_*)\eta_*.$$

$\alpha_1 \in [0, 1]$ and $\alpha_2 \in [0, 1]$, with $\alpha_1 + \alpha_2 \ \leq 1$, are a measure for the importance the agents place on other agents' opinions and the actual price trend in forming

expectations about future price changes. The random variables η and η_* are assumed to be distributed according to $\theta_C(\eta)$ with zero mean and variance σ_C^2, they measure individual deviations to the average behaviour. The function $H(y) \in [0,1]$ characterises the herding behaviour whereas $D_C(y)$ defines the diffusive behaviour. $\Phi(\cdot) \in [-1,1]$ is a normalised value function that models the reaction of individuals towards potential gains and losses in the market (for details see [14], [18]).
This permits to introduce behavioural aspects in the market dynamic and to take into account the influence of irrational parameters on the behaviour of the trading agents.

We define the number density of chartists

$$\rho_C(t) = \int_{-1}^{1} f_C(y,t)\, dy$$

and their mean investment propensity

$$Y_C(t) = \frac{1}{\rho_C(t)} \int_{-1}^{1} f_C(y,t) y\, dy.$$

Using the standard tools of kinetic theory [4], we obtain the following Boltzmann equation for the evolution of the unknown density $f_C(y,t)$:

$$\frac{\partial f_C}{\partial t} = Q_C(f_C, f_C). \tag{2.18}$$

Multiplying (2.18) by a smooth test function φ with compact support in $[-1,1]$, we can conveniently write Q_C in weak form as

$$\int_{-1}^{1} Q_C \varphi(y)\, dy = \int_{[-1,1]^2} \int_{\mathbb{R}^2} B(y,y_*) f_C(y) f_C(y_*) \Big(\varphi(y') - \varphi(y)\Big)\, d\eta\, d\eta_*\, dy_*\, dy.$$

The transition rate for optimistic and pessimistic traders is given by
$B(y,y_*) = \Theta_C(\eta)\Theta_C(\eta_*)\chi(|y'| \le 1)\chi(|y'_*| \le 1)$ with the indicator function $\chi(\cdot)$.
The number density $\rho_C(t)$ is invariant in time so that the total number of chartists remains constant ($\varphi \equiv 1$).

2.2.2. Modelling the Trading Tendency of Fundamentalists

Fundamentalists follow the efficient market hypothesis, so they are of the opinion that prices immediately reflect all available news about future earning prospects in

an unbiased manner. To that effect the best prediction for the future price is the price of today $S(t)$.
These group of traders relies on a fundamental value $S_F(t)$ which takes into account market fundamentals and economic factors, such as dividends, earnings, macroeconomic growth, unemployment rates and others. If the market price $S(t)$ is not equal to the fundamental value $S_F(t)$, the fundamentalists' belief is that $S(t)$ will return to $S_F(t)$ in a short period of time. These traders evaluate their potential gain and loss through the difference $S_F(t) - S(t)$.
If $S_F(t)-S(t) > 0$, the fundamentalists will invest because in their opinion the value of the asset is underrated, the market price $S(t)$ increases until it reaches $S_F(t)$ and selling then yields a profit of $S_F(t) - S(t)$. On the contrary, selling overrated assets at a price of $S_F(t)$ sustains a loss of $S(t) - S_F(t)$, when bought at a price of $S(t)$ with $S_F(t) - S(t) < 0$.

In a similar way to the model of chartists we denote by $f_F(y, t)$ the density function of fundamentalists having propensity to buy or sell at time t, characterised by $y \in [-1, 1]$. These traders are not exposed to herding behaviour.
The dynamic follows this rule

$$y' = (1 - \xi)y + \xi\Psi\left(\frac{S_F(t) - S(t)}{S(t)}\right) + D_F(y)\eta.$$

ξ is a measure for the fact that fundamentalists realise that it can take some time for the market price $S(t)$ to revert to its fundamental value $S_F(t)$.
The random variable η is assumed to be distributed according to $\Theta_F(\eta)$ with zero mean and variance σ_F^2 and measures individual deviations to the average behaviour.
The function $D_F(y) \in [0, 1]$ defines the diffusive behaviour and vanishes in $y = \pm 1$.
The function $\Psi(\cdot) \in [-1, 1]$ is an adequate normalised value function.

We define the number density of fundamentalists

$$\rho_F(t) = \int_{-1}^{1} f_F(y, t)\, dy,$$

additionally their mean investment propensity

$$Y_F(t) = \frac{1}{\rho_F(t)} \int_{-1}^{1} f_F(y, t) y\, dy$$

and with [4] we get

$$\frac{\partial f_F}{\partial t} = Q_F(f_F). \tag{2.19}$$

The linear interaction operator Q_F in weak form is

$$\int_{-1}^{1} Q_F \varphi(y)\,dy = \rho_F \int_{[-1,1]^2} \int_{\mathbb{R}^2} B_F(y) f_F(y) \Big(\varphi(y') - \varphi(y)\Big)\,d\eta\,dy$$

where the transition rate is taken as $B_F(y) = \Theta_F(\eta)\chi(|y'| \leq 1)$.

2.2.3. Modelling the Price

Since the traders are influenced by the dynamics of the price, a coupling with the price dynamic is an obvious choice for a more realistic model.
Therefore we use the mean investment propensity and number density of both trading groups and get the effective demand by multiplying the expected trading propensity $\int_{-1}^{1} f(y,t)y\,dy$ with the corresponding maximum trade volume $T > 0$.
We define the combined effective demand of chartists and fundamentalists as

$$\begin{aligned} ED(t) &= ED_C(t) + ED_F(t) \\ &= T_C \int_{-1}^{1} f_C(y,t)y\,dy + T_F \int_{-1}^{1} f_F(y,t)y\,dy \\ &= \rho_C(t)Y_C(t)T_C + \rho_F(t)Y_F(t)T_F \end{aligned}$$

with $T_C > 0, T_F > 0$.

We assume the market price $S(t)$ to satisfy the following equation

$$S'(t) = \beta ED(t)S(t) = \beta\Big(\rho_C(t)Y_C(t)T_C + \rho_F(t)Y_F(t)T_F\Big)S(t), \tag{2.20}$$

whereupon $\beta > 0$ is the weight of demand.
(2.20) yields

$$\frac{dS}{dt} = \beta(\rho_C Y_C T_C + \rho_F Y_F T_F)S. \tag{2.21}$$

2.2.4. The Dynamics

Chartists and Price

We assume that only chartists and no fundamentalists act on the market. A kinetic model for chartists and the price can be formulated by combining (2.18) with the chartists part of (2.21)

$$\begin{aligned} \frac{\partial f_C}{\partial t} &= Q_C(f_C, f_C) \\ \frac{dS}{dt} &= \beta(\rho_C Y_C T_C)S. \end{aligned} \tag{2.22}$$

The chartists' tendency to trade and the price have an influence on each other. On the one hand the price $S(t)$ is affected by the mean propensity to invest $Y_C(t)$ through the second equation of (2.22) and on the other hand $S(t)$ has an effect on $Y_C(t)$ through the value function $\Phi\left(\frac{S'(t)}{S(t)}\right)$, since $\frac{S'(t)}{S(t)} = \beta(\rho_C Y_C T_C)$, with ρ_C being constant.
If the trading tendency is sufficiently positive at first, the tendency will increase and so will the price. Vice versa will the trading tendency decrease (and also the price) if the tendency value is initially negative.
According to the choices of $H(y)$ and $\Phi(\cdot)$, the market goes towards a boom (exponential growth of the price) or a crash (exponential decay of the price) and agents tend to concentrate in $y = 1$ and $y = -1$ respectively. This is obvious because of the price following nature of the chartists.

Non-interacting Agents and Price

The fundamentalists influence the market price $S(t)$ through their expectation of the fundamental price $S_F(t)$ and for this reason they are pushing the actual price towards the fundamental price. If there is a constant fundamental price $S_F(t)$, the steady state is reached at $S^\infty = S_F$ and $Y_F^\infty = 0$.
If the number of fundamentalists is large in contrast to the number of chartists, they are capable to affect the market and lead the price to the fundamental value. Otherwise the chartists dynamic leads the price.
The presence of fundamentalists can also lead to cycles or oscillations around the fundamental price $S_F(t)$. When booms or crashes appear, a minority of fundamentalists can secure that the chartists do not concentrate in $y = \pm 1$.
If the market price $S(t)$ is equal to the fundamental price $S_F(t)$, we have an equilibrium when chartists and also fundamentalists have the mean propensity to hold, $Y_C(t) = 0$ and $Y_F(t) = 0$. This steady state is independent of the number densities, $\rho_C(t)$ and $\rho_F(t)$.

2.2.5. The Generalised Kinetic Model

From (2.18), (2.19) and (2.21) we get a simple model for chartists and fundamentalists where the traders are coupled through the price:

$$\begin{cases} \frac{\partial f_C}{\partial t} & = Q_C(f_C, f_C) \\ \frac{\partial f_F}{\partial t} & = Q_F(f_F) \\ \frac{dS}{dt} & = \beta(\rho_C Y_C T_C + \rho_F Y_F T_F)S \end{cases}$$

Extended Price Formation

In order to formulate the model in a more realistic way, we characterise the price formation process in a more explicit manner. So far we have neglected any additional source of randomness except the one present in the traders' dynamic, but there could be traders with additional liquidity in the market whose excess demand is stochastic or the value of excess demand is perceived with some inaccuracy by the auctioneer. With the same kinetic setting as before, we introduce the probability density function $V(s,t)$ of a given price s at time t.

The market price $S(t)$ is then defined as the mean value

$$S(t) = \int_0^\infty V(s,t) s \, ds$$

Price changes are modelled as endogenous responses of the market to imbalance between demand and supply characterised by the mean investment propensity according to the following price adjustment

$$s' = s + \beta\big(\rho_C(t) Y_C(t) T_C + \rho_F(t) Y_F(t) T_F\big) s + \beta \eta s,$$

where η is a random variable distributed according to $\Theta_V(\eta)$ with zero mean and variance σ_V^2.

Strategy Exchange

So far the agents are not interacting, but now agents from different groups with trading tendencies (y_c, y_f) meet, compare excess profits from both strategies and with a probability depending on the pay-off differential switch to the more successful strategy. They characterise the success of a given strategy by comparing

$$
\begin{aligned}
X_C(y_c,t) &= y_c\left(\frac{S'(t)+D}{S(t)} - R\right),\\
X_F(y_f,t) &= y_f k \frac{S_F - S(t)}{S(t)}
\end{aligned}
$$

where D is a nominal dividend, R is the average real return of the market with $R = \frac{D}{S_F}$.

The discount factor $k < 1$ is reasonable by the observation that X_F is an expected gain realised only after reversion to S_F.

We add the factor of strategy exchange in the following way to the presented model:

$$
\begin{cases}
(y_c, y_f) \to (y_c, y_c), & \text{with rate} \quad B\big(X_C(y_c), X_F(y_f)\big)\\
(y_c, y_f) \to (y_f, y_f), & \text{with rate} \quad B\big(X_F(y_f), X_C(y_c)\big),
\end{cases}
$$

with $B \geq 0$, $B(x,y) = B(y,x)$ for $S = S_F$, $\partial_x B(x,y) > 0$, $\partial_y B(x,y) < 0$.

That means fundamentalists change their strategy with the rate $B\big(X_C(y_c), X_F(y_f)\big)$ and chartists adopt the strategy of fundamentalists with the rate $B\big(X_F(y_f), X_C(y_c)\big)$.

The related strategy exchange operators read

$$
\begin{aligned}
Q_{FC}(f_C, f_F) &= \mu f_C(y) \int_{-1}^{1} f_F(y_*) \left(B\left(X_C(y), X_F(y_*)\right) - B\left(X_F(y_*), X_C(y)\right)\right) dy_*\\
Q_{CF}(f_F, f_C) &= \mu f_F(y) \int_{-1}^{1} f_C(y_*) \left(B\left(X_F(y), X_C(y_*)\right) - B\left(X_C(y_*), X_F(y)\right)\right) dy_*,
\end{aligned}
$$

where $\mu > 0$ measures the frequency of the exchange rate.

This finally leads to the following model

$$
\begin{cases}
\frac{\partial f_C}{\partial t} &= Q_C(f_C, f_C) + Q_{FC}(f_C, f_F)\\
\frac{\partial f_F}{\partial t} &= Q_F(f_F) + Q_{CF}(f_F, f_C)\\
\frac{\partial V}{\partial t} &= Q_V(V).
\end{cases}
$$

The price formation operator in weak form is

$$
\int_0^\infty Q_V \varphi(s)\, ds = \int_0^\infty \int_{\mathbb{R}} B_V(s) V(s) \big(\varphi(s') - \varphi(s)\big)\, d\eta\, ds
$$

with the transition rate $B_V(s) = \Theta_V(\eta)\chi(s' \geq 0)$.

Our aim is to rewrite the generalised system in form of FPEs. To succeed we observe the limiting asymptotic case of quasi-invariant investment propensity and price variation. For this purpose we introduce a time scaling parameter ξ and define

$$\tau = \xi t, \qquad \tilde{f}_C(y,\tau) = f_C(y,t), \qquad \tilde{f}_F(y,\tau) = f_F(y,t), \qquad \tilde{V}(s,\tau) = V(s,t).$$

For $\xi, \mu, T_C, T_F \to 0$ and $\sigma_C^2, \sigma_F^2, \sigma_V^2 \to 0$ we assume

$$\frac{\alpha_i}{\xi} \to \tilde{\alpha}_i, \qquad \frac{T_C}{\xi} \to \tilde{T}_C, \qquad \frac{T_F}{\xi} \to \tilde{T}_F, \qquad \frac{\mu}{\xi} \to \tilde{\mu}$$

and

$$\frac{\sigma_C^2}{\xi} \to \lambda_C, \qquad \frac{\sigma_F^2}{\xi} \to \lambda_F, \qquad \frac{\sigma_V^2}{\xi} \to \lambda_V.$$

Following standard asymptotic techniques in [8] and for simplicity reverting back to the original notations, we obtain the following *Fokker-Planck system*

$$\begin{cases} \frac{\partial f_C}{\partial t} + \frac{\partial}{\partial y}\left[\alpha_1 \rho_C \left(H(y)Y_C - y\right) f_C + \alpha_2 \rho_C \left(\Phi - y\right) f_C\right] = \frac{\lambda_C}{2}\frac{\partial^2}{\partial y^2}\left(\rho_C D_C^2(y) f_C\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mu \rho_F f_C(y) b(Y_C, Y_F), \\ \frac{\partial f_F}{\partial t} + \frac{\partial}{\partial y}\left[\rho_F \left(\Psi - y\right) f_F\right] = \frac{\lambda_F}{2}\frac{\partial^2}{\partial y^2}\left(\rho_F D_F^2(y) f_F\right) - \mu \rho_C f_F(y) b(Y_C, Y_F), \\ \frac{\partial V}{\partial t} + \frac{\partial}{\partial s}\left[\beta\left(\rho_C Y_C + \rho_F Y_F\right) s V\right] = \frac{\lambda_V}{2}\beta^2 \frac{\partial^2}{\partial s^2}(s^2 V). \end{cases} \tag{2.23}$$

This system of FPEs is useful to study the long time evolution of the agents' behaviour and the price.

Long Time Behaviour of the Price

The equation for the price is of the form

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial s}\left[A(t) s V\right] = \frac{\lambda_V}{2}\beta^2 \frac{\partial^2}{\partial s^2}(s^2 V),$$

with

$$A(t) = \beta\left(\rho_C(t) Y_C(t) + \rho_F(t) Y_F(t)\right).$$

This implies that the long time behaviour is characterised by a self-similar solution of log-normal form [7]

$$V(s,t) = \frac{1}{s(\lambda_V \beta^2 t \pi)^{\frac{1}{2}}} \exp\left(-\frac{\left(\log(s) + \lambda_V \beta^2 t - \log\left(S(t)\right)\right)^2}{\lambda_V \beta^2 t}\right)$$

Long Time Behaviour of the Agents
The explicit computation of the long time behaviour of agents in the general case is difficult due to the presence of the herding term $H(y)$ and the diffusive term $D(y)$. Explicit steady states can be computed for the simplified case $H(y) = 1$ and $D_C(y) = D_F(y) = 1 - y^2$ [26].
In that case we rewrite the first two Fokker-Planck equations of (2.23) into

$$\frac{\lambda_C}{2}\frac{\partial}{\partial y}\left(\left(1-y^2\right)^2 f_C\right) + (y - m_C)\, f_C = 0, \qquad \text{for the chartists and accordingly}$$
$$\frac{\lambda_F}{2}\frac{\partial}{\partial y}\left(\left(1-y^2\right)^2 f_F\right) + (y - m_F)\, f_F = 0, \qquad \text{for the fundamentalists,}$$

with $\quad m_C = \alpha_1 Y_C + \alpha_2 \Phi, \qquad m_F = \Psi.$
For those choices of $H(y)$ and $D(y)$ we obtain the following solutions (cf. [26])

$$f_C^\infty(y) = C_1(1-y)^{-\frac{1}{2}\frac{m_C+4\lambda_C}{\lambda_C}}\,(1+y)^{-\frac{1}{2}\frac{m_C-4\lambda_C}{\lambda_C}}\,\exp\left(\frac{-(m_C y - 1)}{\lambda_C(y-1)(y+1)}\right),$$
$$f_F^\infty(y) = C_2(1-y)^{-\frac{1}{2}\frac{m_F+4\lambda_F}{\lambda_F}}\,(1+y)^{-\frac{1}{2}\frac{m_F-4\lambda_F}{\lambda_F}}\,\exp\left(\frac{-(m_F y - 1)}{\lambda_F(y-1)(y+1)}\right),$$

with C_1 and C_2 related to ρ_C^∞ and ρ_F^∞.
If we make other choices for $D(y)$, we will receive other solutions as well.

2.3. Group Size Model

The last model utilises the trading groups of chartists and fundamentalists as in 2.2 and analyses the evolution of the group size as the system state.

2.3.1. From Ants to a Financial Market

At first view it seems strange to analyse the behaviour of ants in order to understand the performance of agents trading with stocks although the ant model provides a similar phenomena as it has been observed on financial markets. Since we are interested in the collective we are able to draw comparisons. Consecutively we present the fundamental ideas of [15].

Experiments with ant colonies organised by entomologists reveal an interesting behaviour of these insects. Two identical food sources were placed equidistant from

the nest of a colony of ants and the sources were constantly replenished to preserve the condition of equality.
More precisely we call the first food source black and the second one white, the order is not important since the following holds for every order of the two sources. We have N ants and each ant feeds at one of the two sources. The number k of ants feeding at the black source defines the state of the system with $k \in \{1, ..., N\}$. The dynamic of the system is based on the random meetings of two ants and the probability for a switch of the food source's colour is given by $1 - \delta$, that means that an ant feeding at the black (white) food source switches to the white (black) food source with the mentioned probability. It is also possible for an ant to switch the colour independently without meeting another ant. This probability is given by ϵ and it prevents the system from the irremovability in the states $k = 0$ and $k = N$. From the economic point of view an independent switch can be interpreted as a change in the trader's knowledge concerning new information as well as a replacement of an 'old' trader with a 'new' trader who might prefer the other colour of the food source. Furthermore ϵ is assumed to be small and $\epsilon \to 0$ holds for $N \to \infty$.
The evolution of the state is given by

$$k \to \begin{cases} k+1, & \text{with the probability} \quad p_{k,k+1} = \left(1 - \frac{k}{N}\right)\left(\epsilon + (1-\delta)\frac{k}{N-1}\right) \\ k-1, & \text{with the probability} \quad p_{k,k-1} = \frac{k}{N}\left(\epsilon + (1-\delta)\frac{N-k}{N-1}\right) \\ k, & \text{with the probability} \quad p_{k,k} = p_k = 1 - p_{k,k+1} - p_{k,k-1}, \end{cases} \tag{2.24}$$

with $p_{k,k+1} + p_{k,k-1} \leq 1$.
We assume the *Markov property* to be valid, which means that the probability for a change of the system state depends only on the actual state and not on the previous states.
For different choices of δ and ϵ two interesting cases occur:

- $\delta = 1, \epsilon = \frac{1}{2}$: The ants are not interacting, so the food sources are chosen randomly with equal probability.

- $\delta = \epsilon = 0$: We always observe a change of the food source due to the dominant recruiting term which will lead to the state $k = 0$ or $k = N$.

For the long time behaviour one intuitively would expect an even split of the ants between the two food sources but the result of the entomologists' experiments was an asymmetric ant behaviour within a symmetric setting. By specifying this we can say that 80% of the ant colony fed at one food source and 20% at the other.

The presented stochastic process indicates the probability of the movement from one state to another, where the state is defined by the number of ants feeding at a chosen food source. Worth mentioning is that none of these states is an equilibrium. The only meaningful equilibrium is the equilibrium distribution μ_k of (2.24) defining the proportion of time the system spends in state k.
It is given by

$$\mu_k = \sum_{l=0}^{N} \mu_l p_{l,k}.$$

For the evolution of the process with a large number of ants, we have to examine the asymptotic form of μ_k for $N \to \infty$ with $\epsilon \to 0$, more precisely the *mean-field limit* has to be considered. Therefore we choose ϵ for each N such that $\epsilon N < (1-\delta)$ and we approximate μ_k with a continuous distribution $f(x)$ on the unit interval, i.e. $x = \frac{k}{N} \in [0,1]$. Considering $\epsilon = \frac{\alpha}{N}$, $\delta = \frac{2\alpha}{N}$ and $N \to \infty$, we obtain a density f of a symmetric *Beta distribution*

$$f(x) = c \cdot x^{\alpha-1}\,(1-x)^{\alpha-1},$$

where c is a normalisation constant and α is a positive shape parameter. This is a situation with multiple equilibria but not in the regular meaning, since every state is visited more than once. Hence we cannot expect a convergence to any particular state.

Transmission on Financial Markets

Relating this ant model to the behaviour of traders on a financial market, we realise that it is useless to analyse the characteristics of a single isolated trader even though the other $N-1$ traders are identical. Therefore, we consider groups of traders, differentiated by various choices they make, and investigate their interaction involving a herding behaviour. More precisely the findings in the ant model display a recruiting mechanism. On a financial market recruiting appears in terms of conviction due to better information or knowledge of one agent or agreement of one agent with another who already came to a decision. Taking account of fundamentalists as rational sophisticated traders and chartists as noise traders, [20] picture an increase of the proportion of noise traders when they realise higher returns than the sophisticated traders. This can be perceived as a recruitment to the more successful strategy.

In conclusion we recognise that this model is capable to produce just the properties which are needed for the understanding of the traders' behaviour on financial markets. Thus we have learned more about the agents' performance and their interaction than about the ants' feeding manner.

2.3.2. Model with two Groups

Based on the ant model and the ideas in [9] we start observing the evolution process of the group size of chartists, more precisely the optimistic and pessimistic traders. The group size of the fundamentalists N_F remains constant and neglected during this analysis. We have an overall number N_C of agents. With $N_+ = z$ and $N_- = N_C - z$ we define the group size of the optimistic and pessimistic agents excluding the fundamentalists. $S = \{0, ..., N_C\}$ is the discrete set of states, $z \in S$.
The following process is *markovian* and we have

$$\bar{w}(z, t + \Delta t) = \sum_{\tilde{z}} \bar{w}(z, t + \Delta t | \tilde{z}, t) \bar{w}(\tilde{z}, t), \tag{2.25}$$

where $\bar{w}(z, t)$ is the probability distribution with the probability density $w(z, t)$ of a state z at time t describing the probability of z agents being optimistic at time t. $\bar{w}(z, t + \Delta t \,|\tilde{z}, t)$ defines the conditional probability of observing z optimists at time $t + \Delta t$ given $\tilde{z}$ optimists at time t.
We choose Δt as small as necessary in order to ensure that only a switch of one agent at the most is possible. This leads to

$$\begin{aligned} \bar{w}(z, t + \Delta t) =& \bar{w}(z + 1, t) \bar{w}(z, t + \Delta t | z + 1, t) \\ &+ \bar{w}(z - 1, t) \bar{w}(z, t + \Delta t | z - 1, t) \\ &+ \bar{w}(z, t) \bar{w}(z, t + \Delta t | z, t) \end{aligned} \tag{2.26}$$

Next we introduce the probability for a group switch referred to as transition rate

$$\pi_{\tilde{z}z} = \frac{1}{\Delta t} \bar{w}(z, t + \Delta t | \tilde{z}, t).$$

Rewriting this transition rate by using an idiosyncratic propensity a_{ik} to change from group i to k and a parameter b_{ik} related to the herding effect with $i, k \in \{+, -\}$ yields

$$\begin{aligned} \pi_{+-} &= (a_{+-} + (N_C - z) b_{+-}) z, \\ \pi_{-+} &= (a_{-+} + z b_{-+})(N_C - z). \end{aligned}$$

Dividing (2.26) by Δt, using $\bar{w}(z,t|\tilde{z},t) = \delta_{z\tilde{z}}$ with $\delta = \begin{cases} 1, z = \tilde{z} \\ 0, z \neq \tilde{z}, \end{cases}$,
$\sum_z \bar{w}(z, t+\Delta t|\tilde{z},t) = 1$ and the corresponding transition rates we obtain

$$\begin{aligned}\frac{\bar{w}(z,t+\Delta t)}{\Delta t} =& \bar{w}(z+1,t)\pi_{+-}(z+1) + \bar{w}(z-1,t)\pi_{-+}(z-1) \\ &+ \frac{\bar{w}(z,t)}{\Delta t} - \bar{w}(z,t)\pi_{+-}(z) - \bar{w}(z,t)\pi_{-+}(z).\end{aligned}$$

For $\Delta t \to 0$ we get the *master equation*

$$\begin{aligned}\frac{\partial \bar{w}}{\partial t}(z,t) =& \bar{w}(z+1,t)\pi_{+-}(z+1) + \bar{w}(z-1,t)\pi_{-+}(z-1) \\ &- \bar{w}(z,t)\pi_{+-}(z) - \bar{w}(z,t)\pi_{-+}(z).\end{aligned} \tag{2.27}$$

We establish a scaling for the variable z with $n = \frac{z}{N}$, $n \in \Omega$ with the set of states $\Omega = \{0, \frac{1}{N}, ..., \frac{N}{N}\} \subseteq [0,1]$. For $N \to \infty$ the set of states Ω becomes continuous.
The next step is to write (2.27) in terms of the probability density w with the scaled variable n:

$$\begin{aligned}\frac{\partial w}{\partial t}(n,t) =& w(n+\frac{1}{N},t)\pi_{+-}(nN+1) + w(n-\frac{1}{N},t)\pi_{-+}(nN-1) \\ &- w(n,t)\pi_{+-}(nN) - w(n,t)\pi_{-+}(nN).\end{aligned}$$

With the assumption of a symmetric herding effect $b_{+-} = b_{-+} := b$, the definition $a_{+-} := a_1$, $a_{-+} := a_2$ for the tendency of independent group switches and the use of a few mathematical tools we obtain a one-dimensional FPE

$$\frac{\partial w}{\partial t}(n,t) = -\frac{\partial}{\partial n}\left[\bar{A}(n)w(n,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial n^2}\left[\bar{D}(n,t)\right],$$

with the drift coefficient $\bar{A}(n)$ and the diffusion coefficient $\bar{D}(n)$ given by

$$\begin{aligned}\bar{A}(n) &= a_2 - (a_1+a_2)n \\ \bar{D}(n) &= 2bn(1-n) + \frac{1}{N}(a_2(1-n) + a_1 n).\end{aligned}$$

We rewrite $x = \frac{2N_+}{N} - 1 = 2n - 1$ with $x \in [-1,1]$ and transpose the upper FPE. This yields

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}(x,t)D(x) + w(x,t)A(x)\right),$$

with

$$
\begin{aligned}
A(x) &= a_1(x+1) + a_2(x-1) - 2bx \\
D(x) &= b(1-x^2).
\end{aligned}
$$

For more detailed annotations and calculations see [9].

2.3.3. Model with three Groups

The next step is to add the fundamentalists N_F into the interaction with the chartists N_C. This yields an overall number $N = N_F + N_C$ of trading agents. Further we denote by $N_+ = u$ the number of optimistic traders, by $N_- = v$ the number of pessimistic traders and by $N_F = N - u - v$ the fundamentalistic traders with $z = \begin{pmatrix} u \\ v \end{pmatrix}$ representing the system state.
The discrete set of states is given by $\bar{S} = \{u, v \in [0, ..., N]^2 | u+v = N_C\}$. Introducing the probability distribution $\bar{w}(z,t) = \bar{w}(u,v,t)$, which describes the probability of having x optimists and y pessimists at time t, we get

$$
\begin{aligned}
\bar{w}(u,v,t+\Delta t) =& \bar{w}(u+1,v-1,t)\bar{w}(u,v,t+\Delta t|u+1,v-1,t) \\
&+ \bar{w}(u-1,v+1,t)\bar{w}(u,v,t+\Delta t|u-1,v+1,t) \\
&+ \bar{w}(u,v+1,t)\bar{w}(u,v,t+\Delta t|u,v+1,t) \\
&+ \bar{w}(u,v-1,t)\bar{w}(u,v,t+\Delta t|u,v-1,t) \\
&+ \bar{w}(u+1,v,t)\bar{w}(u,v,t+\Delta t|u+1,v,t) \\
&+ \bar{w}(u-1,v,t)\bar{w}(u,v,t+\Delta t|u-1,v,t) \\
&+ \bar{w}(u,v,t)\bar{w}(u,v,t+\Delta t|u,v,t).
\end{aligned}
$$

For $\Delta t \to 0$ multiple group switches during a time step Δt become increasingly unlikely and therefore we assume a switch of one agent only.
Analysing three interacting groups, we require conditions for the idiosyncratic propensity to switch and the herding parameter. We assume symmetric parameters between the optimistic and pessimistic traders. This leads to

$$
\begin{aligned}
a_{+F} &= a_{-F}, \qquad & a_{+-} &= a_{-+}, \qquad & a_{F+} &= a_{F-}, \\
b_{+F} &= b_{-F}, \qquad & b_{+-} &= b_{-+}.
\end{aligned}
$$

Hence we formulate $a_1 = a_{ik}$, $a_2 = a_{iF}$, $a_3 = a_{Fi}$ and $b_1 = b_{ik}$, $b_2 = b_{iF} = b_{Fi}$ for $i, k \in \{+, -\}$. The relation $\frac{a_{+-}}{b_{+-}} = \frac{a_{F-}}{b_{F-}}$ holds as well.
Pursuing the same ideas as in paragraph 2.3.2 and using the scaling $n = \frac{u}{N}$, $m = \frac{v}{N}$ with the set of states $\bar{\Omega} = \{\begin{pmatrix} n \\ m \end{pmatrix} | n, m \in [0,1], n + m \leq 1\}$, we obtain the master equation

$$\begin{aligned}
\frac{\partial w}{\partial t}(n,m,t) =& w(n + \frac{1}{N}, m - \frac{1}{N}, t)\pi_{+-}(nN+1, mN-1) - w(n,m,t)\pi_{+-}(nN, mN) \\
&+ w(n - \frac{1}{N}, m + \frac{1}{N}, t)\pi_{-+}(nN-1, mN+1) - w(n,m,t)\pi_{-+}(nN, mN) \\
&+ w(n, m + \frac{1}{N}, t)\pi_{-F}(nN, mN+1) - w(n,m,t)\pi_{-F}(nN, mN) \\
&+ w(n, m - \frac{1}{N}, t)\pi_{F-}(nN, mN-1) - w(n,m,t)\pi_{F-}(nN, mN) \\
&+ w(n + \frac{1}{N}, m, t)\pi_{+F}(nN+1, mN) - w(n,m,t)\pi_{+F}(nN, mN) \\
&+ w(n - \frac{1}{N}, m, t)\pi_{F+}(nN-1, mN) - w(n,m,t)\pi_{F+}(nN, mN).
\end{aligned}$$

We use a Taylor expansion and insert the transition rates in order to formulate a two-dimensional FPE

$$\frac{\partial w}{\partial t}(n,m,t) = -\sum_{i=1}^{2} \frac{\partial}{\partial i}(\bar{A}_i(n,m)w(n,m,t)) + \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial^2}{\partial i \partial j}(\bar{D}_{ij}(n,m)w(n,m,t)), \tag{2.28}$$

with a two-dimensional drift vector $\bar{A}(n,m)$ and an associated 2×2 diffusion matrix $\bar{D}(n,m)$ given by

$$\begin{aligned}
\bar{A}(n,m) &= \begin{pmatrix} a_1(m-n) - a_2 n + a_3(1-n-m) \\ a_1(n-m) - a_2 m + a_3(1-n-m) \end{pmatrix} \\
\bar{D}(n,m) &= \begin{pmatrix} 2b_1 nm + 2b_2 n(1-n-m) & -2b_1 nm \\ -2b_1 nm & 2b_1 nm + 2b_2 m(1-n-m) \end{pmatrix} \\
&\quad + \frac{1}{N}\begin{pmatrix} a_1(m+n) + a_2 n + a_3(1-n-m) & -a_1(n+m) \\ -a_1(n+m) & a_1(m+n) + a_2 m - a_3(1-n-m) \end{pmatrix}.
\end{aligned}$$

By rewriting (2.28) in divergence form and introducing the new variables $x = \frac{n-m}{n+m}$ and $y = \frac{n+m}{1-n-m}$ we gain

$$\frac{\partial w}{\partial t}(x,y,t) = \nabla \cdot \left(D(x,y)\nabla w(x,y,t) + A(x,y)w(x,y,t)\right),$$

with

$$D(x,y) = \begin{pmatrix} (1-x^2)\frac{b_1 y b_2}{y} & 0 \\ 0 & b_2 y (y+1)^2 \end{pmatrix},$$
$$A(x,y) = \begin{pmatrix} 2x\frac{a_1 y - b_1 y + a_3 - b_2}{y} \\ (y+1)(a_2 y - 2a_3 + 2b_2 - b_2 y) \end{pmatrix}.$$

For further illustrations consult [9].

Special Choice of the Transition Rates

We require the FPE (2.28) and use the definitions in [6] for a special choice of the transition rates π_{ik}, π_{Fi} and π_{iF}, with $i, k \in \{+,-\}$. For this reason we consider

$$\pi_{-+} = \nu_1 \frac{N_C}{N} \exp(U_1), \qquad \pi_{+-} = \nu_1 \frac{N_C}{N} \exp(-U_1),$$

where the parameter ν_1 represents a measure of the frequency of opinion revaluation. The main influence on the chartists' opinion formation is characterised by the forcing term

$$U_1 = \alpha_1 \frac{N_+ - N_-}{N_C} + \frac{\alpha_2}{\nu_1} \frac{dp}{pdt}.$$

α_1 and α_2 measure the importance of the majority opinion of the chartists respectively of the relative price change. The first term is related to herding behaviour, whereas the second term represents the trend following nature of the chartists.

The transition rates involving the fundamentalists are formulated in this manner

$$\pi_{F+} = \nu_2 \frac{N_+}{N} \exp(U_{2,1}), \qquad \pi_{+F} = \nu_2 \frac{N_F}{N} \exp(-U_{2,1}),$$
$$\pi_{F-} = \nu_2 \frac{N_-}{N} \exp(U_{2,2}), \qquad \pi_{-F} = \nu_2 \frac{N_F}{N} \exp(-U_{2,2}),$$

where the parameter ν_2 represents a measure of the frequency of opinion revaluation and the forcing terms $U_{2,1}$ and $U_{2,2}$ depending on the difference between the profits earned by chartists and fundamentalist are given by

$$U_{2,1} = \alpha_3 \left(\underbrace{\frac{r + \frac{1}{\nu_2}\frac{dp}{dt}}{p} - R}_{\textit{profits of optimistic chartists}} - \underbrace{s\left|\frac{p_F - p}{p}\right|}_{fundamentalists' profit} \right),$$

$$U_{2,2} = \alpha_3 \left(\underbrace{R - \frac{r + \frac{1}{\nu_2}\frac{dp}{dt}}{p}}_{profits\ of\ pessimistic\ chartists} - \underbrace{s \left| \frac{p_F - p}{p} \right|}_{fundamentalists' profit} \right).$$

The nominal dividend is represented by r and the average return is characterised by R. The fundamentalists consider the deviation between the price p and the fundamental value p_F. Due to the realisation of gains in the future they require a discount factor $s < 1$.

We attach these information into (2.28) and by following the results of [12], we obtain the rewritten drift vector $\bar{A}(n,m)$ and diffusion matrix $\bar{D}(n,m)$ by

$$\bar{A}(n,m) = \begin{pmatrix} \nu_1(n+m)\exp(-U_1) - \nu_1(n+m)\exp(U_1) + \nu_2\frac{N_F}{N}\exp(-U_{2,1}) - \nu_2 n \exp(U_{2,1}) \\ \nu_1(n+m)\exp(U_1) - \nu_1(n+m)\exp(-U_1) + \nu_2\frac{N_F}{N}\exp(-U_{2,2}) - \nu_2 n \exp(U_{2,2}) \end{pmatrix},$$

$$\bar{D}(n,m) = \nu_1(n+m) \begin{pmatrix} \exp(-U_1) + \exp(U_1) & -\exp(-U_1) - \exp(U_1) \\ -\exp(-U_1) - \exp(U_1) & \exp(-U_1) + \exp(U_1) \end{pmatrix}$$
$$+ \nu_2 \begin{pmatrix} n\exp(-U_{2,1}) + \frac{N_F}{N}\exp(U_{2,1}) & 0 \\ 0 & m\exp(-U_{2,2}) + \frac{N_F}{N}\exp(U_{2,2}) \end{pmatrix}.$$

2.3.4. Mean-Field Consideration

So far, we have introduced an agent-based model which describes the group size development of the optimists in the case of two groups in contrast to the development of the chartists in the case of three groups. We obtained a FPE for the time dependent evolution of the according probability distribution for each case.

Now, with the help of [1] we will investigate the system concerning the increase of the overall number of agents. Hence we focus on the case of two groups, namely the optimistic and pessimistic chartists, with a closer look at the extensive and non-extensive transition rates. The case of three groups with the additional group of fundamentalists yields analogue results.

We remind of the non-extensive transition rates presented in 2.3.2

$$\begin{aligned} \pi_{+-} &= (a_{+-} + (N_C - z)b_{+-})z, \\ \pi_{-+} &= (a_{-+} + zb_{-+})(N_C - z). \end{aligned} \tag{2.29}$$

In the following we abbreviate (2.29) with

$$\pi_1 = (a + (N_C - z)b)z,$$

since we assume the idiosyncratic tendencies for a group change a_{+-} and a_{-+} and the herding parameters b_{+-} and b_{-+} to be symmetric. Furthermore, the transition rates are invariant under the transformation $z \to (N_C - z)$. In order to complete the basics, we establish the extensive transition rate

$$\pi_2 = (a + \frac{(N_C - z)}{N_C}b)z.$$

Both transition rates π_1 and π_2 imply a parameter for independent changes a and a parameter for changes due to recruiting b. The transition rate π_1 includes a herding parameter depending on the overall number of pessimists $N_C - z$ whereas π_2 depends on the fraction of pessimistic traders $\frac{N_C - z}{N_C}$. This difference is essential concerning the increase of the agents, thus taking the limit $N_C \to \infty$, since $b \sim O(1)$ holds in the first case and $b \sim O(\frac{1}{N_C})$ in the second.

Based on $b \sim O(\frac{1}{N_C})$, a decreasing herding behaviour occurs in the limit $N_C \to \infty$ for π_2. Hence the system looses interesting characteristics with an increasing number of agents. It is advisable to use the extensive transition rate in an environment with a 'smaller' number of agents in order to preserve the recruiting mechanism.

Assuming an increase of the number of agents, more precisely considering the *mean-field limit* $N_C \to \infty$ for π_1, we realise, due to $b \sim O(1)$, that the herding influence increases as well and this result seems logical. Increasing the overall number of agents and keeping the area S fixed leads to an increase of the agents' density $w(z,t)$ and therefore to a higher interaction rate. Due to this fact, the overall number of traders can be seen as a force field concerning a herding or recruiting behaviour where every agent is a part of this field and also is influenced by it. It is evident that the non-extensive transition rate is independent of the system size and because of that the interesting features are preserved.

3. Trading Decision Model

So far we have analysed agent-based kinetic models related to financial markets. These models are concerned with various approaches of investigation as the wealth of agents, their trading propensity or the group size of the various types of traders. In this chapter we introduce a novel model that presents a different approach by means of the parameters in section 2.2 and section 2.3. We are interested in the particular conditions that lead to a buying, selling or holding decision of agents in terms of stocks.

3.1. Model Principles

The model is based on the treatment of the trading decision of N agents represented by x. x can take values in $\{-1, 0, 1\}$ whereupon $x = 1$ specifies the buying decision of agent i. On the contrary, the selling decision is characterised by $x = -1$. Agents who are undecided and prefer to hold their stocks come to a holding decision which is represented by $x = 0$. We characterise the decision x_i of agent i, $i \in \{1, ..., N\}$, with reference to the trading tendency $y_i \in \mathbb{T} = (-1, 1)$, the fraction of stocks $z_i \in \mathbb{S} = [0, 1]$ held by agent i and the wealth $w_i \in \mathbb{R}_+$:

$$x_i = (y_i, z_i, w_i) \in \mathbb{T} \times \mathbb{S} \times \mathbb{R}_+.$$

A trading tendency y_i close to one points at a propensity to buy, whereas a value of y_i close to negative one points at a selling tendency. We refer to $y_i = 0$ as the propensity to hold. The trading tendency of agent i is chosen to be independent of agent i's wealth w_i due to the fact that a pure trading behaviour is observed without a monetary restriction. This could appear in form of a strong propensity to buy $y_i \approx 1$ with a given wealth of $w_i = 0$. The fraction of held stocks z_i is characterised by the proportion of the number of possessed stocks z_i^N compared to the overall supply of stocks z, more precisely $\frac{z_i^N}{z}$. If the fraction of stocks held by agent i takes the value $z_i = 0$, the agent is not in possession of any stocks. Contrary the state $z_i = 1$ represents the situation when the agent i owns all the stocks. In this model we assume the wealth w_i only to consist in form of monetary wealth and possessed stocks, this is done by neglecting the material wealth. Additionally, we act on the assumption that all possible sources of monetary wealth are already included in w_i. Furthermore we do not allow debts, i.e. $w_i \in \mathbb{R}_+$.

In the following we identify $x_i(t)$ as the trading decision of agent i at time t, the same holds for $y_i(t)$, $z_i(t)$ and $w_i(t)$.
The next step is to differentiate between agents with homogeneous expectations and agents with heterogeneous expectations about the formation of prices on financial markets.

3.2. Homogeneous Agent-based Model

We denote by x_i^H the trading decision of agent i based on a homogeneous expectation of the price formation, in the same way we refer to y_i^H, z_i^H and w_i^H as the trading tendency, the number of stocks and the wealth of agent i due to a homogeneous price formation expectation.
Referring to this notation we obtain

$$x_i^H = (y_i^H, z_i^H, w_i^H).$$

3.2.1. Components of the Trading Decision

In the following we concentrate on the change $\frac{d}{dt}$ of the trading decision $x_i^H(t)$ within an infinitesimal time step dt. For this purpose we model the change of the independent variables $y_i^H(t)$, $z_i^H(t)$ and $w_i^H(t)$.

- Trading tendency $y_i^H(t)$:
  We formulate the trading propensity y_i^H at the time $t + dt$ in this way

$$y_i^H(t+dt) = y_i^H(t) + dt\, F(y_i^H(t), y_j^H(t), \frac{1}{p}\frac{dp}{dt}),$$

  resulting in

$$\frac{dy_i^H(t)}{dt} = F(y_i^H(t), y_j^H(t), \frac{1}{p}\frac{dp}{dt}),$$

i.e. the variation of the trading tendency within the time interval $[t, t+dt]$, concerning the time step dt, can be written as a function depending on the pairwise trading interaction between agent i and j at the time t given by the corresponding trading tendencies $y_i^H(t)$ and $y_j^H(t) = (y_1^H(t), .., y_{i-1}^H(t), y_{i+1}^H(t), .., y_N^H(t))$. F also depends on the relative price change $\frac{1}{p}\frac{dp}{dt}$, where the actual price is given by p.

We postulate an additive form for F as

$$F(y_i^H(t), y_j^H(t), \frac{1}{p}\frac{dp}{dt}) = \frac{\alpha}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} V_1(y_i^H(t), y_j^H(t)) + \beta g(\frac{1}{p}\frac{dp}{dt}),$$

whereupon V_1 is a function describing the influence of agent j on agent i with respect to their trading tendencies. Hence the average impact of an other agent $j \neq i$ on agent i is expressed through $\frac{1}{N}\sum_{\substack{j=1 \\ j \neq i}}^{N} V_1(y_i^H(t), y_j^H(t))$ with α being a measure for the weight of this impact.
Furthermore the map V_1 is of this shape $V_1 : \mathbb{T} \times \mathbb{T} \to [0,1]$. The impact V_1 increases with a growing convergence of $y_j^H(t)$ approaching $y_i^H(t)$ and accordingly the influence V_1 decreases with a growing difference of $y_i^H(t)$ and $y_j^H(t)$.

The function g characterises the impact of the relative price change $\frac{1}{p}\frac{dp}{dt}$ on the change of the trading tendency $y_i^H(t)$ with a weighting β. We rewrite g by specifying the relative price change

$$\frac{1}{p}\frac{dp}{dt} = \frac{1}{N} \sum_{j=1}^{N} y_j^H(t) \cdot \frac{dz_j^H}{dt}.$$

This can be done due to the identity $\frac{1}{p}\frac{dp}{dt} = ED_H$, where ED_H represents the excess demand of traders with homogeneous expectations concerning the price. More precisely, the excess demand is reflected by an averaged product of the trading propensity of agent j and the according change of held stocks. The latter term relates to the trading volume.

- Fraction of held stocks $z_i^H(t)$:
  For the fraction of stocks z_i^H held by agent i at the time $t + dt$ we obtain

$$z_i^H(t + dt) = z_i^H(t) + dt\, f(y_i^H(t), z_i^H(t), w_i^H(t))$$

hence in the limit
$$\frac{dz_i^H(t)}{dt} = f(y_i^H(t), z_i^H(t), w_i^H(t)) \tag{3.1}$$
with $f : \mathbb{T} \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{S}_1$, $\mathbb{S}_1 \in [-1, 1]$.
Equation (3.1) shows that the change of the proportion of stocks held by agent i within a time step dt can be written in the form of a function f that is dependent on the trading tendency $y_i^H(t)$, the fraction of held stocks $z_i^H(t)$ and the wealth $w_i^H(t)$ before the time step dt.

- Wealth $w_i^H(t)$:
  Based on (3.1) we obtain the following equation for the wealth w_i^H of agent i at the time $t + dt$

$$w_i^H(t + dt) = w_i^H(t) - dt\, p \cdot \frac{dz_i^H(t)}{dt},$$

thus
$$\frac{dw_i^H(t)}{dt} = -p \cdot \frac{dz_i^H(t)}{dt} = -p \cdot f(y_i^H(t), z_i^H(t), w_i^H(t)),$$
i.e. a change in the fraction of stocks forms the basis of a change in wealth.

In case that agent i sells a fraction of $\frac{dz_i^H(t)}{dt}$ stocks at a price p, the wealth of this agent increases about $p \cdot \frac{dz_i^H(t)}{dt}$. Vice versa the wealth of agent i decreases about $p \cdot \frac{dz_i^H(t)}{dt}$ when buying $\frac{dz_i^H(t)}{dt}$ stocks.

We use the form of (3.1) to specify the price function g and define

$$y_j^H(t) \cdot f(y_j^H(t), z_j^H(t), w_j^H(t)) := V_2(y_i^H(t), y_j^H(t), z_j^H(t), w_j^H(t))$$

in order to obtain

$$\begin{aligned} g(\frac{1}{N}\sum_{j=1}^{N} y_j^H(t) \cdot \frac{dz_j^H}{dt}) &= g(\frac{1}{N}\sum_{j=1}^{N} y_j^H(t) \cdot f(y_j^H(t), z_j^H(t), w_j^H(t))) \\ &= g(\frac{1}{N}\sum_{j=1}^{N} V_2(y_i^H(t), y_j^H(t), z_j^H(t), w_j^H(t))). \end{aligned}$$

Since f is in fact independent of the trading propensity $y_i^H(t)$ of agent i, the concrete formulation of V_2 proves to be useful in order to analyse the properties of the evolving system.

As an overall result we obtain an equation for the evolution of the trading decision

$$\begin{aligned}\frac{dx_i^H(t)}{dt} &= (\frac{dy_i^H(t)}{dt}, \frac{dz_i^H(t)}{dt}, \frac{dw_i^H(t)}{dt}) \\ &= \begin{pmatrix} \frac{\alpha}{N}\sum_{\substack{j=1\\ j\neq i}}^N V_1(y_i^H(t), y_j^H(t)) + \beta g(\frac{1}{N}\sum_{j=1}^N V_2(y_i^H(t), y_j^H(t), z_j^H(t), w_j^H(t))) \\ f(y_i^H(t), z_i^H(t), w_i^H(t)) \\ -p \cdot f(y_i^H(t), z_i^H(t), w_i^H(t)) \end{pmatrix}^T .\end{aligned} \tag{3.2}$$

3.2.2. Mean-Field Equation in the Homogeneous Case

Now that we have set up the foundation of the model, we analyse the behaviour of the system in the mean-field limit. Therefore we follow the basic concept of [11].

We start by considering the physical conception of our model. The underlaying problem is this system of ordinary differential equations (ODEs)

$$\dot{x}_i^H = \begin{pmatrix} \frac{\alpha}{N}\sum_{\substack{j=1\\ j\neq i}}^N \nabla_y \tilde{V}_1(y_i^H, y_j^H) + \beta g(\frac{1}{N}\sum_{j=1}^N \nabla_y \tilde{V}_2(y_i^H, y_j^H, z_j^H, w_j^H)) \\ f(y_i^H, z_i^H, w_i^H) \\ -p \cdot f(y_i^H, z_i^H, w_i^H) \end{pmatrix}^T \tag{3.3}$$

where

$$\begin{aligned} V_1 &= \nabla_y \tilde{V}_1, \qquad \tilde{V}_1 : \mathbb{T} \times \mathbb{T} \to [0,1] \qquad \text{and} \\ V_2 &= \nabla_y \tilde{V}_2, \qquad \tilde{V}_2 : \mathbb{T} \times \mathbb{T} \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}, \end{aligned}$$

with $\tilde{V}_1 \in C_b^2(\mathbb{T} \times \mathbb{T})$ and $\tilde{V}_2 \in C_b^2(\mathbb{T} \times \mathbb{T} \times \mathbb{S} \times \mathbb{R}_+)$.

Hence we can perceive $\tilde{V}_1$ as an interaction potential describing an internal force field, respectively the price dependent potential $\tilde{V}_2$ as an external force affecting the traders as well.
Since the fraction of possessed stocks z_j^H of agent j, the overall number of stocks z and the wealth w_j^H in the latter term of the first row of (3.3) are not depending on the trading propensities y_i^H and y_j^H, we reduce (3.3) into

$$\dot{x}_i^H = \begin{pmatrix} \frac{\alpha}{N}\sum_{j=1}^N V_1(y_i^H, y_j^H) + \beta g(\frac{1}{N}\sum_{j=1}^N V_2(y_i^H, y_j^H)) \\ f(y_i^H, z_i^H, w_i^H) \\ -p \cdot f(y_i^H, z_i^H, w_i^H) \end{pmatrix}^T \tag{3.4}$$

obtaining a system which forms the basis for the following analysis realisable with adequate effort. More precisely, we concentrate on

$$\dot{y}_i^H = \frac{\alpha}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} V_1(y_i^H, y_j^H) + \beta g(\frac{1}{N} \sum_{j=1}^{N} V_2(y_i^H, y_j^H)).$$

Special cases occur by considering

$$\dot{y}_i^H \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \dot{y}_{1i}^H \\ \dot{y}_{2i}^H \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

with

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\dot{y}_{1i}^H$ and $\dot{y}_{2i}^H$ are given by

$$\left.\begin{aligned} \dot{y}_{1i}^H &= \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} V_1(y_i^H, y_j^H) = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \nabla_y \tilde{V}_1(y_i^H, y_j^H), \\ \dot{y}_{2i}^H &= g(\frac{1}{N} \sum_{j=1}^{N} V_2(y_i^H, y_j^H)) \end{aligned}\right\} \tag{3.5}$$

Since the following statements hold for both equations in (3.5), we abbreviate this system with $\dot{y}_{ki}^H$, $k \in \{1, 2\}$, until further notice.

The *weak coupling scaling* of the interaction potential $\tilde{V}_1$ and price potential $\tilde{V}_2$ with the factor $\frac{1}{N}$ is required since the exerted interaction and price force on each trader are of order 1 in the mean-field limit $N \to \infty$. This scaling naturally leads to a *mean-field equation.*

By the *Picard-Lindelöf theorem* (cf. Appendix A.1) there exist unique solutions $Y_{kN}^H(t) = (y_{ki}^H)_{1 \leq i \leq N}$ for each initial value $Y_{kN}^{H,0} \in (\mathbb{T} \times \mathbb{T})^N$ to the system of ODEs (3.5) such that $Y_{kN}^H(0) = Y_{kN}^{H,0}$. These solutions are denoted by $Y_{kN}^H \equiv Y_{kN}^H(t, Y_{kN}^{H,0})$ for all t, with $Y_{kN}^H \in C^1(\mathbb{R}_+ \times (\mathbb{T} \times \mathbb{T})^N)$.

For the main convergence theorem we need the following

Definition 1 (Empirical Distribution)**.** *A probability measure on* $\mathbb{T} \times \mathbb{T}$ *of a system* *consisting of N agents in this form*

$$f_{N,Y^{H,0}_{kN}}(t,\cdot) = \frac{1}{N}\sum_{i=1}^{N}\delta_{y^H_{ki}(t,Y^{H,0}_{kN})}(\cdot)$$

is called empirical distribution.

This is a probability measure on the 1-agent phase space, $F_N(t, y^H_1, ..., y^H_N)$ in contrast is a probability measure on the N-agent phase space. Their relation is reasoned in 3.2.3.

Furthermore we introduce the space $C^2_r(\mathbb{T})$ of C^k functions f on $\mathbb{T}$ such that for all $n \geq 0$ and all $p = 1, ..., k$, $|D^p f(x)| = O(|x|^{-n})$ as $|x| \to \infty$ and the space $\mathcal{M}(X)$ of Radon measures on a locally compact space X with a weak-* topology defined by duality with test functions in $C_c(X)$, the space of functions with compact support.

This setting leads to

Theorem 3. *Let $\tilde{V_1}$ and $\tilde{V_2}$ be in $C^2_r(\mathbb{T}\times\mathbb{T})$. Start from sequences $Y^{H,0,N}_{kN}$ of initial configurations of N agents such that*

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{(Y^{H,0,N}_{kN})_i} \to f^0_k \text{ in } w^* - \mathcal{M}(\mathbb{T}\times\mathbb{T}).$$

Then

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{y^H_{ki}(t,Y^{H,0,N}_{kN})} \to f_k(t,\cdot,\cdot) \text{ in } w^* - \mathcal{M}(\mathbb{T}\times\mathbb{T}).$$

uniformly on compact subsets of $\mathbb{R}_+$, where f_1 and f_2 are the solutions to the mean-field equations

$$\partial_t f_1 + \nabla_{y_1}\cdot((V_1 *_{y_1} f_1)f_1) = 0, \qquad f_1|_{t=0} = f^0_1 \tag{3.6}$$

$$\partial_t f_2 + \nabla_{y_2}\cdot(g(V_2 *_{y_2} f_2)f_2) = 0, \qquad f_2|_{t=0} = f^0_2. \tag{3.7}$$

This theorem holds for $k \in \{1, 2\}$. Hence the following analysis is organised for $k = 1$ by mentioning that analogue reasons can be consulted for $k = 2$ and corresponding results are realised with marginal additional effort. We present these results subsequent.

In order to give the proof for the case $k = 1$, we require an essential inequality.

Therefore we need

Definition 2 (Wasserstein Distance)**.** *Let μ and ν be probability measures on $\mathbb{T}$. Then the Wasserstein distance between μ and ν is defined as*

$$\begin{aligned}\mathcal{W}(\mu,\nu) &= \inf\{\mathbb{E}|X-Y||X(\textit{resp. } Y)\textit{has the probability distribution } \mu(\textit{resp. } \nu)\} \\ &= \inf_{P\in E(\mu,\nu)} \int\int_{\mathbb{T}\times\mathbb{T}} |x-y|P(dxdy),\end{aligned}$$

where $E(\mu,\nu)$ is the set of probability measures P on $\mathbb{T}\times\mathbb{T}$ such that the image of P under the projection on X is μ (respectively on Y is ν).
$\mathcal{W}(\mu,\nu)$ is a distance function, comparing the probability distributions μ and ν.

This definition is utilised in the following

Proposition 1 (Dobrushin Inequality)**.** *Let $\mu \equiv \mu(t,dy_1)$ and $\nu \equiv \nu(t,dy_1)$ be two solutions of the partial differential equation* (3.6) *in $C(\mathbb{R}_+, w^* - \mathcal{M}^1(\mathbb{T}))$, where $\mathcal{M}^1(X)$ characterises the set of probability measures in X.*
Then

$$W(\mu(t),\nu(t)) \leq W(\mu^0,\nu^0)\exp(2t\|V_1\|_{Lip}),$$

for $t \geq 0$ applies.

Proof. Cf. [11].

The underlying mean-field equation that needs to be solved is given by

$$\begin{aligned}\partial_t\rho + \nabla_{y_1}\cdot((V_1 *_{y_1} \rho)\rho) = 0,\\ \rho|_{t=0} = \rho^0,\end{aligned}$$

with V_1 being Lipschitz continuous on $\mathbb{T}\times\mathbb{T}$.

$$\begin{aligned}\dot{y}_1^H(t,a,\rho^0) &= (V_1 *_{y_1} \rho(t))(y_1(t,a,\rho^0)) \\ &= \int_{\mathbb{T}} V_1(y_1^H(t,a,\rho^0), y_1^H(t,a',\rho^0))\rho^0(da') \\ y_1^H(0,a,\rho^0) &= a, \qquad \rho(t) = y_1^H(t,\cdot,\rho^0)_*\rho^0\end{aligned}$$

represents the related system of characteristics, where $\nu = f_*\mu$ is called the *image of μ by f* with a given measurable map $f : X \to Y$, the measurable spaces $(X,\mathcal{X})$ and $(Y,\mathcal{Y})$ and the measures μ on $(X,\mathcal{X})$ and ν on $(Y,\mathcal{Y})$.

Now we provide the proof of Theorem 3 in the case for $k = 1$:

Proof of Theorem 3. We define a measure $\nu^{0,N}$ of initial values $y_{1i}^{H,0,N}$ by

$$\nu^{0,N} = \frac{1}{N}\sum_{i=1}^{N}\delta_{y_{1i}^{H,0,N}}.$$

Then $y_{1i}^{H,N}(t, y_{1i}^{H,0,N}, \nu^{0,N})$ solves the system of ODEs

$$\dot{y}_{1i}^{H,N}(t, y_{1i}^{H,0,N}, \nu^{0,N}) = \frac{1}{N}\sum_{j=1}^{N} V_1(y_{1i}^{H,N}(t, y_{1i}^{H,0,N}, \nu^{0,N}), y_{1j}^{H,N}(t, y_{1j}^{H,0,N}, \nu^{0,N}),$$
$$y_{1i}^{H,N}(0, y_{1i}^{H,0,N}, \nu^{0,N}) = y_{1i}^{H,0,N},$$

for each $1 \leq i \leq N$.
In line with the previous setting we define

$$\nu^{N}(t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{y_{1i}^{H,N}(t, y_{1i}^{H,0,N}, \nu^{0,N})}.$$

And if $\mathcal{W}(\mu^0, \nu^{0,N}) \to 0$ we obtain with Dobrushin's inequality

$$\mathcal{W}(\mu(t), \nu^N(t)) \to 0$$

as $N \to \infty$ for each $t > 0$. The weak-* topology of $\mathcal{M}^1(\mathbb{T})$ is metricised by the Wasserstein distance, therefore we obtain the required convergence.

□

The statement of Theorem 3 is crucial for our agent-based kinetic model since it yields the convergence to the solution of the mean-field equation. We utilise these information in order to gain results for the initial formulation with the coefficient α. By referring to the basic ODE

$$\dot{y}_{1i}^{H} = \frac{\alpha}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} V_1(y_i^H, y_j^H),$$

we obtain

$$= \frac{\alpha}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \int_{\mathbb{T}} V_1(y_i^H, y_1)\delta_{y_j^H}(y_1)dy_1$$

by using the convolution property of the δ-distribution. Transposing yields

$$= \int_{\mathbb{T}} V_1(y_i^H, y_1) \frac{\alpha}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{y_j^H}(y_1) dy_1.$$

Exerting the empirical distribution leads to

$$= \int_{\mathbb{T}} V_1(y_i^H, y_1) \alpha f_N(y_1) dy_1$$

and a final application of the convolution definition results in

$$= V_1 *_{y_1} \alpha f_N.$$

In conclusion, we proved that in the mean-field limit $N \to \infty$ the empirical distribution αf_N converges to αf_1 and this is a solution of

$$\begin{aligned} \partial_t \rho + \nabla_{y_1} \cdot ((V_1 *_{y_1} \rho)\rho) = 0, \\ \rho|_{t=0} = \rho^0. \end{aligned}$$

As mentioned above, we obtain an analogue result for $k = 2$ by utilising the same statements and calculations.

$$\begin{aligned} \dot{y}_{2i}^H &= \beta g(\frac{1}{N} \sum_{j=1}^{N} V_2(y_i^H, y_j^H)), \\ &= \beta g(\frac{1}{N} \sum_{j=1}^{N} \int_{\mathbb{T}} V_2(y_i^H, y_2) \delta_{y_j^H}(y_2) dy_2) \\ &= \beta g(\int_{\mathbb{T}} V_2(y_i^H, y_2) f_N(y_2) dy_2) \\ &= \beta g(V_2 *_{y_2} f_N). \end{aligned}$$

Based on the conclusions drawn from the case $k = 1$, we have the mean-field limit $f_N \to f_2$. Where f_2 is a solution to

$$\begin{aligned} \partial_t \rho + \nabla_{y_2} \cdot (\beta g(V_2 *_{y_2} \rho)\rho) = 0, \\ \rho|_{t=0} = \rho^0. \end{aligned}$$

3.2.3. Propagation of Chaos

In the following paragraph we are restricted to the case $k = 1$. The case $k = 2$ is left out due to the complexity of the according analysis considering g.

As mentioned earlier, we consider the empirical distribution f_N as a probability measure on the 1-agent phase space. According to [11] we present the relation of $f_N = f$ and the probability measure on the N-agent phase space F_N in the mean-field limit $N \to \infty$.

Hence we introduce

Lemma 1. *Let $f \in \mathcal{M}^1(\mathbb{T})$ and, for all $N \geq 1$, $F_N \in \mathcal{M}^1(\mathbb{T}^N)$ symmetric in all variables, that means F_N is invariant under the transformation*

$$(y_{11}^H, ..., y_{1N}^H) \longmapsto (y_{1\sigma(1)}^H, ..., y_{1\sigma(N)}^H)$$

for each σ in the symmetric group Σ_N. Then the following statements are equivalent (1)for each $\epsilon > 0$ and each $\phi \in C_c(\mathbb{T})$ applies

$$F_N\left(\left\{(y_{11}^H, ..., y_{1N}^H) \in \mathbb{T}^N | \left|\left\langle \frac{1}{N}\sum_{i=1}^{N} \delta_{y_{1i}^H} - f, \phi\right\rangle\right| \geq \epsilon\right\}\right) \to 0$$

as $N \to \infty$;
(2)$F_{N:n}$ is the image of F_N under the projection $(y_{11}^H, ..., y_{1N}^H) \longmapsto (y_{11}^H, ..., y_{1n}^H)$ and the sequence $F_{N:n}$ of marginal distributions of F_N satisfies

$$F_{N:n} \to f^{\otimes n} \quad \textit{weakly-*}$$

as $N \to \infty$.

Proof. See [11].

Definition 3 (Chaotic Sequence)**.** *A sequence $F_N \in \mathcal{M}^1(\mathbb{T}^N)$ that satisfies either one of the equivalent statements in Lemma 1 is called chaotic.*

These basics of chaotic sequences last in order to formulate the following main statement. It applies the notion of chaotic sequences to the mean-field limit of our N-agent system.

Theorem 4. *V_1 is assumed to be Lipschitz continuous on $\mathbb{T} \times \mathbb{T}$, $f^0 \in \mathcal{M}^1(\mathbb{T})$. Then, for each $n \geq 1$ and each $t > 0$,*

$$F_{N:n}(t) \to f(t)^{\otimes n} \quad \textit{weakly-*}$$

as $N \to \infty$, where $f(t)$ is the solution to the mean-field PDE

$$\partial_t f + \nabla_{y_1} \cdot ((V_1 *_{y_1} f) f) = 0,$$
$$f|_{t=0} = f^0.$$

Proof. Cf. [11].

Summarising the results we conclude for our model

Corollary 1. *Let $f^0 \in \mathcal{M}^1(\mathbb{T})$ and $F_N(t) \in \mathcal{M}^1(\mathbb{T}^N)$, where F_N solves*

$$\partial_t F_N + \sum_{1 \leq i,j \leq N} \nabla_{y_1} \cdot (V_1(y_i^H, y_j^H) F_N) = 0,$$
$$F_N(0) = (f^0)^{\otimes N}.$$

Then, for each $t > 0$, $F_N(t)$ is a chaotic sequence and its marginal distributions $F_{N:n}$ satisfy

$$F_{N:n}(t) \to f(t)^{\otimes n} \quad \textit{weakly-*}$$

as $N \to \infty$, where $f(t)$ solves the mean-field PDE.

This corollary includes the *propagation of chaos*, since $F_N(0)$ being chaotic in the most elementary manner implies that $F_N(t)$ is a chaotic sequence for all $t > 0$.

3.2.4. Conclusion for the Trading Decision

Summing up, we hold a model of the trading decision $x^H = (x_i^H)_{1 \leq i \leq N}$ that was analysed with special regard to the trading tendency $y^H = (y_i^H)_{1 \leq i \leq N}$. We obtained the mean-field equation concerning this parameter due to crucial convergence statements. The next step is to establish similar statements for the initial ODE (3.2).

Referring to equation (3.4), we present results for the trading decision x^H based on the findings of the previous paragraphs. Hence we define the probability density

$\rho(x^H) = \rho(y^H, z^N, w^H)$ and the volume $dx^H = dy^H\, dz^H\, dw^H$ on the 1-agent phase space $\mathbb{T} \times \mathbb{S} \times \mathbb{R}_+$ with $\rho(x^H)\, dx^H$ representing the probability that an agent has a trading decision x^H in a volume dx^H.

We start with the interaction term of the ODE since this part requires more complex analysis than the other two components f and $-p{\cdot}f$. The following equation includes all the essential statements, e.g. Theorem 3. Basically we sum up the findings for the cases $k = 1, 2$ by formulating

$$
\begin{aligned}
&\frac{\alpha}{N}\sum_{j=1}^{N} V_1(y^H, y_j^H) + \beta g(\frac{1}{N}\sum_{j=1}^{N} V_2(y^H, y_j^H))\\
=&\frac{\alpha}{N}\sum_{j=1}^{N}\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} V_1(y^H, x^H)\delta_{y_j^H}(x^H)\, dx^H\\
&+\beta g(\frac{1}{N}\sum_{j=1}^{N}\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} V_2(y^H, x^H)\delta_{y_j^H}(x^H)\, dx^H)\\
=&\alpha\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} V_1(y^H, x^H)\rho_N(x^H)\, dx^H + \beta\, g(\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} V_2(y^H, x^H)\rho_N(x^H)\, dx^H)\\
=&:A(\rho(x^H)).
\end{aligned}
$$

An application of Theorem 3 proves that the empirical distribution $\rho_N(x^H)$ has the mean-field limit $\rho(x^H)$ that solves the following PDE

$$
\begin{aligned}
0 = \partial_t\rho(x^H) &+ \nabla_y\cdot(A(\rho(x^H))\rho(x^H)) + \nabla_z\cdot(f(y^H, z^H, w^H)\rho(x^H))\\
&+ \nabla_w\cdot(-p\cdot f(y^H, z^H, w^H)\rho(x^H))
\end{aligned}
\tag{3.8}
$$

In conclusion we obtain a system of ODEs represented by the trading tendency x^H. Furthermore, the corresponding density of the trading tendency $\rho(x^H)$ is a solution of the mean-field equation (3.8).

3.3. Heterogeneous Agent-based Model

'One of the things that microeconomics teaches you is that individuals are not alike. There is heterogeneity, and probably the most important heterogeneity here is heterogeneity of expectations. If we didn't have heterogeneity, there would be no trade. But developing an analytic model with heterogeneous agents is difficult.'

(Kenneth Arrow, In: D. Colander, R.P.F. Holt and J. Barkley Rosser (eds.), The Changing Face of Economics. Conversations with Cutting Edge Economists. The University of Michigan Press, Ann Arbor, 2004, S. 301)

This quotation insistently identifies the main advantage of heterogeneity concerning traders on financial markets. However, it points out the complexity of generating a model as well.

So far we have achieved a explicit model describing the trading behaviour of only one homogeneous group of agents. The next step is to deliver insight into a more complex design of agents' characteristics on financial markets. Therefore we introduce two already known groups, the chartists and fundamentalists, and try to identify their specific trading decision.

3.3.1. Chartists & Fundamentalists

Based on the notations in section 3.1, we refer to x^C, y^C, z^C and w^C as the trading decision, the trading propensity, the fraction of possessed stocks and the wealth of a chartist. Accordingly x^F, y^F, z^F and w^F represent the trading decision, the trading propensity, the fraction of possessed stocks and the wealth of a fundamentalistic agent. We have an overall number N of traders composed of N_C chartists and $N_F = N - N_C$ fundamentalists.

Chartists

A more detailed characterisation of the chartists' trading decision reveals the distinction between an optimistic trading decision in $x^C = 1 = x^+$ and a pessimistic decision in $x^C = -1 = x^-$. This fact holds since optimists are known for their buying attitude when stock prices increase and agents are called pessimists when they sell stocks in order to avoid losses due to a decreasing stock price.

Chartists are influenced by other traders, therefore their change of the trading tendency depends on a interaction potential as well as on a price function. Considering all these information, we obtain

$$\dot{x}^C = \begin{pmatrix} \frac{\alpha_C}{N_C} \sum_{j=1}^{N_C} V_1(y^C, y_j^C) + \beta_C g_C(\frac{1}{p}\frac{dp}{dt}) \\ f_C(y^C, z^C, w^C) \\ -p \cdot f_C(y^C, z^C, w^C) \end{pmatrix}^T .$$

The price function g_C of the chartists depends on the relative price change $\frac{1}{p}\frac{dp}{dt}$.

Fundamentalists
Since the trading tendency of an agent does not rely on the tendency of other agents, we obtain a trading propensity change only driven by a price function. This results in

$$\dot{x}^F = \begin{pmatrix} \beta_F g_F(\frac{1}{p}\frac{dp}{dt}) \\ f_F(y^F, z^F, w^F) \\ -p \cdot f_F(y^F, z^F, w^F) \end{pmatrix}^T .$$

For a further investigation of a system based on (x^C, x^F), we concretise the relative price change $\frac{1}{p}\frac{dp}{dt}$ and write

$$\begin{aligned} \frac{1}{p}\frac{dp}{dt} &= ED_C + ED_F \\ &= \frac{1}{N_C}\sum_{j=1}^{N_C} y_j \frac{dz_j^C}{dt} + \frac{1}{N-N_C}\sum_{j=N_C+1}^{N} y_j \frac{dz_j^F}{dt} \\ &= \frac{1}{N_C}\sum_{j=1}^{N_C} y_j f_C(y_j^C, z_j^C, w_j^C) + \frac{1}{N-N_C}\sum_{j=N_C+1}^{N} y_j f_F(y_j^F, z_j^F, w_j^F) \end{aligned}$$

with ED_C being the excess demand of chartists and ED_F the excess demand of the fundamentalistic group.

3.3.2. Mean-Field Equation in the Heterogeneous Case

In the homogeneous case we proved the convergence of the underlying system to the solution of the mean-field PDE. Referring to those statements and results we analyse the mean-field limit in the heterogeneous case.

Since the ODE for the trading decision of the fundamentalists holds the same parameters as the ODE for the trading decision of the chartists except that the chartist decision additionally relies on the interaction potential $V_1(y^C, y_j^C)$, we take the chartist's dynamics as a basis for the following investigation. The results can be transferred to the fundamentalistic system with little effort by excluding the statements for the interaction potential.

Inserting the rewritten relative price change into the system of the chartist decision by using the identities $x^C = (y^C, z^C, w^C)$ and $x^C = (y^C, z^C, w^C)$ we obtain

$$\dot{x}^C = \begin{pmatrix} \frac{\alpha_C}{N_C}\sum_{j=1}^{N_C} V_1(y^C, y_j^C) + \beta_C g_C(\frac{1}{N_C}\sum_{j=1}^{N_C} y_j f_C(x_j^C) + \frac{1}{N-N_C}\sum_{j=N_C+1}^{N} y_j f_F(x_j^F)) \\ f_C(x^C) \\ -p \cdot f_C(x^C) \end{pmatrix}^T$$

The next step is to insert the convolution property of the δ-distribution with the empirical distribution of this form

$$\rho_N(x^C, x^F) = \frac{1}{N_C}\sum_{j=1}^{N_C} \delta_{x_j^C}(x^C) \otimes \frac{1}{N-N_C}\sum_{j=N_C+1}^{N} \delta_{x_j^F}(x^F).$$

Thus we obtain for the first row of $\dot{x}^C$

$$\begin{aligned}
&\frac{\alpha_C}{N_C}\sum_{j=1}^{N_C} V_1(y^C, y_j^C) + \beta_C g_C(\frac{1}{N_C}\sum_{j=1}^{N_C} y_j f_C(x_j^C) + \frac{1}{N-N_C}\sum_{j=N_C+1}^{N} y_j f_F(x_j^F)) \\
&= \frac{\alpha_C}{N_C}\sum_{j=1}^{N_C} \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} V_1(y^C, y)\delta_{y_j^C}(x)\, dx \\
&\quad + \beta_C\, g_C(\frac{1}{N_C}\sum_{j=1}^{N_C} \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} y_j f_C(x)\delta_{x_j^C}(x)\, dx \\
&\qquad + \frac{1}{N-N_C}\sum_{j=N_C+1}^{N} \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} y_j f_F(x)\delta_{x_j^F}(x)\, dx) \\
&= \alpha_C \iint\limits_{(\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+)^2} V_1(y^C, y)\rho_N(x, x^F)\, dx dx^F \\
&\quad + \beta_C\, g_C(\iint\limits_{(\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+)^2} y f_C(x)\rho_N(x, x^F)\, dx dx^F + \iint\limits_{(\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+)^2} y f_F(x)\rho_N(x^C, x)\, dx^C dx)
\end{aligned}$$

The application of Theorem 3 for the system of chartists and fundamentalists yields the limit $\rho(x^C, x^F)$. This probability density with the related state (x^C, x^F) solves the following mean-field equation

$$\begin{aligned}
0 = \partial_t \rho(x^C, x^F) &+ \nabla_y \cdot \left(\begin{pmatrix} A_1(y^C) \\ A_2 \end{pmatrix} \rho(x^C, x^F) \right) \\
&+ \nabla_z \cdot \left(\begin{pmatrix} B_1(x^C) \\ B_2(x^F) \end{pmatrix} \rho(x^C, x^F) \right) + \nabla_w \cdot \left(\begin{pmatrix} -p \cdot B_1(x^C) \\ -p \cdot B_2(x^F) \end{pmatrix} \rho(x^C, x^F) \right), \qquad (3.9)
\end{aligned}$$

with the boundary condition for the probability density $\rho(x^C, x^F)$

$$\rho(x^C, x^F)_{|\partial(\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+)} = 0. \tag{3.10}$$

$A_1(y^C)$ and A_2 are given by

$$\begin{aligned} A_1(y^C) = \alpha_C & \iint\limits_{(\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+)^2} V_1(y^C, y)\rho(x, x^F)\, dx dx^F \\ & + \beta_C\, g_C(\iint\limits_{(\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+)^2} y f_C(x)\rho(x, x^F)\, dx dx^F \\ & \qquad + \iint\limits_{(\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+)^2} y f_F(x)\rho(x^C, x)\, dx^C dx), \\ A_2 = \beta_F\, g_F(& \iint\limits_{(\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+)^2} y f_C(x)\rho(x, x^F)\, dx dx^F \\ & + \iint\limits_{(\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+)^2} y f_F(x)\rho(x^C, x)\, dx^C dx) \end{aligned}$$

and $B_1(x^C)$, $B_2(x^F)$ are defined by

$$\begin{pmatrix} B_1(x^C) \\ B_2(x^F) \end{pmatrix} = \begin{pmatrix} f_C(x^C) \\ f_F(x^F) \end{pmatrix}.$$

Starting from the equation (3.9) we derive the chartist's density $\rho_C(x)$ and the fundamentalist's density $\rho_F(x)$. This is done by integrating the PDE with respect to x^F in the first case and x^C afterwards.

Integration with respect to x^F

We rewrite equation (3.9) into

$$\begin{aligned} 0 = & \partial_t \rho(x^C, x^F) + \partial_{y^C}[A_1(y^C)\rho(x^C, x^F)] + \partial_{y^F}[A_2\rho(x^C, x^F)] \\ & + \partial_{z^C}[B_1(x^C)\rho(x^C, x^F)] + \partial_{z^F}[B_2(x^F)\rho(x^C, x^F)] \\ & + \partial_{w^C}[-pB_1(x^C)\rho(x^C, x^F)] + \partial_{w^F}[-pB_2(x^F)\rho(x^C, x^F)] \end{aligned} \tag{3.11}$$

and the integration of the PDE with respect to x^F yields

$$\begin{aligned} 0 = & \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_t \rho(x^C, x^F)\, dx^F \\ & + \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{y^C}[A_1(y^C)\rho(x^C, x^F)]\, dx^F + \underbrace{\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{y^F}[A_2\rho(x^C, x^F)]\, dx^F}_{\overset{(3.10)}{=} 0} \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{z^C}[B_1(x^C)\rho(x^C,x^F)]\,dx^F + \underbrace{\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{z^F}[B_2(x^F)\rho(x^C,x^F)]\,dx^F}_{\overset{(3.10)}{=}0} \\
&+\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{w^C}[-pB_1(x^C)\rho(x^C,x^F)]\,dx^F + \underbrace{\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{w^F}[-pB_2(x^F)\rho(x^C,x^F)]\,dx^F}_{\overset{(3.10)}{=}0} \\
=\,&\partial_t \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C,x^F)\,dx^F + \partial_{y^C}\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} [A_1(y^C)\rho(x^C,x^F)]\,dx^F \\
&+\partial_{z^C}\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} [B_1(x^C)\rho(x^C,x^F)]\,dx^F + \partial_{w^C}\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} [-pB_1(x^C)\rho(x^C,x^F)]\,dx^F \\
=\,&\partial_t \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C,x^F)\,dx^F + \partial_{y^C}[A_1(y^C)\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C,x^F)\,dx^F] \\
&+\partial_{z^C}[B_1(x^C)\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C,x^F)\,dx^F] + \partial_{w^C}[-pB_1(x^C)\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C,x^F)\,dx^F]
\end{aligned}
$$

With the chartists' probability density

$$\rho_C(x^C) = \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C,x^F)\,dx^F$$

we finally obtain the mean-field equation

$$0 = \partial_t\rho_C(x^C) + \partial_{y^C}[A_1(y^C)\rho_C(x^C)] + \partial_{z^C}[B_1(x^C)\rho_C(x^C)] + \partial_{w^C}[-pB_1(x^C)\rho_C(x^C)].$$

Integration with respect to x^C

Using equation (3.11) and proceeding as in the case of the derivation of the mean-field equation for the chartists' probability density yields

$$
\begin{aligned}
0 =\,&\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_t\rho(x^C,x^F)\,dx^C \\
&+\underbrace{\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{y^C}[A_1(y^C)\rho(x^C,x^F)]\,dx^C}_{\overset{(3.10)}{=}0} + \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{y^F}[A_2\rho(x^C,x^F)]\,dx^C \\
&+\underbrace{\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{z^C}[B_1(x^C)\rho(x^C,x^F)]\,dx^C}_{\overset{(3.10)}{=}0} + \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{z^F}[B_2(x^F)\rho(x^C,x^F)]\,dx^C \\
&+\underbrace{\int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{w^C}[-pB_1(x^C)\rho(x^C,x^F)]\,dx^C}_{\overset{(3.10)}{=}0} + \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \partial_{w^F}[-pB_2(x^F)\rho(x^C,x^F)]\,dx^C
\end{aligned}
$$

$$
\begin{aligned}
&= \partial_t \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C, x^F)\, dx^C + \partial_{y^F} \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} [A_2 \rho(x^C, x^F)]\, dx^C \\
&\quad + \partial_{z^F} \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} [B_2(x^F)\rho(x^C, x^F)]\, dx^C + \partial_{w^F} \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} [-pB_2(x^F)\rho(x^C, x^F)]\, dx^C \\
&= \partial_t \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C, x^F)\, dx^C + \partial_{y^F} [A_2 \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C, x^F)\, dx^C] \\
&\quad + \partial_{z^F} [B_2(x^F) \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C, x^F)\, dx^C] + \partial_{w^F} [-pB_2(x^F) \int_{\mathbb{T}\times\mathbb{S}\times\mathbb{R}_+} \rho(x^C, x^F)\, dx^C]
\end{aligned}
$$

This analysis results in the PDE

$$
0 = \partial_t \rho_F(x^F) + \partial_{y^F} [A_2 \rho_F(x^F)] + \partial_{z^F} [B_2(x^F)\rho_F(x^F)] + \partial_{w^F} [-pB_2(x^F)\rho_F(x^F)],
$$

where the solution $\rho_F(x^F)$ is the probability density of the fundamentalists.

4. Summary and Outlook

First we gave an outline of kinetic models designed with differing parameters.
However the main result of this thesis was the derivation of a novel kinetic model respectively mean-field model based on trading agents of financial markets. Even though the model utilises approaches of the presented models in chapter 2, it is more complex due to its detailed characterisations of the basics.

We started with the introduction of a homogeneous model by assuming that the traders have the same expectations about their price formation process, this situation became more sophisticated in the heterogeneous case. In order to yield results for the long time behaviour of the trading decision in the mean-field limit $N \to \infty$, we required essential statements concerning the convergence of the empirical distribution to the solution of the mean-field equation. These statements were proved for a special case of the trading tendency of an agent and they were transferred to the general case of the trading tendency and to the trading decision in the homogeneous and in the heterogeneous case afterwards.
Additionally, we showed proof of the validity of the propagation of chaos in the homogeneous case for the special case of the trading propensity.
The model of heterogeneous agents is based on the existence of chartists and fundamentalists. In this situation we carried over the previous results in order to obtain a mean-field limit as the solution of a PDE.
As a last result, we formulated the probability density of chartists and fundamentalists in the heterogeneous case.

The numerical simulation of the achieved mean-field equations for the homogeneous and for the heterogeneous case is of further interest.

Furthermore several model extensions would result in a more detailed comprehension of the trading activities of agents. Therefore exchange rates could be introduced in order to picture the coherence in a more realistic manner. A stochastic noise in form of a diffusion term could be added as well, with the result that the homogeneous

mean-field equations turn into inhomogeneous equations.
The corresponding analysis of the diversified problems is a potential basis for continuative researches.

A. Appendix

A.1. Picard-Lindelöf Theorem

Let $G \in \mathbb{R}^{n+1}$ *be a domain and let* $f : G \to \mathbb{R}^n$ *be a continuous function satisfying a Lipschitz condition*

$$\|f(x,u) - f(x,v)\| \leq L\|u - v\|,$$

for all $(x,u),(x,v) \in G$ *and some constant* $L > 0$*, which is called the Lipschitz constant.*
Then for each initial data pair $(x_0, u_0) \in G$ *there exists an interval* $[x_0 - a, x_0 + a]$ *with* $a > 0$ *such that the initial value problem*

$$u' = f(x,u),$$
$$u(x_0) = u_0$$

has a unique solution in this interval.

Proof. See [16].

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Affirmation

Hereby I, *Veronika Penner*, affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken directly or indirectly from other sources are indicated as such.

Münster, 29. October 2010

Veronika Penner