Image restoration and simultaneous edge detection by optimal control methods

Lucas Franek[†], *Marzena Franek*[‡], *Helmut Maurer*[*] *and Marcus Wagner*[**]

Abstract. The present paper is concerned with the numerical solution of multidimensional control problems of Dieudonné-Rashevsky type by discretization methods and large-scale optimization techniques. We prove first a convergence theorem wherein the difference of the minimal value and the objective values along a minimizing sequence is estimated by the mesh size of the underlying triangulations. Then we apply the proposed method to the problem of the simultaneous restoration of noisy image data and edge detection therein. Instead of using an Ambrosio-Tortorelli type energy functional, we reformulate the image restoration problem as a multidimensional control problem. The edge detector can be immediately built from the control variables. The quality of our numerical results competes well with the results given by variational techniques.

Key words. Image restoration, edge detection, PDE constrained optimization, optimal control problem, direct methods, convergence theorem.

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1. Introduction.

The present paper is concerned with the numerical solution of multidimensional control problems of the following type:

$$(\mathrm{P}):\quad F(x,u)=\int_\Omega f(s,x(s),u(s))\,ds \longrightarrow \inf!\,;\quad (x,u)\in W_0^{1,p}(\Omega,\mathbb{R}^n)\times L^p(\Omega,\mathbb{R}^{nm})\,; \tag{1.1}$$

$$Jx(s)=\begin{pmatrix}\dfrac{\partial x_1}{\partial s_1}(s) & \dots & \dfrac{\partial x_1}{\partial s_m}(s)\\ \vdots & & \vdots\\ \dfrac{\partial x_n}{\partial s_1}(s) & \dots & \dfrac{\partial x_n}{\partial s_m}(s)\end{pmatrix}=B(s)\,u(s)\ \ (\forall)\,s\in\Omega\,; \tag{1.2}$$

$$u\in \mathrm{U}=\big\{\,u\in L^p(\Omega,\mathbb{R}^{nm})\ \big|\ u(s)\in \mathrm{K}\ \ (\forall)\,s\in\Omega\,\big\}\,. \tag{1.3}$$

Problems of this kind, referred as Dieudonné-Rashevsky type problems, are connected with the study of BVP's for nonlinear first-order PDE's, [01] they arise from optimization problems for convex bodies under geometrical restrictions, [02] in elasticity theory (torsion problems) [03] and population dynamics (models with comprehension of an age structure). [04] Recently, problems from image processing have been studied under consideration of gradient constraints as well. [05]

Although Dieudonné-Rashevsky type problems have been as yet less intensely investigated than optimal control problems with second-order PDE's, [06] necessary optimality conditions as well as duality theorems have been derived in the linear-convex case (with state equations $Jx(s)-u(s)=0$ resp. $Jx(s)-A(s)\,x(s)-B(s)\,u(s)=0$). [07] A numerical approach, however, has been proposed first in the context of the applications from image processing as a direct method: [08] the problem (P) will be discretized first, and the arising large-scale finite-dimensional problems will be numerically solved by interior-point methods.

In the present paper, we prove first a convergence theorem for this discretization method (*Section 2*). By use of the global minimizers of the discretized problems, we construct a minimizing sequence $\{\,(\widetilde{x}_N,\widetilde{u}_N)\,\}$ of admissible solutions of (P) with $|\,F(\widetilde{x}_N,\widetilde{u}_N)-m\,|\leqslant C\,\sigma_N$ (m and σ_N denote the minimal value of (P)

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01) [Dacorogna/Marcellini 97], [Dacorogna/Marcellini 98] and [Dacorogna/Marcellini 99].

02) [Andrejewa/Klötzler 84a] and [Andrejewa/Klötzler 84b], p. 149 f.

03) [Funk 62], pp. 531 ff., [Lur'e 75], pp. 240 ff., [Ting 69a], p. 531 f., [Ting 69b] and [Wagner 96], pp. 76 ff.

04) [Brokate 85], [Feichtinger/Tragler/Veliov 03].

05) [Wagner 06], pp. 108 ff., [Wagner 07].

06) See e. g. [Tröltzsch 05].

07) The corresponding theory has been developed substantially by Klötzler, Pickenhain and Wagner, cf. the bibliography in [Wagner 07], pp. 2 and 5.

08) [Brune/Maurer/Wagner 08], [Franek 07a], [Franek 07b]. The only precursor is [Dewess/Helbig 95] where a transportation flow problem as the *dual* problem to a Dieudonné-Rashevsky type problem has been solved by methods of combinatorial optimization.

and the mesh size of the triangulation in the discretized problem, respectively). Within this minimizing sequence, there exist a subsequence, which converges uniformly w. r. to x and L^p-weakly ($1 < p < \infty$) w. r. to u. Then in *Section 3*, we apply the method to a well-known problem from image processing: the problem of edge detection within noisy image data ("image restoration / image smoothing with simultaneous edge detection").

To the best of our knowledge, within mathematical image processing the effects of gradient restrictions have not been studied yet, and thus are of particular interest. In order to compare our newly proposed approach with the existing ones, its application to a well-examined basic problem is desirable. The quality of our numerical results competes well with and may even exceed the best existing variational method (edge detection via Ambrosio-Tortorelli functionals). On the other hand, the present studies may be considered as a preparatory work for the treatment of situations where image restoration and simultaneous edge detection must be embedded as a subtask into a more comprehensive problem. Even in this case, the optimal control formulation provides considerable advantages. [09)]

Notations.

Assume that $\Omega \subset \mathbb{R}^m$ is the closure of a bounded strongly Lipschitz domain. Then $C^k(\Omega, \mathbb{R}^r)$ denotes the space of r-dimensional vector functions $f\colon\ \Omega \to \mathbb{R}^r$, whose components are continuous ($k = 0$) resp. k-times continuously differentiable ($k = 1, \dots, \infty$). $L^p(\Omega, \mathbb{R}^r)$ denotes the space of r-dimensional vector functions $f\colon\ \Omega \to \mathbb{R}^r$ with components, which are integrable in the pth power ($1 \leqslant p < \infty$) resp. are measurable and essentially bounded ($p = \infty$). $W_0^{1,p}(\Omega, \mathbb{R}^r)$ denotes the Sobolev space of compactly supported $L^p(\Omega, \mathbb{R}^r)$ (vector) functions $f\colon\ \Omega \to \mathbb{R}^r$, whose components possess first-order weak partial derivatives in $L^p(\Omega, \mathbb{R})$ ($1 \leqslant p < \infty$). $W_0^{1,\infty}(\Omega, \mathbb{R}^r)$ is understood as the Sobolev space of all r-vector functions $f\colon\ \Omega \to \mathbb{R}^r$ with Lipschitz continuous components and zero boundary values. [10)] Jx denotes the Jacobi matrix of the vector function $x \in W_0^{1,p}(\Omega, \mathbb{R}^r)$. The diameter of a set $\Omega \subset \mathbb{R}^m$ is defined as $\mathrm{Diam}\,(\Omega) = \sup\,\{\, |\, x' - x''\,| \;\big|\; x',\, x'' \in \Omega\,\}$ while $|\,\Omega\,|$ denotes the m-dimensional Lebesgue measure of Ω. The abbreviation "$(\forall)\, t \in \mathrm{A}$" has to be read as "for almost all $s \in \mathrm{A}$" resp. "for all $s \in \mathrm{A}$ except a Lebesgue null set". Finally, the symbol $\mathfrak{o}$ denotes, depending on the context, the zero element of the underlying space.

2. Convergence of a discretization method for (P).

a) Discretization of the problem (P).

We specify the basic assumptions about the problem (P) as follows: Let $n \geqslant 1$, $m = 2$ and $1 < p < \infty$. We choose a rectangular domain $\Omega \subset \mathbb{R}^2$ with edge lengths $a,\, b \in \mathbb{N}$, $a \geqslant b > 0$. The integrand $f(s, \xi, v)\colon\ \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times 2} \to \mathbb{R}$ is measurable and essentially bounded w. r. to s and continuously differentiable w. r. to all ξ_i and v_{ij}. For all $s \in \Omega$, $B(s)$ is the (n, n)-unit matrix, and $\mathrm{K} \subset \mathbb{R}^{n\times 2}$ is the norm body

$$\mathrm{K} \,=\, \Big\{\, v \in \mathbb{R}^{n\times 2} \;\Big|\; \sum_{i=1}^{n} \sum_{j=1}^{2} |\, v_{ij}\,|^q \leqslant R^q \,\Big\} \tag{2.1}$$

with $q \in \mathbb{N}$, $q \geqslant 1$. From these assumptions, we derive immediately the existence of a feasible solution (the pair $(\mathfrak{o},\, \mathfrak{o})$). For any admissible pair (x, u), the restriction $Jx(s) \in \mathrm{K}$ $(\forall)\, s \in \Omega$ implies $x \in W_0^{1,p}(\Omega, \mathbb{R}^n) \,\cap$

[09)] For example, we mention the image registration problem in absence of a greyscale value correspondence. Then the image matching must be based on the gradient resp. edge information only, cf. [Haber/Modersitzki 07].

[10)] [Evans/Gariepy 92], p. 131, Theorem 5.

$W_0^{1,\infty}(\Omega, \mathbb{R}^n)$. As a consequence, x possesses a Lipschitz continuous representative even in the case $1 < p \leqslant m$. If c_q denotes the constant of equivalence between the 2- and q-norm in $\mathbb{R}^2$ then all components x_i have the Lipschitz constant $C_1 = R/c_q$.

Let now $K = L = 2^N$. We construct a triangulation of Ω by introducing first a grid of $(K \times L)$ rectangles $\mathrm{Q}_{k,l}$ with edge lengths $a/2^N$ and $b/2^N$ and splitting then every rectangle along the principal diagonal into triangles $\Delta'_{k,l} = \Delta(s_{k-1,l-1}, s_{k,l-1}, s_{k,l})$ and $\Delta''_{k,l} = \Delta(s_{k-1,l-1}, s_{k,l}, s_{k-1,l})$. Thus we arrive at a regular triangulation $\mathcal{T}_N$ [11] with vertices $s_{k,l}$ and mesh size $\sigma_N = \sqrt{a^2+b^2}\,/2^{N+1}$. For every triangle, its interior angles ϑ obey $\sin\vartheta \geqslant \sin\vartheta_0 = b/\sqrt{a^2+b^2} > 0.$ [12] The space of all piecewise affine functions with zero boundary values on $\partial\Omega$, which are adapted to the triangulation $\mathcal{T}_N$, will be denoted by X_0^N. Then we may restrict the problem (P) in the following way:

$$(\mathrm{P})_N: \quad F(x,u) = \int_\Omega f(s, x(s), u(s))\, ds \longrightarrow \inf!\,; \quad (x,u) \in \Big(W_0^{1,p}(\Omega, \mathbb{R}^n) \cap X_0^N \Big) \times L^p(\Omega, \mathbb{R}^{n\times 2})\,; \tag{2.2}$$

$$Jx(s) = u(s) \ (\forall)\, s \in \Omega\,; \tag{2.3}$$

$$u(s) \in \mathrm{K} = \Big\{ v \in \mathbb{R}^{n\times 2} \ \Big|\ \sum_{i=1}^{n} \sum_{j=1}^{2} |\, v_{ij}\,|^q \leqslant R^q \Big\} \ (\forall)\, s \in \Omega\,. \tag{2.4}$$

For the admissible pairs (x,u) in $(\mathrm{P})_N$, it holds that

$$\int_{\mathrm{Q}_{k,l}} f(s, x(s), u(s))\, ds = \int_{\Delta'_{k,l}} f(s, x(s), u(s))\, ds + \int_{\Delta''_{k,l}} f(s, x(s), u(s))\, ds \tag{2.5}$$

$$= \int_{\Delta'_{k,l}} f\Bigg(s,\ x(s), \begin{pmatrix} \dfrac{x_1(s_{k,l-1}) - x_1(s_{k-1,l-1})}{(a/2^N)} & \dfrac{x_1(s_{k,l}) - x_1(s_{k,l-1})}{(b/2^N)} \\ \vdots & \vdots \\ \dfrac{x_n(s_{k,l-1}) - x_n(s_{k-1,l-1})}{(a/2^N)} & \dfrac{x_n(s_{k,l}) - x_n(s_{k,l-1})}{(b/2^N)} \end{pmatrix} \Bigg)$$

$$+ \int_{\Delta''_{k,l}} f\Bigg(s,\ x(s), \begin{pmatrix} \dfrac{x_1(s_{k,l}) - x_1(s_{k-1,l})}{(a/2^N)} & \dfrac{x_1(s_{k-1,l}) - x_1(s_{k-1,l-1})}{(b/2^N)} \\ \vdots & \vdots \\ \dfrac{x_n(s_{k,l}) - x_n(s_{k-1,l})}{(a/2^N)} & \dfrac{x_n(s_{k-1,l}) - x_n(s_{k-1,l-1})}{(b/2^N)} \end{pmatrix} \Bigg)\,. \tag{2.6}$$

With the abbreviation $x_i(s_{k,l}) = \xi_{k,l}^{(i)}$, we obtain as the discretized problem related to $(\mathrm{P})_N$:

$$(\mathrm{D})_N: \quad \widetilde{F}\big(\xi_{1,1}^{(1)}, \ldots, \xi_{K,L}^{(n)}, v_{1,1}^{(1,1)}, \ldots, v_{K,L}^{(n,4)} \big) = \frac{1}{2} \cdot \frac{a\,b}{4^N}$$

$$\cdot \sum_{k=1}^{K} \sum_{l=1}^{L} \Bigg(f\Bigg(s_{k-1,l-1}, \begin{pmatrix} \xi_{k-1,l-1}^{(1)} \\ \vdots \\ \xi_{k-1,l-1}^{(n)} \end{pmatrix}, \begin{pmatrix} v_{k,l}^{(1,1)} & v_{k,l}^{(1,2)} \\ \vdots & \vdots \\ v_{k,l}^{(n,1)} & v_{k,l}^{(n,2)} \end{pmatrix} \Bigg) + f\Bigg(s_{k,l}, \begin{pmatrix} \xi_{k,l}^{(1)} \\ \vdots \\ \xi_{k,l}^{(n)} \end{pmatrix}, \begin{pmatrix} v_{k,l}^{(1,3)} & v_{k,l}^{(1,4)} \\ \vdots & \vdots \\ v_{k,l}^{(n,3)} & v_{k,l}^{(n,4)} \end{pmatrix} \Bigg) \Bigg) \tag{2.7}$$

$$\longrightarrow \inf!\,; \quad \big(\xi_{0,0}^{(1)}, \ldots, \xi_{K,L}^{(n)}, v_{1,1}^{(1,1)}, \ldots, v_{K,L}^{(n,4)} \big) \in \mathbb{R}^{n(K+1)(L+1)} \times \mathbb{R}^{4nKL}\,; \tag{2.8}$$

$$\xi_{0,l}^{(i)} = \xi_{K,l}^{(i)} = 0\,, \quad 1 \leqslant i \leqslant n\,,\ 0 \leqslant l \leqslant L\,; \tag{2.9}$$

$$\xi_{k,0}^{(i)} = \xi_{k,L}^{(i)} = 0\,, \quad 1 \leqslant i \leqslant n\,,\ 0 \leqslant k \leqslant K\,; \tag{2.10}$$

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[11] We adopt the terminology from [Goering/Roos/Tobiska 93], pp. 28 and 40, (Z1) – (Z4), and p. 138, (Z5).

[12] Consequently, the sequence $\{\, \mathcal{T}_N \,\}$ of the triangulations satisfies the Zlámal condition, cf. [Ciarlet 87], pp. 124 and 130.

$$v^{(i,1)}_{k,l} \;=\; \frac{\xi^{(i)}_{k,l-1} - \xi^{(i)}_{k-1,l-1}}{(a/2^N)}\,; \quad v^{(i,2)}_{k,l} \;=\; \frac{\xi^{(i)}_{k,l} - \xi^{(i)}_{k,l-1}}{(b/2^N)}\,,\ 1 \leqslant i \leqslant n\,,\ 1 \leqslant k \leqslant K\,,\ 1 \leqslant l \leqslant L\,; \tag{2.11}$$

$$v^{(i,3)}_{k,l} \;=\; \frac{\xi^{(i)}_{k,l} - \xi^{(i)}_{k-1,l}}{(a/2^N)}\,; \quad v^{(i,4)}_{k,l} \;=\; \frac{\xi^{(i)}_{k-1,l} - \xi^{(i)}_{k-1,l-1}}{(b/2^N)}\,,\ 1 \leqslant i \leqslant n\,,\ 1 \leqslant k \leqslant K\,,\ 1 \leqslant l \leqslant L\,; \tag{2.12}$$

$$\sum_{i=1}^{n} \Big(\,|\, v^{(i,1)}_{k,l} \,|^q \,+\, |\, v^{(i,2)}_{k,l} \,|^q \Big) \;\leqslant\; R^q\,,\ 1 \leqslant k \leqslant K\,,\ 1 \leqslant l \leqslant L\,; \tag{2.13}$$

$$\sum_{i=1}^{n} \Big(\,|\, v^{(i,3)}_{k,l} \,|^q \,+\, |\, v^{(i,4)}_{k,l} \,|^q \Big) \;\leqslant\; R^q\,,\ 1 \leqslant k \leqslant K\,,\ 1 \leqslant l \leqslant L\,. \tag{2.14}$$

In consequence of the possible discontinuity of the control variables within $(\mathrm{P})_N$, the number of the corresponding discretization variables in $(\mathrm{D})_N$ has been doubled. For the same reason, the classical two-dimensional Newton-Cotes cubature formula [13] had to be modified.

b) Existence of global minimizers for the problems (P), $(\mathrm{P})_N$ and $(\mathrm{D})_N$.

Theorem 2.1. (Global minimizers for (P) and $(\mathrm{P})_N$) *Consider the problem* (P) *under the assumptions of Section 2.a). Assume further that the integrand $f(s,\xi,v)$ is convex as a function of v for almost all $s \in \Omega$ and all $\xi \in \mathbb{R}^n$, and a growth condition*

$$\big|\, f(s,\xi,v) \,\big| \;\leqslant\; \varphi_1(s) \,+\, \varphi_2(\,|\,\xi\,|\,,\,|\,v\,|\,) \quad (\forall)\, s \in \Omega \quad \forall\, (\xi, v) \in \mathbb{R}^n \times \mathrm{K} \tag{2.15}$$

with $\varphi_1 \in L^1(\Omega,\mathbb{R})$, $\varphi_1(s) \geqslant 0$ $(\forall)\, s \in \Omega$ and $\varphi_2 \in C^0(\mathbb{R}^n \times \mathrm{K}\,,\,\mathbb{R})$, $\varphi_2(\,|\,\xi\,|\,,\,|\,v\,|\,) \geqslant 0$ $\forall\,(\xi,v) \in \mathbb{R}^n \times \mathrm{K}$ is satisfied where φ_2 is a monotonically increasing function in $|\,\xi\,|$ as well as in $|\,v\,|$.

1) [14] *Then* (P) *admits a global minimizer $(\hat{x},\hat{u})$.*

2) Every problem $(\mathrm{P})_N$, $N \in \mathbb{N}$, admits a global minimizer $(\hat{x}_N,\hat{u}_N)$.

Proof. 2) Although the feasible domain is restricted to $\big(W^{1,p}_0(\Omega,\mathbb{R}^n) \cap X^N_0 \,\big) \times L^p(\Omega,\mathbb{R}^{n\times 2})$, the arguments from [Pickenhain/Wagner 00] , pp. 222 – 224, and [Dacorogna 08] , p. 378, Theorem 8.8., remain valid. ■

Theorem 2.2. *Under the assumptions of Theorem 2.1., every problem $(\mathrm{D})_N$, $N \in \mathbb{N}$, admits a global minimizer $\big(\, \hat{\xi}^{(1)}_{0,0},\ \dots,\ \hat{\xi}^{(n)}_{K,L},\ \hat{v}^{(1,1)}_{1,1},\ \dots,\ \hat{v}^{(n,4)}_{K,L} \,\big)$.*

Proof. The feasible domain of $(\mathrm{D})_N$ is nonempty, convex and compact since the equations (2.11) − (2.12) for $v^{(i,j)}_{k,l}$ and the assignment $\xi^{(i)}_{k,0} = \xi^{(i)}_{k,L} = \xi^{(i)}_{0,l} = \xi^{(i)}_{K,l} = 0$ at the boundary vertices imply the boundedness of the variables $\xi^{(i)}_{k,l}$ as well. The objective $\widetilde{F}$ is continuous w. r. to all variables; consequently, it takes its global minimum on the feasible domain. ■

c) A convergence theorem for the discretization method.

Theorem 2.3. (Convergence of the discretization method for (P)) *Consider the problems* (P), $(\mathrm{P})_N$ *and* $(\mathrm{D})_N$ *under the assumptions of Theorem 2.1. Let $1 < p < \infty$ and assume further that $4\,C_1\,\sigma_N/\sin\vartheta_0 \leqslant 1$ as well as $8\,n\,C_1\,(1+C_1)^{q-1}\,\sigma_N/\sin\vartheta_0 < 1$ with $C_1 = R/c_q$.*

1) **(Relations between (P) and $(\mathrm{P})_N$)** *Let $N \in \mathbb{N}$. For all global minimizers $(\hat{x},\hat{u})$ of* (P) *and $(\hat{x}_N,\hat{u}_N)$ of $(\mathrm{P})_N$, the inequality*

$$F(\hat{x},\hat{u}) \;\leqslant\; F(\hat{x}_N,\hat{u}_N) \;\leqslant\; F(\hat{x},\hat{u}) \,+\, C_2\,\sigma_N \tag{2.16}$$

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[13] Cf. [Maess 88] , p. 238 f.

[14] [Brune/Maurer/Wagner 08] , p. 4, Theorem 3.1.

holds with a constant $C_2 > 0$*, which does not depend on* $N \in \mathbb{N}$*. Moreover, every sequence* $\{ (\hat{x}_N, \hat{u}_N) \}$ *of global minimzers of* $(\mathrm{P})_N$ *contains a subsequence* $\{ (\hat{x}_{N'}, \hat{u}_{N'}) \}$ *with*

$$\| \hat{x}_{N'} - \hat{x} \|_{C^0(\Omega,\mathbb{R}^n)} \to 0 \quad \text{and} \quad \big(\hat{u}_{N'} - \hat{u} \big) \rightharpoonup \mathfrak{o}_{L^p(\Omega,\mathbb{R}^{n\times 2})} \tag{2.17}$$

where $(\hat{x}, \hat{u})$ *is a global minimizer of* (P)*, and the entire sequence satisfies*

$$\big| F(\hat{x}_N, \hat{u}_N) - F(\hat{x}, \hat{u}) \big| \to 0 \,. \tag{2.18}$$

2) **(Relations between** (P) **and** $(\mathrm{D})_N$ **)** *Let* $N \in \mathbb{N}$*. Assume that for all feasible solutions* (x, u) *of* $(\mathrm{P})_N$ *and all indices* $1 \leqslant k \leqslant K$*,* $1 \leqslant l \leqslant L$ *the conditions*

$$\big| f(s, x(s), u(s)) - f(s_{k-1,l-1}, x(s), u(s)) \big| \leqslant C_3 \cdot \sigma_N \quad (\forall)\, s \in \Delta'_{k,l} \quad \text{and} \tag{2.19}$$

$$\big| f(s, x(s), u(s)) - f(s_{k,l}, x(s), u(s)) \big| \leqslant C_3 \cdot \sigma_N \quad (\forall)\, s \in \Delta''_{k,l} \tag{2.20}$$

hold. Consider a global minimizer $(\hat{\xi}, \hat{v})$ *of* $(\mathrm{D})_N$ *and the related feasible solution* $(\widetilde{x}_N, \widetilde{u}_N)$ *of* $(\mathrm{P})_N$ *defined by*

$$\widetilde{x}_{N,i}(s_{k,l}) = \hat{\xi}^{(i)}_{k,l}\,, \quad 1 \leqslant i \leqslant n\,,\ 0 \leqslant k \leqslant K\,,\ 0 \leqslant l \leqslant L\,. \tag{2.21}$$

Then for all global minimizers $(\hat{x}, \hat{u})$ *of* (P)*, the inequality*

$$F(\hat{x}, \hat{u}) \leqslant F(\widetilde{x}_N, \widetilde{u}_N) \leqslant F(\hat{x}, \hat{u}) + C_4\, \sigma_N \tag{2.22}$$

holds with a constant $C_4 > 0$*, which does not depend on* $N \in \mathbb{N}$*. Moreover, the sequence* $\{ (\widetilde{x}_N, \widetilde{u}_N) \}$ *contains a subsequence* $\{ (\widetilde{x}_{N'}, \widetilde{u}_{N'}) \}$ *with*

$$\| \widetilde{x}_{N'} - \hat{x} \|_{C^0(\Omega,\mathbb{R}^n)} \to 0 \quad \text{and} \quad \big(\widetilde{u}_{N'} - \hat{u} \big) \rightharpoonup \mathfrak{o}_{L^p(\Omega,\mathbb{R}^{n\times 2})} \tag{2.23}$$

where $(\hat{x}, \hat{u})$ *is a global minimizer of* (P)*, and the entire sequence satisfies*

$$\big| F(\widetilde{x}_N, \widetilde{u}_N) - F(\hat{x}, \hat{u}) \big| \to 0 \,. \tag{2.24}$$

Remarks. 1. The additional assumption in Part 2) of the theorem reflects the fact that the integrand f is only a measurable and essentially bounded function of s.

2. If (P) admits an uniquely determined global minimizer then the convergence relations (2.17) and (2.23) hold for the entire sequences $\{ (\hat{x}_N, \hat{u}_N) \}$ resp. $\{ (\widetilde{x}_N, \widetilde{u}_N) \}$, since, in consequence of (2.18) and (2.24), all weakly convergent subsequences possess the same limit element. [15)]

Proof. The proof will be divided into seven steps.

• **Step 1.** C^∞*-approximation of a global minimizer of* (P). From Theorem 2.1., we know that (P) possesses a global minimizer $(\hat{x}, \hat{u}) \in W^{1,\infty}_0(\Omega, \mathbb{R}^n) \times L^\infty(\Omega, \mathbb{R}^{n\times 2})$. We rely to the following approximation theorem:

Theorem 2.4. [16)] *Assume that* $\Omega \subset \mathbb{R}^m$ *is the closure of a bounded strongly Lipschitz domain,* $\mathrm{K} \subset \mathbb{R}^{nm}$ *is a convex, compact set with* $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$ *and* $\hat{x} \in W^{1,\infty}_0(\Omega, \mathbb{R}^n)$ *is a function with* $J\hat{x}(s) \in \mathrm{K}$ *for almost all* $s \in \Omega$*. Then* $\hat{x}$ *can be approximated by a sequence of functions* $z^N \in C^\infty_0(\Omega, \mathbb{R}^n)$ *with the following properties:*

15) [Gajewski/Gröger/Zacharias 74], p. 10, Lemma 5.4.

16) [Wagner 99], p. 2, Theorem 1.5., with $\mathrm{S}(s) \equiv \mathrm{K}$, $\Gamma = \partial\Omega$ and $c = \mathfrak{o}$.

1) $\lim_{N\to\infty} \| z^N - \hat{x} \|_{C^0(\Omega,\mathbb{R}^n)} = 0\,,$ (2.25)

2) $\lim_{N\to\infty} \| Jz^N - J\hat{x} \|_{L^1(\Omega,\mathbb{R}^{nm})} = 0\,,$ (2.26)

3) $Jz^N(s) \in \mathrm{K}$ *for all* $s \in \Omega$. (2.27)

Consequently, in relation to $\hat{x}$ we may choose a function $z \in C_0^\infty(\Omega, \mathbb{R}^{n\times 2})$ with

$$\| z - \hat{x} \|_{C^0(\Omega,\mathbb{R}^n)} + \| Jz - J\hat{x} \|_{L^1(\Omega,\mathbb{R}^{n\times 2})} \leqslant \sigma_N \tag{2.28}$$

and $Jz(s) \in \mathrm{K}$ for all $s \in \Omega$. The Lipschitz constant of the components of z amounts to $C_1 = R/c_q$ as well, and the zero boundary conditions for $\hat{x}$ and z imply

$$\| \hat{x}_i \|_{C^0(\Omega,\mathbb{R})} \leqslant C_1 \cdot \mathrm{Diam}\,(\Omega) = \frac{C_1}{2}\sqrt{a^2+b^2} \quad \text{and} \quad \| z_i \|_{C^0(\Omega,\mathbb{R})} \leqslant C_1 \cdot \mathrm{Diam}\,(\Omega)\,,\ 1 \leqslant i \leqslant n\,. \tag{2.29}$$

• **Step 2.** *Piecewise affine approximation of z.* We approximate z by that continuous, piecewise affine function $y \in X_0^N$, which coincides with z in all vertices of the the triangulation $\mathcal{T}_N$. Let us note first that

$$\| y_i \|_{C^0(\Omega,\mathbb{R})} \leqslant \| z_i \|_{C^0(\Omega,\mathbb{R})} \leqslant C_1 \cdot \mathrm{Diam}\,(\Omega)\,,\ 1 \leqslant i \leqslant n\,. \tag{2.30}$$

Moreover, the following inequalities hold:

Lemma 2.5. [17] *Let a function* $z \in C_0^\infty(\Omega, \mathbb{R}^{n\times 2})$ *with Lipschitz constant* $C_1 > 0$ *for all components be given. Then for* $1 \leqslant i \leqslant n$*, the following estimates hold:*

$$\Big| z_i(s) - y_i(s) \Big| \leqslant C_1 \cdot C_5 \cdot \sigma_N \quad \textit{with} \quad C_5 = 6 + \frac{32}{\sin\vartheta_0}\,; \tag{2.31}$$

$$\Big| \frac{\partial z_i}{\partial s_j}(s) - \frac{\partial y_i}{\partial s_j}(s) \Big| \leqslant C_1 \cdot C_6 \cdot \sigma_N \quad \textit{with} \quad C_6 = \frac{4}{\sin\vartheta_0}\,. \tag{2.32}$$

From Lemma 2.5. we conclude that, for all sufficiently large $N \in \mathbb{N}$, every pair $\big((1-D)\,y\,,\,(1-D)\,Jy \big)$ with

$$0 < D = 2\,n\,C_1\,(1+C_1)^{q-1}\,C_6\,\sigma_N < 1 \tag{2.33}$$

is admissible in $(\mathrm{P})_N$. To see this, we enlarge N until

$$C_1 \cdot C_6 \cdot \sigma_N \leqslant 1 \tag{2.34}$$

is satisfied. Then it follows that

$$\Big| \frac{\partial y_i}{\partial s_j}(s) \Big|^q \leqslant \Big(\Big| \frac{\partial z_i}{\partial s_j}(s) \Big| + \Big| \frac{\partial y_i}{\partial s_j}(s) - \frac{\partial z_i}{\partial s_j}(s) \Big| \Big)^q$$

$$= \Big| \frac{\partial z_i}{\partial s_j}(s) \Big|^q + \sum_{k=1}^{q-1} \binom{q}{k} \cdot \Big| \frac{\partial z_i}{\partial s_j}(s) \Big|^k \cdot \Big| \frac{\partial y_i}{\partial s_j}(s) - \frac{\partial z_i}{\partial s_j}(s) \Big|^{q-k} \tag{2.35}$$

$$\leqslant \Big| \frac{\partial z_i}{\partial s_j}(s) \Big|^q + \Big| \frac{\partial y_i}{\partial s_j}(s) - \frac{\partial z_i}{\partial s_j}(s) \Big| \cdot \sum_{k=1}^{q-1} \binom{q}{k} \cdot \Big| \frac{\partial z_i}{\partial s_j}(s) \Big|^k \cdot 1^{q-1-k} \tag{2.36}$$

$$\leqslant \Big| \frac{\partial z_i}{\partial s_j}(s) \Big|^q + \Big| \frac{\partial y_i}{\partial s_j}(s) - \frac{\partial z_i}{\partial s_j}(s) \Big| \cdot \sum_{k=1}^{q-1} \binom{q}{k} \cdot L^k \cdot 1^{q-k} \tag{2.37}$$

$$\leqslant \Big| \frac{\partial z_i}{\partial s_j}(s) \Big|^q + C_1\,(1+C_1)^{q-1}\,C_6\,\sigma_N \quad \Longrightarrow \tag{2.38}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{2} \Big| \frac{\partial y_i}{\partial s_j}(s) \Big|^q \leqslant R^q + 2\,n\,C_1\,(1+C_1)^{q-1}\,C_6\,\sigma_N \quad \Longrightarrow \tag{2.39}$$

$$(1-D) \sum_{i=1}^{n} \sum_{j=1}^{2} \Big| \frac{\partial y_i}{\partial s_j}(s) \Big|^q \leqslant (1-D)\,R^q + 2\,n\,C_1\,(1+C_1)^{q-1}\,C_6\,\sigma_N\,. \tag{2.40}$$

[17] [Wagner 03], p. 41, Lemma 0.1. and 0.2., modified from [Ekeland/Témam 99], p. 309, Proposition 2.1.

From the requirement

$$(1-D)\,R^q \;+\; 2\,n\,C_1\,(1+C_1)^{q-1}\,C_6\,\sigma_N \;\leqslant\; R^q\,, \tag{2.41}$$

we obtain

$$D \;=\; 2\,n\,C_1\,(1+C_1)^{q-1}\,C_6\,\sigma_N \tag{2.42}$$

as well as

$$\begin{aligned} \Big|\,z_i(s)-(1-D)\,y_i(s)\,\Big| \;&\leqslant\; C_1\cdot C_5\cdot\sigma_N \;+\; D\cdot\Big|\,y_i(s)\,\Big| \\ &\leqslant\; \Big(\,C_1\,C_5 \;+\; 2\,n\,C_1\,(1+C_1)^{q-1}\,C_6\cdot C_1\,\mathrm{Diam}\,(\Omega)\,\Big)\cdot\sigma_N \;=\; C_7\,\sigma_N\,; \qquad (2.43)\\ \Big|\,\frac{\partial z_i}{\partial s_j}(s)-(1-D)\,\frac{\partial y_i}{\partial s_j}(s)\,\Big| \;&\leqslant\; C_1\cdot C_6\cdot\sigma_N \;+\; D\,\Big|\,\frac{\partial y_i}{\partial s_j}(s)\,\Big| \\ &\leqslant\; \Big(\,C_1\,C_6 \;+\; 2\,n\,C_1\,(1+C_1)^{q-1}\,C_6\cdot R\,\Big)\cdot\sigma_N \;=\; C_8\,\sigma_N\,. \qquad (2.44) \end{aligned}$$

Now we define $\widetilde{y}=(1-D)\,y$ and $\widetilde{w}=(1-D)\,Jy$.

• **Step 3.** *Estimation of the objective values in* (P) *and* $(\mathrm{P})_N$. From the assumed differentiability of the integrand $f(s,\xi,v)$ w. r. to ξ_i and v_{ij}, we obtain together with (2.29):

$$\begin{aligned} \big|\,f(s,\xi',v)\;-\;f(s,\xi'',v)\,\big| \;&\leqslant\; \Big(\,\underset{(s,\,\xi,\,v)\,\in\,\Omega\,\times\,\mathrm{A}\,\times\,\mathrm{K}}{\mathrm{Max}}\big|\,\nabla_\xi\,f(s,\xi,v)\,\big|\,\Big)\cdot\big|\,\xi'-\xi''\,\big| \\ &=\; C_9\cdot\big|\,\xi'-\xi''\,\big| \quad (\forall)\,s\in\Omega\;\;\forall\,\xi',\,\xi''\in\mathrm{A}\;\;\forall\,v\in\mathrm{K}\,; \qquad (2.45)\\ \big|\,f(s,\xi,v')\;-\;f(s,\xi,v'')\,\big| \;&\leqslant\; \Big(\,\underset{(s,\,\xi,\,v)\,\in\,\Omega\,\times\,\mathrm{A}\,\times\,\mathrm{K}}{\mathrm{Max}}\big|\,\nabla_v\,f(s,\xi,v)\,\big|\,\Big)\cdot\big|\,v'-v''\,\big| \\ &=\; C_{10}\cdot\big|\,v'-v''\,\big| \quad (\forall)\,s\in\Omega\;\;\forall\,\xi\in\mathrm{A}\;\;\forall\,v',v''\in\mathrm{K}\,. \qquad (2.46) \end{aligned}$$

Here $\mathrm{A}\subset\mathbb{R}^n$ denotes the closed ball with center $\mathfrak{o}$ and radius $(\,C_1\cdot\mathrm{Diam}\,(\Omega)\,)$ while the vector $\nabla_v\,f(s,\xi,v)$ has been assembled from the columns of the Jacobi matrix with entries $\partial f(s,\xi,v)/\partial v_{ij}$. It follows that

$$\begin{aligned} F(\hat{x},\hat{u}) \;\leqslant\; F(\hat{x}_N,\hat{u}_N) \;\leqslant\; F(\widetilde{y},\widetilde{w}) \;\leqslant\; F(\hat{x},\hat{u}) \;+\; \big|\,F(z,Jz)-F(\hat{x},\hat{u})\,\big| \;+\; \big|\,F(\widetilde{y},\widetilde{w})-F(z,Jz)\,\big| \qquad (2.47)\\ \leqslant\; F(\hat{x},\hat{u}) \;+\; \int_\Omega\Big(\,\big|\,f(s,z(s),Jz(s))-f(s,\hat{x}(s),Jz(s))\,\big| \;+\; \big|\,f(s,\hat{x}(s),Jz(s))-f(s,\hat{x}(s),\hat{u}(s))\,\big|\,\Big)\,ds \\ +\;\int_\Omega\Big(\,\big|\,f(s,\widetilde{y}(s),\widetilde{w}(s))-f(s,z(s),\widetilde{w}(s))\,\big| \;+\; \big|\,f(s,z(s),\widetilde{w}(s))-f(s,z(s),Jz(s))\,\big|\,\Big)\,ds \qquad (2.48)\\ \leqslant\; F(\hat{x},\hat{u}) \;+\; \int_\Omega\Big(\,C_9\,\big|\,z(s)-\hat{x}(s)\,\big| \;+\; C_{10}\,\big|\,Jz(s)-\hat{u}(s)\,\big|\,\Big)\,ds \\ +\;\int_\Omega\Big(\,C_9\,\big|\,\widetilde{y}(s)-z(s)\,\big| \;+\; C_{10}\,\big|\,\widetilde{w}(s)-Jz(s)\,\big|\,\Big)\,ds \qquad (2.49)\\ \leqslant\; F(\hat{x},\hat{u}) \;+\; \Big(\,|\,\Omega\,|\,C_9\,\sqrt{n} \;+\; C_{10}\,\sqrt{2\,n} \;+\; |\,\Omega\,|\,C_7\,C_9\,\sqrt{n} \;+\; |\,\Omega\,|\,C_8\,C_{10}\,\sqrt{2\,n}\,\Big)\cdot\sigma_N\,. \qquad (2.50) \end{aligned}$$

The first and the second member have been estimated by use of (2.28); the third and the fourth one by use of (2.43) and (2.44).

• **Step 4.** *Analysis of the sequence* $\{\,(\hat{x}_N,\hat{u}_N)\,\}$. For every $N\in\mathbb{N}$, we choose a global minimizer $(\hat{x}_N,\hat{u}_N)$ of $(\mathrm{P})_N$. All pairs $(\hat{x}_N,\hat{u}_N)$ are feasible in (P) as well, and by Step 3, the sequence $\{\,(\hat{x}_N,\hat{u}_N)\,\}$, $W_0^{1,p}(\Omega,$

$\mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{n\times 2})$ is a minimizing sequence for (P). Due to (1.3) and (2.29), the feasible domain of (P) is bounded. Consequently, $\{(\hat{x}_N, \hat{u}_N)\}$ admits a weakly convergent subsequence whose limit $(\hat{x}, \hat{u})$ is feasible in (P) as well (this holds in analogy to [WAGNER 96], p. 60, Lemma 4.2–10). Since $\hat{x}_N \in W_0^{1,p}(\Omega, \mathbb{R}^n) \cap W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ for all $N \in \mathbb{N}$, we may assume that $p > 2$, and from the Rellich-Kondrachev embedding theorem [18] we get the existence of a subsequence with uniform convergence w. r. to x. The proof of Part 1) is complete.

• **Step 5.** *Relations between feasible solutions of* $(\mathrm{P})_N$ *and* $(\mathrm{D})_N$. The proof of Part 2) starts with the following

Lemma 2.6. *Consider the problems* $(\mathrm{P})_N$ *and* $(\mathrm{D})_N$ *under the assumptions of Theorem 2.1. Assume further that the conditions*

$$\big|\, f(s, x(s), u(s)) - f(s_{k-1,l-1}, x(s), u(s)) \,\big| \leqslant C_3 \cdot \sigma_N \ \ (\forall)\, s \in \Delta'_{k,l} \quad \textit{and} \tag{2.51}$$

$$\big|\, f(s, x(s), u(s)) - f(s_{k,l}, x(s), u(s)) \,\big| \leqslant C_3 \cdot \sigma_N \ \ (\forall)\, s \in \Delta''_{k,l} \tag{2.52}$$

hold for all feasible solutions (x, u) *of* $(\mathrm{P})_N$ *and all indices* $1 \leqslant k \leqslant K$, $1 \leqslant l \leqslant L$. *Then for every feasible solution* (x, u) *of* $(\mathrm{P})_N$ *there exists a feasible solution* $\big(\xi_{0,0}^{(1)}, \ldots, \xi_{K,L}^{(n)}, v_{1,1}^{(1,1)}, \ldots, v_{K,L}^{(n,4)}\big)$ *of* $(\mathrm{D})_N$ *with*

$$\big|\, F(x, u) - \widetilde{F}\big(\xi_{0,0}^{(1)}, \ldots, \xi_{K,L}^{(n)}, v_{1,1}^{(1,1)}, \ldots, v_{K,L}^{(n,4)}\big) \big| \leqslant C_{11} \cdot \sigma_N\,. \tag{2.53}$$

Conversely, for every feasible solution $\big(\xi_{0,0}^{(1)}, \ldots, \xi_{K,L}^{(n)}, v_{1,1}^{(1,1)}, \ldots, v_{K,L}^{(n,4)}\big)$ *of* $(\mathrm{D})_N$ *there exists a feasible solution* (x, u) *of* $(\mathrm{P})_N$ *with*

$$\big|\, \widetilde{F}\big(\xi_{0,0}^{(1)}, \ldots, \xi_{K,L}^{(n)}, v_{1,1}^{(1,1)}, \ldots, v_{K,L}^{(n,4)}\big) - F(x, u) \big| \leqslant C_{11} \cdot \sigma_N\,. \tag{2.54}$$

Proof. Let a feasible solution (x, u) of $(\mathrm{P})_N$ be given. The objective in $(\mathrm{P})_N$ may be represented as

$$F(x, u) = \sum_{k=1}^{K} \sum_{l=1}^{L} \Big(\int_{\Delta'_{k,l}} f(s, x(s), u(s))\, ds + \int_{\Delta''_{k,l}} f(s, x(s), u(s))\, ds \Big). \tag{2.55}$$

With the setting

$$\xi_{k,l}^{(i)} = x_i(s_{k,l})\,, \ \ 1 \leqslant i \leqslant n\,, \ \ 0 \leqslant k \leqslant K\,, \ \ 0 \leqslant l \leqslant L\,; \tag{2.56}$$

$$v_{k,l}^{(i,1)} = \frac{x_i(s_{k,l-1}) - x_i(s_{k-1,l-1})}{(a/2^N)}\,; \ \ v_{k,l}^{(i,2)} = \frac{x_i(s_{k,l}) - x_i(s_{k,l-1})}{(b/2^N)}\,, \ \ 1 \leqslant i \leqslant n\,, \ \ 1 \leqslant k \leqslant K\,, \ \ 1 \leqslant l \leqslant L\,;$$

$$v_{k,l}^{(i,3)} = \frac{x_i(s_{k,l}) - x_i(s_{k-1,l})}{(a/2^N)}\,; \qquad v_{k,l}^{(i,4)} = \frac{x_i(s_{k-1,l} - x_i(s_{k-1,l-1})}{(b/2^N)}\,, \ \ 1 \leqslant i \leqslant n\,, \ \ 1 \leqslant k \leqslant K\,, \ \ 1 \leqslant l \leqslant L\,,$$

we obtain a feasible solution $\big(\xi_{0,0}^{(1)}, \ldots, \xi_{K,L}^{(n)}, v_{1,1}^{(1,1)}, \ldots, v_{K,L}^{(n,4)}\big)$ of $(\mathrm{D})_N$, and the difference of the objectives can be estimated as follows:

$$\Big|\, F(x, u) - \widetilde{F}\big(\xi_{0,0}^{(1)}, \ldots, \xi_{K,L}^{(n)}, v_{1,1}^{(1,1)}, \ldots, v_{K,L}^{(n,4)}\big) \Big|$$

$$\leqslant \sum_{k=1}^{K} \sum_{l=1}^{L} \Big(\int_{\Delta'_{kl}} \Big| f\Big(s,\, x(s), \begin{pmatrix} v_{k,l}^{(1,1)} & v_{k,l}^{(1,2)} \\ \vdots & \vdots \\ v_{k,l}^{(n,1)} & v_{k,l}^{(n,2)} \end{pmatrix} \Big) - f\Big(s_{k-1,l-1},\, x(s), \begin{pmatrix} v_{k,l}^{(1,1)} & v_{k,l}^{(1,2)} \\ \vdots & \vdots \\ v_{k,l}^{(n,1)} & v_{k,l}^{(n,2)} \end{pmatrix} \Big) \Big|\, ds$$

$$+ \int_{\Delta'_{kl}} \Big| f\Big(s_{k-1,l-1}, \begin{pmatrix} x_1(s) \\ \vdots \\ x_n(s) \end{pmatrix}, \begin{pmatrix} v_{k,l}^{(1,1)} & v_{k,l}^{(1,2)} \\ \vdots & \vdots \\ v_{k,l}^{(n,1)} & v_{k,l}^{(n,2)} \end{pmatrix} \Big) - f\Big(s_{k-1,l-1}, \begin{pmatrix} \xi_{k-1,l-1}^{(1)} \\ \vdots \\ \xi_{k-1,l-1}^{(n)} \end{pmatrix}, \begin{pmatrix} v_{k,l}^{(1,1)} & v_{k,l}^{(1,2)} \\ \vdots & \vdots \\ v_{k,l}^{(n,1)} & v_{k,l}^{(n,2)} \end{pmatrix} \Big) \Big|\, ds$$

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[18] [ADAMS/FOURNIER 07], p. 168, Theorem 6.3.

$$+ \int_{\Delta''_{kl}} \Big| f\Big(s,\, x(s),\, \begin{pmatrix} v_{k,l}^{(1,3)} & v_{k,l}^{(1,4)} \\ \vdots & \vdots \\ v_{k,l}^{(n,3)} & v_{k,l}^{(n,4)} \end{pmatrix} \Big) - f\Big(s_{k,l},\, x(s),\, \begin{pmatrix} v_{k,l}^{(1,3)} & v_{k,l}^{(1,4)} \\ \vdots & \vdots \\ v_{k,l}^{(n,3)} & v_{k,l}^{(n,4)} \end{pmatrix} \Big) \Big| \, ds$$

$$+ \int_{\Delta''_{kl}} \Big| f\Big(s_{k,l},\, \begin{pmatrix} x_1(s) \\ \vdots \\ x_n(s) \end{pmatrix},\, \begin{pmatrix} v_{k,l}^{(1,3)} & v_{k,l}^{(1,4)} \\ \vdots & \vdots \\ v_{k,l}^{(n,3)} & v_{k,l}^{(n,4)} \end{pmatrix} \Big) - f\Big(s_{k,l},\, \begin{pmatrix} \xi_{k,l}^{(1)} \\ \vdots \\ \xi_{k,l}^{(n)} \end{pmatrix},\, \begin{pmatrix} v_{k,l}^{(1,3)} & v_{k,l}^{(1,4)} \\ \vdots & \vdots \\ v_{k,l}^{(n,3)} & v_{k,l}^{(n,4)} \end{pmatrix} \Big) \Big| \, ds \Big) \tag{2.57}$$

$$\leqslant \sum_{k=1}^{K} \sum_{l=1}^{L} \Big(\big| \Delta'_{kl} \big| \cdot C_3 \cdot \sigma_N + \big| \Delta'_{kl} \big| \cdot C_9 \cdot \Big| \begin{pmatrix} x_1(s) - \xi_{k-1,l-1}^{(1)} \\ \vdots \\ x_n(s) - \xi_{k-1,l-1}^{(n)} \end{pmatrix} \Big|$$

$$+ \big| \Delta''_{kl} \big| \cdot C_3 \cdot \sigma_N + \big| \Delta''_{kl} \big| \cdot C_9 \cdot \Big| \begin{pmatrix} x_1(s) - \xi_{k,l}^{(1)} \\ \vdots \\ x_n(s) - \xi_{k,l}^{(n)} \end{pmatrix} \Big| \Big) \tag{2.58}$$

$$\leqslant \Big(\big| \Omega \big| \cdot C_3 + \big| \Omega \big| \cdot C_9 \cdot C_1 \cdot \sqrt{n} \cdot 2 \Big) \cdot \sigma_N = C_{11}\, \sigma_N\,. \tag{2.59}$$

Conversely, to every feasible solution $\big(\xi_{0,0}^{(1)},\, \dots,\, \xi_{K,L}^{(n)},\, v_{1,1}^{(1,1)},\, \dots,\, v_{K,L}^{(n,4)} \big)$ of $(\mathrm{D})_N$, we may relate immediately a feasible solution (x, u) of $(\mathrm{P})_N$ defined by $x_i(s_{kl}) = \xi_{kl}^{(i)}$, $1 \leqslant i \leqslant n$, $0 \leqslant k \leqslant K$, $0 \leqslant l \leqslant L$. Then for the objective values the same estimate holds as above. ■

• **Step 6.** *Estimation of the objective values of* (P) *and* $(\mathrm{D})_N$. We consider a global minimizer $\big(\hat{\xi}_{0,0}^{(1)},\, \dots,\, \hat{\xi}_{K,L}^{(n)},\, \hat{v}_{1,1}^{(1,1)},\, \dots,\, \hat{v}_{K,L}^{(n,4)} \big)$ of $(\mathrm{D})_N$ and the related feasible solution $(\widetilde{x}_N, \widetilde{u}_N)$ of $(\mathrm{P})_N$ defined by $\widetilde{x}_{N,i}(s_{k,l}) = \hat{\xi}_{k,l}^{(i)}$ and $\widetilde{u}_{N,ij}(s) = \partial \widetilde{x}_{N,i}(s) / \partial s_j$. Further, let $(\hat{x}, \hat{u})$ be a global minimizer of (P) and $(\hat{x}_N, \hat{u}_N)$ a global minimizer of $(\mathrm{P})_N$. By (ξ, v) we denote the feasible solution of $(\mathrm{D})_N$, which is defined in analogy to (2.56) by $\xi_{k,l}^{(i)} = \hat{x}_{N,i}(s_{k,l})$. Then from Part 1) and Lemma 2.6. it follows that

$$F(\hat{x}, \hat{u}) \leqslant F(\hat{x}_N, \hat{u}_N) \leqslant F(\widetilde{x}_N, \widetilde{u}_N) \leqslant \widetilde{F}(\hat{\xi}, \hat{v}) + C_{11}\, \sigma_N \leqslant \widetilde{F}(\xi, v) + C_{11}\, \sigma_N$$

$$\leqslant F(\hat{x}_N, \hat{u}_N) + 2\, C_{11}\, \sigma_N \leqslant F(\hat{x}, \hat{u}) + \big(C_2 + 2\, C_{11} \big) \cdot \sigma_N = F(\hat{x}, \hat{u}) + C_4\, \sigma_N\,. \tag{2.60}$$

• **Step 7.** *Analysis of the sequence* $\{ (\widetilde{x}_n, \widetilde{u}_N) \}$. All pairs $(\widetilde{x}_N, \widetilde{u}_N)$ are feasible in (P), and from (2.60) we may conclude that $\{ (\widetilde{x}_N, \widetilde{u}_N) \}$, $W_0^{1,p}(\Omega, \mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{n \times 2})$ forms a minimizing sequence in (P) as well. The existence of a subsequence with the claimed properties can be ensured as in Step 4, and the proof of Part 2) is complete. ■

3. Application to the image restoration problem.

a) The image restoration problem.

Throughout this section, we describe *greyscale images* through at least measurable functions x defined on a rectangle $\Omega \subset \mathbb{R}^2$ with values $0 \leqslant x(s) \leqslant 1$ $(\forall)\, s \in \Omega$. The most commonly used model for the capture of an original scene $x\colon\ \Omega \to [\,0\,,\,1\,]$ is the equation

$$I(s) = \mathcal{S}(\, x(s)\,) + \mathcal{N}(s) \tag{3.1}$$

where the systematical errors (e. g. blur) are described by an operator $\mathcal{S}$, and $\mathcal{N}(s)$ is a noise term.[19] The formal solution of (3.1),

$$x(s) = \mathcal{S}^{-1}(\, I(s)\,) - \mathcal{S}^{-1}(\, \mathcal{N}(s)\,)\,, \tag{3.2}$$

[19] Cf. [Aubert/Kornprobst 06], pp. 68 ff., and [Chambolle 00], pp. 7 ff.

however, leads to an ill-posed problem. Since the 90s, in generalization of the “scale-space” approach where the noisy image data $I(s)$ are subjected to a smoothing diffusion process, [20] the image restoration problem has been extensively treated by variational methods. The objectives within these problems minimize the defect $\mathcal{S}\big(x(s)\big) - I(s)$ together with a regularization term involving the first generalized partial derivatives of x. For the diffusion methods, one has to assume $x \in C^2(\Omega, \mathbb{R})$ whereas the variational problems can be formulated within Sobolev spaces $W^{1,p}(\Omega, \mathbb{R})$ resp. the space $BV(\Omega, \mathbb{R})$ [21] of the functions of bounded variation. [22] The original scenes x, however, belong in general even not to $BV(\Omega, \mathbb{R})$, the most capacious of the mentioned spaces. [23] It must be emphasized that, for this reason, all methods proposed as yet are based on a compromise: instead of the original scene x, one searches for a representative with a prescribed grade of additional smoothness (“image smoothing”). In the following, we assume that $\mathcal{S}(\,x(s)\,) = x(s)$, and the variational problem will be formulated within the Sobolev space $W_0^{1,p}(\Omega, \mathbb{R})$:

$$(\mathrm{V})^{(1)}: \quad F(x) = \int_\Omega \big(\,x(s) - I(s)\,\big)^2\,ds \,+\, \mu \cdot \int_\Omega f(\,|\,\nabla x(s)\,|\,)\,ds \;\longrightarrow\; \inf!\,; \quad x \in W_0^{1,p}(\Omega, \mathbb{R}) \tag{3.3}$$

with $1 \leqslant p < \infty$, $I \in L^\infty(\Omega, \mathbb{R})$, $0 \leqslant I(s) \leqslant 1$ $(\forall)\, s \in \Omega$, a regularization parameter $\mu > 0$ and an integrand $f \in C^2(\mathbb{R}, \mathbb{R})$, which obeys an appropriate growth condition.
An important goal in image restoration/image smoothing is to conserve as far as possible the edge structure within the image data.[24] For this purpose, the use of an anisotropic regularization term of the shape

$$\int_\Omega \sqrt{\,|\,\nabla x(s)\,|^2 + \eta^2}\;ds \tag{3.4}$$

is advisable. For sufficiently small values $\eta > 0$, this term may be understood as an approximation for the L^1- resp. total variation norm of ∇x but avoids its main disadvantages since the integrand in (3.4) is differentiable in $\mathfrak{o}$ and produces only a reduced staircasing effect.

b) Edge detection within noisy image data.

In order to perform an image restoration with simultaneous edge detection, two different strategies may be pursued. The first possibility is to replace the cost functional in $(\mathrm{V})^{(1)}$ by a functional of Ambrosio-Tortorelli type. Besides the denoised image x, this functional depends on an additional variable k as a “sketch” for the edges with $k(s) \approx 0$ resp. $k(s) \approx 1$, depending on whether the point $s \in \Omega$ belongs to an edge within x or not. We obtain the following rather complicated variational problem:

$$\begin{aligned}(\mathrm{V})^{(2)}: \quad F(x,k) \;=\; & c_1(\varepsilon) \cdot \int_\Omega \big(\,x(s) - I(s)\,\big)^2\,ds \,+\, c_2(\varepsilon) \cdot \int_\Omega \big|\,\nabla x(s)\,\big|^p \cdot \big(\,k(s)^2 + c_4(\varepsilon)\,\big)\,ds \\ & + c_3(\varepsilon) \cdot \int_\Omega \Big(\,\varepsilon \cdot \big|\,\nabla k(s)\,\big|^2 + \frac{1}{4\,\varepsilon} \cdot \big(\,k(s) - 1\,\big)^2\,\Big)\,ds \longrightarrow \inf!\,; \quad (x,k) \in W_0^{1,p}(\Omega, \mathbb{R}) \,\times\, W_0^{1,2}(\Omega, \mathbb{R})\end{aligned} \tag{3.5}$$

[20] [Aubert/Kornprobst 06] , pp. 94 ff., [Weickert 96] , pp. 2 – 18.

[21] Cf. [Evans/Gariepy 92] , pp. 166 ff.

[22] For the treatment of image restoration problems within the space $BV(\Omega, \mathbb{R})$, we refer e. g. to [Aujol/Aubert/Blanc-Féraud/Chambolle 05] , [Chambolle 00] , [Chambolle/Lions 97] , [Hinterberger/Hintermüller/Kunisch/Oehsen/Scherzer 03] , [Osher/Burger/Goldfarb/Xu/Yin 05] and the seminal paper [Rudin/Osher/Fatemi 92] .

[23] This conclusion will be suggested by [Gousseau/Morel 01] .

[24] Here again, the inherent compromise within Sobolev space methods comes forward: “The theory seems to adopt again what it tried to avoid.” ([Catté/Lions/Morel/Coll 92] , p. 183).

with image data I as above, $\varepsilon > 0$ and weights $c_i(\varepsilon) > 0$, $1 \leqslant i \leqslant 4$. [25] Within the objective, the first term is the classical fidelity term. The second one replaces the regularization term in $(\mathrm{V})^{(1)}$ and realizes a coupling between x and k, which favors values $k(s) \approx 0$ in points $s \in \Omega$ with large magnitudes of $\nabla x(s)$. Within the third term, the first member effects a quadratical regularization of k while the second member enforces $k(s) \approx 1$ except a subset of Ω of small measure. The interpretation of k as an edge detector is heuristically clear and may be rigorously justified by proving the Γ-convergence of the solutions of $(\mathrm{V})^{(2)}$ towards a solution of a variational problem with an Mumford-Shah type objective. [26]

As an alternative to the study of $(\mathrm{V})^{(2)}$, we may add convex restrictions for ∇x to the problem $(\mathrm{V})^{(1)}$, thus converting the original variational problem into a multidimensional control problem of Dieudonné-Rashevsky type. Under comprehension of the regularization term (3.4), this problem reads as follows:

$$(\mathrm{P})^{(1)}: \quad F(x,u) = \int_\Omega \big(x(s) - I(s) \big)^2 \, ds \, + \, \mu \cdot \int_\Omega \sqrt{u_1(s)^2 + u_2(s)^2 + \eta^2} \, ds \longrightarrow \inf!\,; \tag{3.6}$$

$$(x,u) \in W_0^{1,p}(\Omega,\mathbb{R}) \times L^p(\Omega,\mathbb{R}^2)\,; \tag{3.7}$$

$$\nabla x(s) = \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} \quad (\forall)\, s \in \Omega\,; \tag{3.8}$$

$$u \in \mathrm{U} = \big\{ u \in L^p(\Omega,\mathbb{R}^2) \;\big|\; |\,u_1(s)\,|^q + |\,u_2(s)\,|^q \leqslant R^q \;\; (\forall)\, s \in \Omega \big\}\,. \tag{3.9}$$

Here we assume $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, $\mu > 0$, $\eta > 0$, $R > 0$ and I as above. The edge detector k is immediately constructed from the control variables u_1 and u_2, e. g. [27]

$$k(s) = 1 - \frac{1}{R^q} \Big(\big|\, u_1(s) \,\big|^q + \big|\, u_2(s) \,\big|^q \Big)\,. \tag{3.10}$$

In this optimal control reformulation of the problem, we interpret as "edges" those subsets of Ω where the control restriction becomes nearly active.

For $(\mathrm{P})^{(1)}$, the assumptions of Theorem 2.1. are satisfied: Since $0 \leqslant I(s) \leqslant 1$ for all $s \in \Omega$ it follows that

$$f(s,\xi,v) = \big(\xi - I(s) \big)^2 + \mu \sqrt{v_1^2 + v_2^2 + \eta^2} \quad \Longrightarrow \quad \big|\, f(s,\xi,v) \,\big| \leqslant \xi^2 + 2\,|\,\xi\,| + \mu \sqrt{|\,v\,|^2 + \eta^2}\,, \tag{3.11}$$

and the growth condition holds with $\varphi_1(s) \equiv 0$ and $\varphi_2(\,|\,\xi\,|\,, |\,v\,|\,) = |\,\xi\,|^2 + 2\,|\,\xi\,| + \mu \sqrt{|\,v\,|^2 + \eta^2}$.

c) Numerical solution of the discretized control problem.

In the present paper, we pursue the second approach and solve the image restoration problem with simultaneous edge detection as a multidimensional control problem $(\mathrm{P})^{(1)}$, applying the discretization method from Section 2. We choose $a = b = 128$ and $N = 7$ and decompose $\Omega = [\,0\,,\,128\,]^2$ into $(K \times L)$ pixels $\mathrm{Q}_{k,l}$ with edge length 1 and northeastern vertex $s_{k,l}$. Then the mesh size amounts to $\sigma_N = \sqrt{2}/2$, and the noisy image data $I(s)$ are given as a pixelwise constant function. For this reason, the assumptions (2.19) – (2.20) of Theorem 2.3., 3) are satisfied with $C_3 = 0$, and we may assume that $I(s)\,\big|\,\mathrm{Q}_{k,l} \equiv I(s_{k,l})$, $1 \leqslant k \leqslant K$,

[25] In [Ambrosio/Tortorelli 92], p. 111, resp. [Bellettini/Coscia 94], p. 205, (2.1), this functional has been proposed as an approximation for the Mumford-Shah functional. See also [Aubert/Kornprobst 06], pp. 166 – 173.

[26] [Bellettini/Coscia 94], p. 205 f., Theorem 2.1., for $p = 2$.

[27] Cf. [Franek 07a], p. 65.

$1 \leqslant l \leqslant L$. We may state the following discretized problem:

$$(\mathrm{D})_N^{(1)}: \quad \widetilde{F}\big(\,\xi_{11}^{(1)},\,\ldots\,,\xi_{KL}^{(1)},\,v_{11}^{(1,1)},\,\ldots\,,\,v_{KL}^{(1,4)}\,\big) \;=\; \frac{1}{2}\cdot\frac{a\,b}{4^N}\cdot\sum_{k=1}^{K}\sum_{l=1}^{L}\Bigg(\big(\,\xi_{k-1,l-1}^{(1)}-I(s_{k,l})\,\big)^2+\big(\,\xi_{k,l}^{(1)}-I(s_{k,l})\,\big)^2$$

$$+\,\mu\sqrt{\big(\,v_{k,l}^{(1,1)}\,\big)^2+\big(\,v_{k,l}^{(1,2)}\,\big)^2+\eta^2}\,+\,\mu\sqrt{\big(\,v_{k,l}^{(1,3)}\,\big)^2+\big(\,v_{k,l}^{(1,4)}\,\big)^2+\eta^2}\,\Bigg) \longrightarrow \text{inf!}\,; \quad (3.12)$$

$$\big(\,\xi_{0,0}^{(1)},\,\ldots\,,\xi_{K,L}^{(1)},\,v_{1,1}^{(1,1)},\,\ldots\,,\,v_{K,L}^{(1,4)}\,\big) \,\in\, \mathbb{R}^{(K+1)(L+1)}\times\mathbb{R}^{4KL}\,; \quad (3.13)$$

$$\xi_{0,l}^{(1)} \;=\; \xi_{K,l}^{(1)} \;=\; 0\,,\;\; 0\leqslant l\leqslant L\,; \quad (3.14)$$

$$\xi_{k,0}^{(1)} \;=\; \xi_{k,L}^{(1)} \;=\; 0\,,\;\; 0\leqslant k\leqslant K\,; \quad (3.15)$$

$$v_{k,l}^{(1,1)} \;=\; \frac{\xi_{k,l-1}^{(1)}-\xi_{k-1,l-1}^{(1)}}{(a/2^N)}\,; \quad v_{k,l}^{(1,2)} \;=\; \frac{\xi_{k,l}^{(1)}-\xi_{k,l-1}^{(1)}}{(b/2^N)}\,,\;\; 1\leqslant k\leqslant K\,,\;\; 1\leqslant l\leqslant L\,; \quad (3.16)$$

$$v_{k,l}^{(1,3)} \;=\; \frac{\xi_{k,l}^{(1)}-\xi_{k-1,l}^{(1)}}{(a/2^N)}\,; \qquad v_{k,l}^{(1,4)} \;=\; \frac{\xi_{k-1,l}^{(1)}-\xi_{k-1,l-1}^{(1)}}{(b/2^N)}\,,\;\; 1\leqslant k\leqslant K\,,\;\; 1\leqslant l\leqslant L\,; \quad (3.17)$$

$$\big|\,v_{k,l}^{(1,1)}\,\big|^q\,+\,\big|\,v_{k,l}^{(1,2)}\,\big|^q \;\leqslant\; R^q\,,\;\; 1\leqslant k\leqslant K\,,\;\; 1\leqslant l\leqslant L\,; \quad (3.18)$$

$$\big|\,v_{k,l}^{(1,3)}\,\big|^q\,+\,\big|\,v_{k,l}^{(1,4)}\,\big|^q \;\leqslant\; R^q\,,\;\; 1\leqslant k\leqslant K\,,\;\; 1\leqslant l\leqslant L\,. \quad (3.19)$$

As the discretization of the edge detector, we obtain

$$k(s_{k,l}) \;=\; 1-\frac{1}{R^q}\,\mathrm{Max}\Big(\,\big|\,v_{k,l}^{(1,1)}\,\big|^q\,+\,\big|\,v_{k,l}^{(1,2)}\,\big|^q,\,\big|\,v_{k,l}^{(1,3)}\,\big|^q\,+\,\big|\,v_{k,l}^{(1,4)}\,\big|^q\Big)\,. \quad (3.20)$$

The evaluation of the necessary optimality conditions (Karush-Kuhn-Tucker conditions) results in a large-scale system of nonlinear equations and inequalities, which may be solved with high precision and efficiency by interior point methods. As input/output platform, MATLAB has been used while the discretized problem has been formulated with the aid of the modelling language AMPL [28] and then transferred to the interior-point solver IPOPT. [29] The results have been represented, evaluated and archived with MATLAB again.

d) Image data and evaluation of the results.

For the numerical experiments, we used a segment of the Lena image [30] with $K = L = 128$ with or without artificial addition of white noise [31] (Figs. 3.1. – 3.2.). The quality of the image restoration will be evaluated by means of the SNR indicator

$$SNR\,(\hat{x},x) \;=\; -10\log_{10}\Bigg(\frac{\sum_{k=1}^{K}\sum_{l=1}^{L}\big(\,x_{kl}-\hat{x}(s_{kl})\,\big)^2}{\sum_{k=1}^{K}\sum_{l=1}^{L}\big(\,\hat{x}(s_{kl})\,\big)^2}\Bigg) \;\approx\; -10\log_{10}\Bigg(\frac{\int_\Omega\big(\,x(s)-\hat{x}(s)\,\big)^2\,ds}{\int_\Omega\hat{x}(s)^2\,ds}\Bigg)\,. \quad (3.21)$$

---

28) AMPL is a commercially distributed modelling language with easily comprehensible syntax, which allows a formal description of optimization problems, their transfer to a solver and the further processing of the output data. Cf. [Fourer/Gay/Kernighan 02] .

29) [Laird/Wächter 07] , [Wächter/Biegler 06] .

30) Accessible via http://www.am.uni-duesseldorf.de/∼witsch/html/lehre/bild-06/lena_gray.tif (last access: 14. 07. 2008).

31) With standard deviation zero and variance $\sigma = 0.01$.

For the evaluation of the edge sketches k, however, the literature does not provide a generally accepted criterion as yet. For this reason, we propose the following error measure IEE ("intensity edge error"): [32)]

$$IEE\,(\hat{x}, x) \;=\; \frac{1}{KL} \sum_{k=1}^{K} \sum_{l=1}^{L} \big(k(s_{kl}) - \hat{k}(s_{kl}) \big)^2 \;\approx\; \frac{1}{|\,\Omega\,|} \int_{\Omega} \big(k(s) - \hat{k}(s) \big)^2 \, ds \tag{3.22}$$

with reference to an "ideal" edge sketch $\hat{k}$ (Fig. 3.3.), which has been obtained by application of the Ambrosio-Tortorelli functional (3.5) to the (noiseless) original image data.

Fig. 3.1.

Fig. 3.2.

Segment of the Lena image: original (left) and noisy version (right) with $SNR\,(\hat{x}, x)$ $= 12.4569$

Fig. 3.3.

Variational problem $(\mathrm{V})^{(2)}$ with Ambrosio-Tortorelli objective: edge sketch $\hat{k}$ for the original Lena image

$p = 2$, $\varepsilon = 0.5$, $c_1(\varepsilon) = 1275$, $c_2(\varepsilon) = 10$, $c_3(\varepsilon) = 0.5$, $c_4(\varepsilon) = 0$

e) Numerical results.

In order to compare the proposed discretization approach with the Ambrosio-Tortorelli method, the variational problem $(\mathrm{V})^{(2)}$ has been solved first. [33)] With the parameters $p = 2$ and $\varepsilon = 0.5$, we arrive at the results depicted in Figs. 3.4. – 3.5.

For the numerical solution of the multidimensional control problem $(\mathrm{P})^{(1)}$, we take the values $p = q = 2$, $\mu = 0.5$, $\eta = 0.01$ and vary the parameter R. The edge sketches have been generated with the aid of the discretized edge detector (3.20). Figs. 3.6. – 3.13. show the results for the original Lena image from Fig. 3.1.; the second series (Figs. 3.14. – 3.21.) has been calculated with the noisy Lena image from Fig. 3.2.

32) Cf. [Brune/Maurer/Wagner 08], p. 8, Definition 4.2.

33) Following [Bourdin 99], a direct method has been applied to $(\mathrm{V})^{(2)}$ as well.

Fig. 3.4.

Fig. 3.5.

Variational problem $(\mathrm{V})^{(2)}$ with Ambrosio-Tortorelli objective: restored version (left) and edge sketch (right) for the noisy Lena image

$p = 2$, $\varepsilon = 0.5$, $c_1(\varepsilon) = 3$, $c_2(\varepsilon) = 30$, $c_3(\varepsilon) = 1.2$, $c_4(\varepsilon) = 0$

$SNR = 16.1037$, $IEE = 18.7680$

f) Discussion and conclusion.

Among the solutions of the *control problems*, the best results both with respect to the visual comparison as well as for the IEE criterion have been obtained for $R \approx 0.06$. There are no significant differences between the SNR as well as the IEE values for the original and the noisy data. If the value of the parameter R has been selected too small then "overcrowded" edge sketches develop (Figs. 3.13. und 3.21.). On the other hand, an enlargement of R beyond 0.25 produced no change of the edge sketches. Although the assumptions of the convergence theorem (Theorem 2.3.) are satisfied for

$$R < \operatorname{Min}\left(\frac{1}{4}\,,\, \frac{\sqrt{3}-\sqrt{2}}{2\sqrt{2}}\right) = 0.112372\ldots \qquad (3.23)$$

only, reasonable results have been obtained even for $R \in [\,0.11\,,\,0.25\,]$.

A *visual comparison* between the variational and optimal control method shows that the optimal control method supplies the clearer edge sketches but tends to loose some fine details. To the contrary, the edge sketch $\hat{k}$ resulting from the variational method (Fig. 3.3.) seems to contain a lot of unnecessary details. Among the experiments with the original data from Fig. 3.1., the eyelashes have been reproduced only by the variational method while the detail at the margin of the hat above the center of the image has been recognized by the control method as well. For the noisy data, the control method provides a better reconstruction of this detail (Fig. 3.19.) than the variational method (Fig. 3.5.).

The *quantitative comparison* between variational and optimal control method was possible only for the noisy data since the IEE values had to be related to $\hat{k}$. The best result obtained by the optimal control method (Figs. 3.18. – 3.19.) and the result from the variational method have nearly the same SNR and comparable IEE values. The difference in the latter is possibly caused even by the "unnecessary" details in $\hat{k}$, which are suppressed by the optimal control method.

We may summarize that the presented variational and optimal control method supply results of comparable quality. Consequently, our experiments demonstrate that the treatment of the image restoration problem with simultaneous edge detection as a multidimensional control problem offers a real alternative to the existing variational methods.

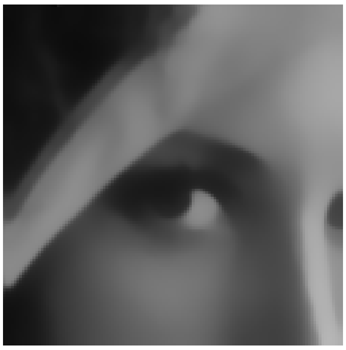

Fig. 3.6.

Fig. 3.7.

Control problem $(P)^{(1)}$ with robust TV regularization term: original Lena image

$R = 0.25$
$SNR = 16.3235$, $IEE = 28.7146$

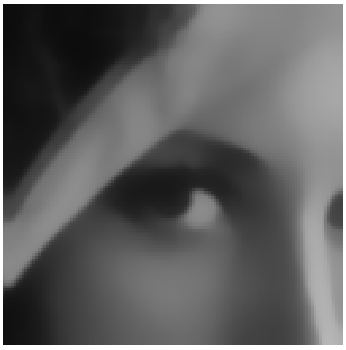

Fig. 3.8.

Fig. 3.9.

Control problem $(P)^{(1)}$ with robust TV regularization term: original Lena image

$R = 0.125$
$SNR = 16.2890$, $IEE = 24.9645$

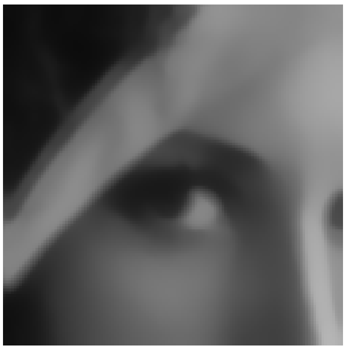

Fig. 3.10.

Fig. 3.11.

Control problem $(P)^{(1)}$ with robust TV regularization term: original Lena image

$R = 0.0625$
$SNR = 15.7875$, $IEE = 21.9830$

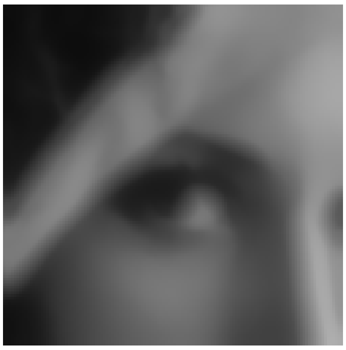

Fig. 3.12.

Fig. 3.13.

Control problem $(P)^{(1)}$ with robust TV regularization term: original Lena image

$R = 0.03125$
$SNR = 14.4561$, $IEE = 34.6860$

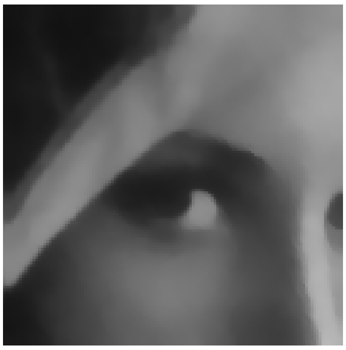

Fig. 3.14.

Fig. 3.15.

Control problem $(P)^{(1)}$ with robust TV regularization term: noisy Lena image

$R = 0.25$
$SNR = 16.7465$, $IEE = 29.0117$

Fig. 3.16.

Fig. 3.17.

Control problem $(P)^{(1)}$ with robust TV regularization term: noisy Lena image

$R = 0.125$
$SNR = 16.6978$, $IEE = 25.2298$

Fig. 3.18.

Fig. 3.19.

Control problem $(P)^{(1)}$ with robust TV regularization term: noisy Lena image

$R = 0.0625$
$SNR = 16.1393$, $IEE = 22.8745$

Fig. 3.20.

Fig. 3.21.

Control problem $(P)^{(1)}$ with robust TV regularization term: noisy Lena image

$R = 0.03125$
$SNR = 14.6684$, $IEE = 35.0349$

References.

1. [ADAMS/FOURNIER 07] Adams, R. A.; Fournier, J. J. F.: *Sobolev Spaces.* Academic Press / Elsevier; Amsterdam etc. 2007, 2. Aufl.
2. [AMBROSIO/TORTORELLI 92] Ambrosio, L.; Tortorelli, V. M.: *On the approximation of free discontinuity problems.* Boll. Un. Mat. Ital. B (7) **6** (1992), 105 – 123
3. [ANDREJEWA/KLÖTZLER 84A] Andrejewa, J. A.; Klötzler, R.: *Zur analytischen Lösung geometrischer Optimierungsaufgaben mittels Dualität bei Steuerungsproblemen. Teil I.* Z. Angew. Math. Mech. **64** (1984), 35 – 44
4. [ANDREJEWA/KLÖTZLER 84B] Andrejewa, J. A.; Klötzler, R.: *Zur analytischen Lösung geometrischer Optimierungsaufgaben mittels Dualität bei Steuerungsproblemen. Teil II.* Z. Angew. Math. Mech. **64** (1984), 147 – 153
5. [AUJOL/AUBERT/BLANC-FÉRAUD/CHAMBOLLE 05] Aujol, J.-F.; Aubert, G.; Blanc-Féraud, L.; Chambolle, A.: *Image decomposition into a bounded variation component and an oscillating component.* J. Math. Imaging Vision **22** (2005), 71 – 88
6. [AUBERT/KORNPROBST 06] Aubert, G.; Kornprobst, P.: *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations.* Springer; New York etc. 2006, 2nd ed.
7. [BELLETTINI/COSCIA 94] Bellettini, G.; Coscia, A.: *Discrete approximation of a free discontinuity problem.* Numer. Funct. Anal. Optim. **15** (1994), 201 – 224
8. [BOURDIN 99] Bourdin, B.: *Image segmentation with a finite element method.* M2AN Mathematical Modelling and Numerical Analysis **33** (1999), 229 – 244
9. [BROKATE 85] Brokate, M.: *Pontryagin's principle for control problems in age-dependent population dynamics.* J. Math. Biology **23** (1985), 75 – 101
10. [BRUNE/MAURER/WAGNER 08] Brune, C.; Maurer, H.; Wagner, M.: *Edge detection within optical flow via multidimensional control.* BTU Cottbus, Preprint-Reihe Mathematik, Preprint Nr. M-02/2008. Submitted: SIAM Journal on Imaging Sciences
11. [CATTÉ/LIONS/MOREL/COLL 92] Catté, F.; Lions, P.-L.; Morel, J.-M.; Coll, T.: *Image selective smoothing and edge detection by nonlinear diffusion.* SIAM J. Numer. Anal. **29** (1992), 182 – 193
12. [CHAMBOLLE 00] Chambolle, A.: *Mathematical problems in image processing. Inverse problems in image processing and image segmentation: some mathematical and numerical aspects.* ICTP Lecture Notes, II. Abdus Salam International Centre for Theoretical Physics, Trieste 2000 (electronic)
13. [CHAMBOLLE/LIONS 97] Chambolle, A.; Lions, P.-L.: *Image recovery via total variation minimization and related problems.* Numer. Math. **76** (1997), 167 – 188
14. [CIARLET 87] Ciarlet, P.: *The Finite Element Methods for Elliptic Problems.* North-Holland; Amsterdam - New York - Oxford - Tokyo 1987, 2nd ed.
15. [DACOROGNA 08] Dacorogna, B.: *Direct Methods in the Calculus of Variations.* Springer; New York etc. 2008, 2nd ed.
16. [DACOROGNA/MARCELLINI 97] Dacorogna, B.; Marcellini, P.: *General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial case.* Acta Mathematica **178** (1997), 1 – 37
17. [DACOROGNA/MARCELLINI 98] Dacorogna, B.; Marcellini, P.: *Cauchy-Dirichlet problem for first order nonlinear systems.* J. Funct. Anal. **152** (1998), 404 – 446
18. [DACOROGNA/MARCELLINI 99] Dacorogna, B.; Marcellini, P.: *Implicit Partial Differential Equations.* Birkhäuser; Boston - Basel - Berlin 1999
19. [DEWESS/HELBIG 95] Deweß, G.; Helbig, P.: *Einschätzen und Optimieren von Verkehrsflüssen.* In: Bachem, A.; Jünger, M.; Schrader, R. (Eds.): Mathematik in der Praxis. Fallstudien aus Industrie, Wirtschaft, Naturwissenschaften und Medizin. Springer; Berlin etc. 1995, 473 – 492
20. [EKELAND/TÉMAM 99] Ekeland, I.; Témam, R.: *Convex Analysis and Variational Problems.* SIAM; Philadelphia 1999, 2nd ed.
21. [EVANS/GARIEPY 92] Evans, L. C.; Gariepy, R. F.: *Measure Theory and Fine Properties of Functions.* CRC Press; Boca Raton etc. 1992

22. [Feichtinger/Tragler/Veliov 03] Feichtinger, G.; Tragler, G.; Veliov, V. M.: *Optimality conditions for age-structured control systems.* J. Math. Anal. Appl. **288** (2003), 47 – 68

23. [Fourer/Gay/Kernighan 02] Fourer, R.; Gay, D. M.; Kernighan, B. W.: *AMPL: A Modeling Language for Mathematical Programming.* Duxbury Press / Brooks/Cole; 2002, 2nd ed.

24. [Franek 07a] Franek, L.: *Anwendung optimaler Steuerungsprobleme mit L^∞-Steuerbeschränkung auf ein Modellproblem der Bildverarbeitung.* Diploma thesis; University of Münster 2007

25. [Franek 07b] Franek, M.: *Bildentrauschung und Kantenerkennung mit L^p-Regularisierung und Gradientenbeschränkung bei Graustufenbildern.* Diploma thesis; University of Münster 2007

26. [Funk 62] Funk, P.: *Variationsrechnung und ihre Anwendung in Physik und Technik.* Springer; Berlin - Göttingen - Heidelberg 1962 (Grundlehren 94)

27. [Gajewski/Gröger/Zacharias 74] Gajewski, H.; Gröger, K.; Zacharias, K.: *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen.* Akademie-Verlag; Berlin 1974

28. [Goering/Roos/Tobiska 93] Goering, H.; Roos, H.-G.; Tobiska, L.: *Finite-Element-Methode.* Akademie-Verlag; Berlin 1993, 3rd ed.

29. [Gousseau/Morel 01] Gousseau, Y.; Morel, J.-M.: *Are natural images of bounded variation?* SIAM J. Math. Anal. **33** (2001), 634 – 648

30. [Haber/Modersitzki 07] Haber, E.; Modersitzki, J.: *Intensity gradient based registration and fusion of multi-modal images.* Methods of Information in Medicine **46** (2007), 292 – 299

31. [Hinterberger/Hintermüller/Kunisch/Oehsen/Scherzer 03] Hinterberger, W.; Hintermüller, M.; Kunisch, K.; von Oehsen, M.; Scherzer, O.: *Tube methods for BV regularization.* J. Math. Imaging Vision **19** (2003), 219 – 235

32. [Laird/Wächter 07] Laird, C.; Wächter, A.: *Introduction to IPOPT: A tutorial for downloading, installing, and using IPOPT. Revision No. 1024.* Electronically published: http://www.coin-or.org/Ipopt/documentation/ (accessed at 29. 01. 2008)

33. [Lur'e 75] Lur'e, K. A. (Лурье, К. А.): *Оптимальное управление в задачах математической физики.* Наука; Москва 1975

34. [Maess 88] Maess, G.: *Vorlesungen über numerische Mathematik II.* Akademie-Verlag; Berlin 1988

35. [Osher/Burger/Goldfarb/Xu/Yin 05] Osher, S.; Burger, M.; Goldfarb, D.; Xu, J.; Yin, W.: *An iterative regularization method for total variation-based image restoration.* Multiscale Model. Simul. **4** (2005), 460 – 489

36. [Pickenhain/Wagner 00] Pickenhain, S.; Wagner, M.: *Critical points in relaxed deposit problems.* In: Ioffe, A.; Reich, S.; Shafrir, I. (Eds.): Calculus of Variations and Optimal Control, Technion 98, Vol. II (Research Notes in Mathematics, Vol. 411). Chapman & Hall / CRC Press; Boca Raton etc. 2000, 217 – 236

37. [Rudin/Osher/Fatemi 92] Rudin, L. I.; Osher, S.; Fatemi, E.: *Nonlinear total variation based noise removal algorithms.* Physica D **60** (1992), 259 – 268

38. [Ting 69a] Ting, T. W.: *Elastic-plastic torsion of convex cylindrical bars.* J. Math. Mech. **19** (1969), 531 – 551

39. [Ting 69b] Ting, T. W.: *Elastic-plastic torsion problem III.* Arch. Rat. Mech. Anal. **34** (1969), 228 – 244

40. [Tröltzsch 05] Tröltzsch, F.: *Optimale Steuerung partieller Differentialgleichungen — Theorie, Verfahren und Anwendungen.* Vieweg; Wiesbaden 2005

41. [Wächter/Biegler 06] Wächter, A.; Biegler, L. T.: *On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming.* Math. Program., Ser. A **106** (2006), 25 – 57

42. [Wagner 96] Wagner, M.: *Erweiterungen des mehrdimensionalen Pontrjaginschen Maximumprinzips auf meßbare und beschränkte sowie distributionelle Steuerungen.* PhD thesis; University of Leipzig 1996

43. [Wagner 99] Wagner, M.: *Erweiterungen eines Satzes von F. Hüseinov über die C^∞-Approximation von Lipschitzfunktionen.* BTU Cottbus, Preprint M-11/1999.

44. [Wagner 03] Wagner, M.: *Variationsrechnung.* Course at the BTU Cottbus in the winter term 2003/04

45. [Wagner 06] Wagner, M.: *Mehrdimensionale Steuerungsprobleme mit quasikonvexen Integranden.* Habilitation thesis; BTU Cottbus 2006

46. [Wagner 07] Wagner, M.: *Pontryagin's maximum principle for multidimensional control problems in image processing.* BTU Cottbus, Preprint-Reihe Mathematik, Preprint Nr. M-10/2007. Accepted for publication: J. Optim. Theory Appl.

47. [Weickert 96] Weickert, J.: *Anisotropic diffusion in image processing.* PhD thesis; University of Kaiserslautern 1996

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