Locally Sparse Reconstruction Using $\ell^{1,\infty}$-Norms

Applied Inverse Problem Conference 2013, Daejeon, Korea
Introduction

Variational Model

Some Computational Results

Summary & Outlook
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Summary & Outlook
Inverse Problem

Consider the discrete **time-dependent** inverse problem

\[ AZ = W \]

- \( A \in \mathbb{R}^{L \times M} \) is a discretized operator
- \( Z \in \mathbb{R}^{M \times T} \) is the unknown dynamic image
- \( W \in \mathbb{R}^{L \times T} \) is the measured data
Inverse Problem

Assume that every pixel $m$ at time step $t$ of the image $Z$ can be written as a linear combination of known basis vectors $b_t$ with coefficient vectors $u_m$, i.e.

$$z_{mt} = \sum_{n=1}^{N} u_{mn} b_{tn} \implies Z = UB^T.$$ 

Thus we obtain

$$A UB^T = W$$

with $A \in \mathbb{R}^{L \times M}$, $U \in \mathbb{R}^{M \times N}$, $B \in \mathbb{R}^{T \times N}$ and $W \in \mathbb{R}^{L \times T}$. 
Example: Dynamic Positron-Emission-Tomography

- PET is an **imaging technique** in nuclear medicine
- It visualizes the **distribution** of a weak radioactive labeled substance (**tracer**) in order to image functional processes in the body

**Figure:** J. Langner, M.Sc. Thesis, 2003
Example: Dynamic Positron-Emission-Tomography

Visualize the **perfusion** of the **cardiac muscle**

**Figure:** U.S. National Heart Lung and Blood Institute
Example: Dynamic Positron-Emission-Tomography

Visualize the **perfusion** of the cardiac muscle

Model the **tracer exchange** in the capillaries

**Figure:** U.S. National Heart Lung and Blood Institute

**Figure:** U.S. National Cancer Institute
Example: Dynamic Positron-Emission-Tomography

- obtain basis functions from a kinetic blood flow model\(^1\)
- can compute basis vectors \(b_n\) in advance
- want to pick out one \(b_n\), which fits best per pixel

There are other applications which lead to a similar problem, e.g. unmixing problems.

\(^1\) [3, Wernick & Aarsvold, Emission Tomography, 2004]

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Knowledge about the Basis

- **every pixel** should consist of **only one** or at most **very few** of the given basis vectors

- consider the operator to be **coherent**, i.e. the coherence parameter

\[
\mu := \max_{i \neq j} |\langle b_i, b_j \rangle|
\]

for \( b_i, b_j \) being distant basis vectors, is large. In other words, the **basis vectors are very similar**.

Note that orthogonalization does not help because the coefficients would not be sparse and we would lose our prior knowledge.
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A Priori Knowledge

Variational Problem

\[
\frac{1}{2} \| AUB^T - W \|_F^2 + \alpha \mathcal{R}(U) \rightarrow \min_U
\]

Prior Knowledge:
(At best) we would like to have just one (or only a few) basis vector, which fits best in the considered pixel.

\[ \Rightarrow \text{only one coefficient unequal to zero per pixel.} \]

\[ \Rightarrow \text{want to promote sparsity in every pixel} \]
Which Regularization?

Idea: $\ell^{0,\infty}$-Regularization

$$\min_U \|U\|_{0,\infty} = \min_U \left\{ \max_i \sum_{j=1}^{N} |u_{ij}|^0 \right\}$$

[1, Donoho & Elad, Optimally Sparse Representation in General (non-Orthogonal) Dictionaries via l1 Minimization, 2003]
Which Regularization?

Idea: $\ell^1,\infty$-Regularization as Relaxation

$$\min_U \|U\|_{1,\infty} = \min_U \left\{ \max_i \sum_{j=1}^N |u_{ij}| \right\}$$

$^{[1, \text{Donoho & Elad, Optimally Sparse Representation in General (non-Orthogonal) Dictionaries via l1 Minimization, 2003}]$
Which Regularization?

Idea: $\ell^{1,\infty}$-Regularization as Relaxation

$$\min_U \|U\|_{1,\infty} = \min_U \left\{ \max_i \sum_{j=1}^N |u_{ij}| \right\}$$

Variational Model

$$\min_U \frac{1}{2} \| AUB^T - W \|_F^2 + \alpha \|U\|_{\ell^{1,\infty}}$$

$^[1, \text{Donoho & Elad, Optimally Sparse Representation in General (non-Orthogonal) Dictionaries via l1 Minimization, 2003}]$
Reconstruction with Local Sparsity

Implementation

- The implementation of the $\ell^{1,\infty}$-term is not that easy
- Reformulation of the problem is necessary
- Simplified assumption: *Nonnegativity of the coefficients* (reasonable in many applications)
Reformulation

\[
\min_{U} \frac{1}{2} \|AUB^T - W\|_F^2 + \alpha \max_i \sum_{j=1}^{N} |u_{ij}|
\]
Reformulation

Add a nonnegativity constraint

\[
\min_U \frac{1}{2} \|AUB^T - W\|_F^2 + \alpha \max_i \sum_{j=1}^N u_{ij} \quad \text{s. t.} \quad u_{ij} \geq 0 \quad \forall \ i, j
\]
Reformulation

Add a nonnegativity constraint

\[
\min_U \frac{1}{2} \| AUB^T - W \|_F^2 + \alpha \max_i \sum_{j=1}^N u_{ij} \quad \text{s. t.} \quad u_{ij} \geq 0 \quad \forall i, j
\]

Maximum is a problem!
Reformulation

Add a nonnegativity constraint

\[
\min_U \frac{1}{2} \|AUB^T - W\|_F^2 + \alpha \max_i \sum_{j=1}^N u_{ij} \quad \text{s. t.} \quad u_{ij} \geq 0 \quad \forall i, j
\]

Maximum is a problem!
\(\Rightarrow\) Use equivalent formulation, i. e.

\[
\min_{U, \tilde{v}} \frac{1}{2} \|AUB^T - W\|_F^2 + \tilde{v} \quad \text{s. t.} \quad \alpha \sum_{j=1}^N u_{ij} \leq \tilde{v}, \ u_{ij} \geq 0 \quad \forall i, j
\]
Reformulation

Add a nonnegativity constraint

\[
\min_U \frac{1}{2} \| AUB^T - W \|_F^2 + \alpha \max_i \sum_{j=1}^N u_{ij} \quad \text{s. t.} \quad u_{ij} \geq 0 \quad \forall i, j
\]

Maximum is a problem!

\[\mapsto\text{Use equivalent formulation, i. e.}\]

\[
\min_{U, \tilde{v}} \frac{1}{2} \| AUB^T - W \|_F^2 + \tilde{v} \quad \text{s. t.} \quad \sum_{j=1}^N u_{ij} \leq \frac{\tilde{v}}{\alpha}, \quad u_{ij} \geq 0 \quad \forall i, j
\]

Instead of regularizing with \(\alpha\) and minimizing over \(\tilde{v}\), we can choose \(v\) in advance and thus regularize with \(v\) instead. Note that we make a systematic error and only obtain the support.

Include additional \(\ell_1, \ell_\infty\) regularization.
Reformulation

Add a nonnegativity constraint

\[
\min_U \frac{1}{2} \|AUB^T - W\|_F^2 + \alpha \max_i \sum_{j=1}^N u_{ij} \quad \text{s. t.} \quad u_{ij} \geq 0 \quad \forall i, j
\]

Maximum is a problem!

\Rightarrow \text{Use equivalent formulation, i. e.}

\[
\min_U \frac{1}{2} \|AUB^T - W\|_F^2 \quad \text{s. t.} \quad \sum_{j=1}^N u_{ij} \leq v, \ u_{ij} \geq 0 \quad \forall i, j
\]

Instead of regularizing with \(\alpha\) and minimizing over \(\tilde{v}\), we can choose \(v\) in advance and thus regularize with \(v\) instead.

Note that we make a systematic error and only obtain the support.
Reformulation

Add a nonnegativity constraint

$$\min_U \frac{1}{2} \|AUB^T - W\|_F^2 + \alpha \max_i \sum_{j=1}^N u_{ij} \quad \text{s. t.} \quad u_{ij} \geq 0 \quad \forall \ i, j$$

Maximum is a problem!

$$\Rightarrow$$ Use equivalent formulation, i.e.

$$\min_U \frac{1}{2} \|AUB^T - W\|_F^2 + \beta \|U\|_{\ell^1,1} \quad \text{s. t.} \quad \sum_{j=1}^N u_{ij} \leq \bar{v}, \ u_{ij} \geq 0 \quad \forall \ i, j$$

Instead of regularizing with $\alpha$ and minimizing over $\bar{v}$, we can choose $v$ in advance and thus regularize with $v$ instead.

Note that we make a systematic error and only obtain the support.

Include additional $\ell^{1,1}$-regularization.
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Algorithm

Splitting Approach:

\[
\min_{U, Z, D} \frac{1}{2} \|AZ - W\|_F^2 + \beta \sum_{i,j} d_{ij} \quad \text{s. t.} \quad \sum_{j=1}^{N} d_{ij} \leq v, \quad d_{ij} \geq 0 \quad \forall i
\]

\[Z = UBT, \quad D = U\]

Solve via **Alternating Direction Method of Multipliers**\(^2\) (ADMM), i.e.

- state Augmented Lagrangian
- compute optimality conditions
- solve subproblems successively

\(^2[2, \text{D. Gabay, Applications of the Method of Multipliers to Variational Inequalities}]\)
Exact Coefficients

- Use simple 3D matrix $\hat{U}$ containing the exact coefficients
- Define 2 regions where coefficients are nonzero for one basis vector
- Thus the corresponding coefficients for the most basis vectors are zero
Construction of Artificial Data

Apply $A$ and $B^T$ to the exact coefficients $\hat{U}$

we obtain the artificial “measured” data $W$ via

$$W = A\hat{U}B^T.$$ 

(For simplicity) $A$ is a 2D convolution with

$$\frac{1}{16} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 12 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which works on the pixels for every basis function.

Start the reconstruction process using $W$ as measured data.
Basic Idea

Strong regularization $\Rightarrow$ very good reconstruction of the support

**Figure:** Reconstruction with $\nu = 0.1$ and $\beta = 0.1$

Every value larger than $\nu$ is projected down to $\nu$

$\leadsto$ we are not really close to the exact data
Basic Idea

- **First run** with $\ell^1,\infty$- and $\ell^1$-regularization to obtain the support
Basic Idea

- **First run** with $\ell_1,\infty$- and $\ell_1$-regularization to obtain the support
- **Second run** without regularization *only on the known support* to reduce the distance to the exact data
Basic Idea

- **First run** with $\ell_1,\infty$- and $\ell_1$-regularization to obtain the support
- **Second run** without regularization *only on the known support* to reduce the distance to the exact data

**Figure:** Reconstruction with $\nu = 0.1$ and $\beta = 0.1$ including second run  

→ very good results
Example including Noise

Figure: Reconstruction of a $200 \times 200 \times 8$ image with $\nu = 0.01$ and $\beta = 0.1$ and standard deviation $\sigma = 0.01$
Example including Noise

More regularization

$\Rightarrow$ prior knowledge is fulfilled
Example including Noise

More regularization $\Rightarrow$ prior knowledge is fulfilled

Basis functions are very similar $\Rightarrow$ in some pixels we obtain the “wrong” basis function

Reminder:

Input Curve $C_A(t)$ and Kinetic Modeling Basis Functions $b_n(t)$

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Summary

- Similar problem in different applications (dPET, FLIM, ECG, ...)
  - use specific operator and basis functions

- Reformulated problem is easier to implement and leads to the same solution

- Use ADMM for the double splitting

- Exact recovery of the support under certain circumstances
Outlook

- Use more difficult data, i.e. more and smaller regions
- Use larger data
- Use additional problem-specific regularization in space
Thank you for your attention!

Questions?
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