More on Locally Sparse Reconstruction
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Summary & Outlook
Myocardial Infraction

- Many people suffer from coronary heart disease
- Heart attacks are life-threatening and often lead to death
- Necessity to know how well the cardiac muscle is perfused and where exactly the problems are located

Heart With Muscle Damage and a Blocked Artery

Figure: U.S. National Heart Lung and Blood Institute
Visualize the Myocardial Perfusion

- Visualize the perfusion of the cardiac muscle
- Locate damaged areas after a heart attack
- Find arteries which might become a problem to prevent an infarction

Figure: U.S. National Heart Lung and Blood Institute
Use Dynamic Positron-Emission-Tomography

PET is an imaging technique in nuclear medicine

It visualizes the distribution of a weak radioactive labeled substance (tracer) in order to image functional processes in the body.

Figure: J. Langner, M.Sc. Thesis, 2003
Inverse Problem

From PET we obtain the following discrete *time-dependent* inverse problem

\[ AZ = W \]

where

- \( A \in \mathbb{R}^{L \times M} \) is the *PET-Matrix*
- \( Z \in \mathbb{R}^{M \times T} \) is the *unknown dynamic image*, i.e. the distribution of the tracer over the time
- \( W \in \mathbb{R}^{L \times T} \) is the *measured data*
Myocardial Perfusion with Dynamic PET

- **Tracer:** $H_2^{15}O$ (short half-life $\rightarrow$ bad statistics)
- **Data:** partitioned in temporal bins
- **First approach:** generate sequence of images from the temporal bins

**Figure:** M. Benning, Improved Dynamic PET via Kinetic Models and Operator Splitting, 2010

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Problem

Due to the **bad statistics**, we obtain **bad quality images**.

We also **lose** **temporal information**.

**Figure:** Transversal Cut
Include Kinetic Modeling¹

- Model the **tracer exchange** in the capillaries
- Obtain equation for the **dynamic image**, which depends on some parameters
- **Parameters** are in a **certain range**

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¹[3, Wernick & Aarsvold, Emission Tomography, 2004]
Kinetic Modeling Basis Functions

- Choose different parameters out of the reasonable interval
- Obtain basis vectors from different parameters in the blood flow model
- Basis vectors $b_n$ are computable in advance
Inverse Problem including Dictionary

Assume that every pixel \( m \) at time step \( t \) of the image \( Z \) can be written as a **linear combination of known basis vectors** \( b_t \) with coefficient vectors \( u_m \), i.e.

\[
z_{mt} = \sum_{n=1}^{N} u_{mn} b_{tn} \quad \Rightarrow \quad Z = U B^T.
\]

Thus we obtain

\[
A U B^T = W
\]

with \( A \in \mathbb{R}^{L \times M} \), \( U \in \mathbb{R}^{M \times N} \), \( B \in \mathbb{R}^{T \times N} \) and \( W \in \mathbb{R}^{L \times T} \).
Knowledge about the Basis

- **every pixel** should consist of **only one** or at most **very few** of the given basis vectors.

- Consider the matrix $B$ to be **coherent**, i.e. the coherence parameter

$$\mu := \max_{i \neq j} |\langle b_i, b_j \rangle|$$

for $b_i, b_j$ being distant basis vectors, is large. In other words, the **basis vectors are very similar**.

Note that orthogonalization does not help because the coefficients would not be sparse and we would lose our prior knowledge.
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A Priori Knowledge

Variational Problem

\[ \frac{1}{2} \|AUB^T - W\|_F^2 + \alpha \mathcal{R}(U) \rightarrow \min_U \]

Prior Knowledge:
(At best) we would like to have just one (or only a few) basis vector, which fits best in the considered pixel.

\[ \Rightarrow \text{only one coefficient unequal to zero per pixel}. \]

\[ \Rightarrow \text{want to promote sparsity in every pixel} \]
Which Regularization?

Idea: $\ell^{0,\infty}$-Regularization

$$\min_U \|U\|_{0,\infty} = \min_U \left\{ \max_i \sum_{j=1}^N |u_{ij}|^0 \right\}$$
Which Regularization?

Idea: $\ell^{1,\infty}$-Regularization as **Relaxation**\(^1\)

$$\min_U \|U\|_{1,\infty} = \min_U \left\{ \max_i \sum_{j=1}^{N} |u_{ij}| \right\}$$

\(^1[1, \text{Donoho & Elad, Optimally Sparse Representation in General (non-Orthogonal) Dictionaries via } l_1 \text{ Minimization, 2003}]\)
Which Regularization?

Idea: $\ell^{1,\infty}$-Regularization as Relaxation

$$\min_U \|U\|_{1,\infty} = \min_U \left\{ \max_i \sum_{j=1}^N |u_{ij}| \right\}$$

Variational Model

$$\min_U \frac{1}{2} \|AUB^T - W\|_F^2 + \alpha \|U\|_{\ell^{1,\infty}}$$

$^[1, \text{Donoho & Elad, Optimally Sparse Representation in General (non-Orthogonal) Dictionaries via l1 Minimization, 2003}]$
Reconstruction with Local Sparsity

Implementation

- The implementation of the $\ell^{1,\infty}$-term and also its analysis is not that easy

- Reformulation of the problem is necessary

- Simplified assumption: *Nonnegativity of the coefficients* (reasonable in many applications)
Reformulation

\[
\min_U \frac{1}{2} \|AUB^T - W\|_F^2 + \alpha \max_i \sum_{j=1}^{N} |u_{ij}|
\]
Reformulation

Add a nonnegativity constraint

$$\min_U \frac{1}{2} \| AUB^T - W \|_F^2 + \alpha \max_i \sum_{j=1}^N u_{ij} \quad \text{s. t.} \quad u_{ij} \geq 0 \quad \forall i, j$$
Reformulation

Add a nonnegativity constraint

$$\min_U \frac{1}{2} \|AUB^T - W\|_F^2 + \alpha \max \sum_{j=1}^{N} u_{ij}$$

s. t. $$u_{ij} \geq 0 \quad \forall \, i, j$$

Maximum is a problem!
Reformulation

Add a nonnegativity constraint

$$\min_{U} \frac{1}{2} \|AUB^T - W\|_F^2 + \alpha \max_{i} \sum_{j=1}^{N} u_{ij} \quad \text{s. t.} \quad u_{ij} \geq 0 \quad \forall \ i, j$$

Maximum is a problem!

$\Rightarrow$ Use equivalent formulation, i. e.

$$\min_{U, \tilde{v}} \frac{1}{2} \|AUB^T - W\|_F^2 + \tilde{v} \quad \text{s. t.} \quad \alpha \sum_{j=1}^{N} u_{ij} \leq \tilde{v}, \ u_{ij} \geq 0 \quad \forall \ i, j$$
Reformulated Problems

Analog to

\[
\min_{U, \tilde{v}} \frac{1}{2} \|AUB^T - W\|_F^2 + \tilde{v} \quad \text{s. t.} \quad \alpha \sum_{j=1}^{N} u_{ij} \leq \tilde{v}, \quad u_{ij} \geq 0 \quad \forall \ i, j
\]

we obtain the constrained problem

\[
\min_{U, \tilde{v}} \tilde{v} \quad \text{s. t.} \quad \alpha \sum_{j=1}^{N} u_{ij} \leq \tilde{v}, \quad u_{ij} \geq 0 \quad \forall \ i, j, \ AUB^T = W
\]

on which we did some analysis about exact recovery.
Locally 1-Sparse Exact Recovery

Sufficient Condition (*Constrained Problem*)

In case

1. the exact solution has only one non-zero entry per pixel,

2. the matrix $B$ is normalized with a strict inequality for distinct basis vectors,

3. we include an additional $\ell^{1,1}$-regularization,

then we are able to reconstruct the support exactly, i.e. we only have to deal with loss of contrast.
Further Modification of the Problem

\[
\begin{align*}
\min_{U, \tilde{v}} & \quad \frac{1}{2} \|AUB^T - W\|_F^2 + \tilde{v} \\
\text{s. t.} & \quad \alpha \sum_{j=1}^{N} u_{ij} \leq \tilde{v}, \quad u_{ij} \geq 0 \quad \forall i, j
\end{align*}
\]
Further Modification of the Problem

Add $\ell^{1,1}$-regularization.

\[
\begin{align*}
\min_{U,\tilde{v}} & \quad \frac{1}{2} \|AUB^T - W\|_F^2 + \tilde{v} + \beta \|U\|_{\ell^{1,1}} \\
\text{s. t.} & \quad \alpha \sum_{j=1}^{N} u_{ij} \leq \tilde{v}, \quad u_{ij} \geq 0 \quad \forall \ i,j
\end{align*}
\]
Further Modification of the Problem

Add $\ell^{1,1}$-regularization.

The problem is still difficult to implement.

$$
\min_{U,\tilde{v}} \quad \frac{1}{2}\|AUB^T - W\|_F^2 + \tilde{v} + \beta \|U\|_{\ell^{1,1}} \quad \text{s. t.} \quad \sum_{j=1}^{N} u_{ij} \leq \frac{\tilde{v}}{\alpha}, \quad u_{ij} \geq 0 \quad \forall \ i, j
$$
Further Modification of the Problem

Add $\ell^{1,1}$-regularization.

The problem is still difficult to implement.

$$\min_U \frac{1}{2} \|AUB^T - W\|_F^2 + \beta \|U\|_{\ell^{1,1}} \quad \text{s. t.} \quad \sum_{j=1}^{N} u_{ij} \leq v, \quad u_{ij} \geq 0 \quad \forall \ i, j$$

Instead of regularizing with $\alpha$ and minimizing over $\tilde{v}$, we can choose $v$ in advance and thus regularize with $v$ instead.

Note that we make a systematic error and only obtain the support.

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Algorithm

Splitting Approach:

$$\min_{U, Z, D} \frac{1}{2} \| AZ - W \|_F^2 + \beta \sum_{i,j} d_{ij} \quad \text{s. t.} \quad \sum_{j=1}^N d_{ij} \leq v, \quad d_{ij} \geq 0 \quad \forall i$$

$Z = U B^T, \quad D = U$

Solve via **Alternating Direction Method of Multipliers**\(^2\) (ADMM), i.e.

- state Augmented Lagrangian
- compute optimality conditions
- solve subproblems successively

\(^2[2, \text{D. Gabay, Applications of the Method of Multipliers to Variational Inequalities}]\)
Exact Coefficients

- Use simple 3D matrix $\hat{U}$ containing the exact coefficients
- Define 2 regions where coefficients are nonzero for one basis vector
- Thus the corresponding coefficients for the most basis vectors are zero

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Construction of Artificial Data

Apply $A$ and $B^T$ to the exact coefficients $\hat{U}$
$\leadsto$ we obtain the artificial “measured” data $W$ via

$$W = A\hat{U}B^T.$$  

(For simplicity) $A$ is a 2D convolution with

$$\frac{1}{16} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 12 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which works on the pixels for every basis function.

Start the reconstruction process using $W$ as measured data.
Basic Idea

Strong regularization $\Rightarrow$ very good reconstruction of the support

Figure: Reconstruction with $\nu = 0.1$ and $\beta = 0.1$

Every value larger than $\nu$ is projected down to $\nu$
$
\Rightarrow$ we are not really close to the exact data
Basic Idea

► First run with $\ell^{1,\infty}$- and $\ell^{1,1}$-regularization to obtain the support
Basic Idea

- **First run** with $\ell^{1,\infty}$- and $\ell^{1,1}$-regularization to obtain the support
- **Second run** without regularization *only on the known support* to reduce the distance to the exact data
Basic Idea

- **First run** with $\ell^{1,\infty}$- and $\ell^{1,1}$-regularization to obtain the support
- **Second run** without regularization *only on the known support* to reduce the distance to the exact data

Figure: Reconstruction with $\nu = 0.1$ and $\beta = 0.1$ including second run → very good results
Example including Noise

Figure: Reconstruction of a $200 \times 200 \times 8$ image with $\nu = 0.01$ and $\beta = 0.1$ and standard deviation $\sigma = 0.01$
Example including Noise

More regularization

⇝ prior knowledge is fulfilled
Example including Noise

More regularization
\(\Rightarrow\) prior knowledge is fulfilled

Basis functions are very similar
\(\Rightarrow\) in some pixels we obtain the “wrong” basis function

Reminder:

Input Curve \(C_A(t)\) and Kinetic Modeling Basis Functions \(b_n(t)\)
How to make further improvements?

Use **additional** problem-specific **regularization in space**

\[ \min_U \frac{1}{2} \| AUB^T - W \|_F^2 + \alpha \| U \|_{\ell^1,\infty} + \beta \| U \|_{\ell^{1,1}} + \gamma \| UB^T \|_{TV} \]

\( \sim \sim \) still **work in progress**
Summary

- Similar problem in different applications (dPET, FLIM, ECG, ...) → use specific operator and basis functions

- Reformulated problem is easier to implement and leads to the same solution

- Use ADMM for the double splitting

- Exact recovery of the support under certain circumstances
Outlook

- Get TV-regularization to work properly
- Use more difficult data, i.e. more and smaller regions
- Use larger data
Thank you for your attention!
Questions?
Bibliography

D. L. Donoho and M. Elad.
Optimally sparse representation in general (non-orthogonal) dictionaries via $l_1$ minimization.


D. Gabay.
Applications of the method of multipliers to variational inequalities.


M. N. Wernick and J. N. Aarsvold, editors.

*Emission Tomography: The Fundamentals of PET and SPECT.*
Sufficient Condition for Locally 1-Sparse Exact Recovery

Consider the constrained problem again and include an additional $\ell^{1,1}$-regularization term, i.e

$$\min_{U, \tilde{v}} \beta \|U\|_{1,1} + \tilde{v} \quad \text{s.t.} \quad \alpha \sum_{j=1}^{N} u_{ij} \leq \tilde{v}, \quad u_{ij} \geq 0 \quad \forall \ i, j, \quad AUB^T = W. \quad (1)$$

Want to use the *strong* scaling condition

$$\|b_n\|_2 = 1 \quad \text{and} \quad |\langle b_n, b_m \rangle| < 1 \quad \forall \ n \neq m \quad (2)$$

as sufficient condition for exact recovery of locally 1-sparse data.
Sufficient Condition for Locally 1-Sparse Exact Recovery

Set the exact solution of (1) as before. Choose the Lagrange parameter \( \lambda \) as before, and now \( \eta \) as

\[ \eta_{lk} = b_{kj(i)} (\varphi_l + \beta) \quad \text{with} \quad A^T \varphi = \alpha \lambda. \]

Let the strong scaling condition (2) hold. Then

\[ u_{ij} = \begin{cases} \tilde{c}_i = c_i + r_i, & \text{if } j = J(i) \\ 0, & \text{if } j \neq J(i) \end{cases} \]

with \( r_i \in \mathbb{R} \) is a solution of (1).
Sufficient Condition for Locally 1-Sparse Exact Recovery

Sketch of Proof

- Set up the Lagrange-Functional
- Compute optimality and complementary conditions
- Satisfy optimality conditions
  - case-by-case analysis for \( j = J(i) \) and \( j \neq J(i) \)
  - insert Lagrange parameters
  - insert scaling condition to compute the last Lagrange parameter \( \mu_{ij} \)
  - use complementary condition to obtain the reconstruction \( u \)
Sufficient Condition for Locally 1-Sparse Exact Recovery

Sketch of Proof

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  - insert scaling condition to compute the last Lagrange parameter \( \mu_{ij} \)
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- \( \ell^{1,1} \)-regularization to exclude dividing by zero
- strict inequality to deduce that \( \mu_{ij} \neq 0 \) for one Lagrange parameter. Thus \( u_{ij} = 0 \) holds for all \( j \neq J(i) \)