Numerical solution of inverse evolution problems via the nonlinear Levitan equation

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1 Introduction

We consider a second order evolution problem

$$u_{tt}^{\varphi} = Lu^{\varphi} \quad , \quad u^{\varphi}(0) = 0 \quad , \quad u_t^{\varphi}(0) = \varphi \tag{1.1}$$

where L is a selfadjoint operator in a Hilbert space H. The problem is to recover L (i.e. all or some of its coefficients) from the observations

$$g_{\varphi,\psi}(t) = (\psi, u^{\varphi}(t)) \ , \quad 0 \leq t \leq T \ , \tag{1.2}$$

for sufficiently many φ, ψ. We shall show that we have the identity

$$(u^{\psi}(s), u^{\varphi}(t)) = G_{\varphi,\psi}(s,t) \ , \quad 0 \leq s \leq t \ , \tag{1.3}$$

$$G_{\varphi,\psi}(s,t) = \frac{1}{2} \int\limits_0^s \left(g_{\varphi,\psi}(t+s') + g_{\varphi,\psi}(t-s')\right) ds' \ .$$

In the $1D$ case and $L = \frac{d^2}{dx^2} - q(x)$, (1.3) will turn out to be Levitan's nonlinear integral equation. We shall extend this to the case $L = \frac{d}{dx}(c^2 \frac{d}{dx}) - q$ and suggest a numerical method based on the Cholesky decomposition. Finally we indicate how (1.3) can be used for the numerical solution of nD problems.

2 Derivation of the identity

Let $E_0(t)$, $E_1(t)$ be solution operators for (1.1), i.e. $v_{tt} = Lv$ with initial values $v(0) = \varphi_0$, $v_t(0) = \varphi_1$ has the solution

$$v(t) = E_0(t)\varphi_0 + E_1(t)\varphi_1 \ .$$

We extend E_0, E_1 to all of $\mathbb{R}^1$ as even and odd functions, respectively. We have $E_1' = E_0$, and $L^* = L$ implies $E_0^* = E_0$ and $E_1^* = E_1$.

Now let $0 \leq s \leq t$. Exploiting the time invariance of (1.1) we get

$$\begin{aligned} g_{\varphi,\psi}(t+s) &= (\psi, u^{\varphi}(t+s)) \\ &= (\psi, E_0(s)u^{\varphi}(t) + E_1(s)u_t^{\varphi}(t)) \\ &= (E_0(s)\psi, u^{\varphi}(t)) + (E_1(s)\psi, u_t^{\varphi}(t)) \\ &= (E_0(s)\psi, E_1(t)\varphi) + (E_1(s)\psi, E_1'(t)\varphi) \ . \end{aligned}$$

Likewise, since E_0 is even and E_1 is odd,

$$g_{\varphi,\psi}(t-s) = (E_0(s)\psi, E_1(t)\varphi) - (E_1(s)\psi, E_1'(t)\varphi) \ .$$

Adding the last two equations yields

$$\frac{1}{2}(g_{\varphi,\psi}(t+s) + g_{\varphi,\psi}(t-s)) = (E_0(s)\psi, E_1(t)\varphi) \ .$$

The identity (1.3) is now obtained by integrating with respect to s.

We remark that for a non-selfadjoint operator L (1.3) reads

$$(v^{\psi}(s), u^{\varphi}(t)) = G_{\varphi,\psi}(s,t)$$

where v^{ψ} is the solution of

$$v_{tt}^{\psi} = L^* v^{\psi} \quad , \quad v^{\psi}(0) = 0 \quad , \quad v_t^{\psi}(0) = \psi \ .$$

Thus the method of the following sections can be carried out also in the non-selfadjoint case, with the Cholesky decomposition replaced by the LU-factorization.

3 The nonlinear Levitan equation

Now consider L to be the differential operator

$$Lu = u'' - q(x)u \quad \text{in} \quad (0, \infty)$$

with boundary conditions $u'(0) = 0$ and let $\varphi = \psi = \delta$. In slightly simplified notation (1.3) reads

$$\int\limits_0^s u(s,x)u(t,x)dx = G(s,t) \ , \quad s \leq t \ , \tag{3.1}$$

where now $g(t) = u(0,t)$. The restriction of the integration to $[0,s]$ is due to causality, i.e. $u(x,t) = 0$ for $x > t$. (3.1) is clearly the well known nonlinear Levitan equation. It leads immediately to a numerical algorithm: Let

$$x_i = i\Delta x \ , \ \ t_\ell = \ell \Delta t \ \ , \ \ u_{i\ell} = u(t_\ell, x_i) \ , \ \ G_{k\ell} = \frac{1}{\Delta x} G(t_\ell, t_k) \ .$$

Let $U = (U_{i\ell})_{i,\ell=0,\dots,n}$, $G = (G_{k\ell})_{k,\ell=0,\dots,n}$. Then, (3.1) reads approximately

$$U^* U = G \ . \tag{3.2}$$

For $\Delta t = \Delta x$, U is upper triangular. From Kirchhoff's formula we get

$$u(x,t) = \frac{1}{2}(H(x+t) - H(x-t)) - \frac{1}{2} \int\limits_{\Delta(x,t)} qudx'dt'$$

with H the Heavyside function and $\Delta(x,t)$ the triangle with vertex (x,t) and base $(x-t, x+t)$ on the x-axes. It follows that $u = 1/2$ along $x = t$. Hence the diagonal of U in positive. Thus, U can be computed from (3.2) simply by a Cholesky decomposition. This leads to an approximation to the function u, which in turn provides an approximation to q by the differential equation. Of course all this is well known, see e.g. [2].

This procedure can easily be extended to the differential operator

$$Lu = (c^2(x)u')' - q(x)u \ .$$

Assume we know an upper bound $\overline{c}$ for c, i.e. $0 < c(x) \leq \overline{c}$. Then, by causality, $u(x,t) = 0$ for $|x| > \overline{c}t$. Choosing $\Delta x = \overline{c}\Delta t$ leads again to an upper triangular matrix U, and (3.2) is still valid. However, the diagonal of U can now longer be assumed to be positive, and the Cholesky process may break down. Fortunately we have the following

Lemma 3.1 *Let U be an upper triangular matrix with the property that $u_{k,\ell} = 0$ implies $u_{k+1,\ell+1} = 0$ and the non vanishing element in each column (if any) with largest row index is positive. Then U is uniquely determined by $G = U^*U$.*

Proof: Let $m_\ell = \mathrm{Max}\{k : u_{k,\ell} \neq 0\}$, $\ell = 1, \ldots, n$, and $m_\ell = 0$ if column ℓ of U is zero. Since U is upper triangular, $m_\ell \leq \ell$. Since $u_{k+1,\ell+1} = 0$ when ever $u_{k,\ell} = 0$, $m_{\ell+1} \leq m_\ell + 1$. Moreover, $u_{m_\ell,\ell} > 0$ for $\ell = 1, \ldots, n$. We show by induction on ℓ that the rows m_ℓ, $\ell = 1, \ldots, n$, of U are uniquely determined by G.

Let $\ell = 1$. If $m_\ell = 0$ there is nothing to prove. If $m_\ell = 1$ then $u_{1,1}^2 = g_{1,1}$ and $u_{1,1}$ is determined since $u_{1,1} > 0$. The rest of row m_1 is now determined by $u_{1,\ell} = g_{1,\ell}/u_{1,1}$, $\ell = 2, \ldots, n$. Thus row m_1 is determined.

Assume that rows $m_1, \ldots, m_{\ell-1}$ are already determined for some $\ell \leq n$. The equation for $g_{\ell,\ell}$ reads

$$\sum_{k=1}^{m_{\ell-1}} u_{k,\ell}^2 + \sum_{k=m_{\ell-1}+1}^{m_\ell} u_{k,\ell}^2 = g_{\ell,\ell}\ .$$

The first of these sums is known. Because of $m_\ell \leq m_{\ell-1} + 1$ the second sum is either empty (this is the case $m_\ell < m_{\ell-1} + 1$) or consists of the single term $u_{m_\ell,\ell}^2$ (case $m_\ell = m_{\ell-1} + 1$)). In the former case, $m_\ell \leq m_{\ell-1}$, hence row m_ℓ is already known. In the latter case, $u_{m_\ell,\ell}$ is uniquely determined since $u_{m_\ell,\ell} > 0$. The remaining elements in row m_ℓ are computed from the elements $g_{m_\ell,k}$ exactly as in the Cholesky algorithm.

□

The condition for the assumptions of the lemma to hold is essentially that the discretization be sufficiently fine. The solution $u(x,t)$ is continuous for $t \geq \tau(x)$ where $\tau(x)$ is the travel time of a signal travelling from 0 to x, and $u(x, \tau(x) + 0) > 0$. Since τ is continuous (provided c is), U satisfies the assumptions of the lemma if the discretization is sufficiently fine.

4 Extension to higher dimensions

We start with the simple example

$$\begin{aligned} u_{tt} &= \Delta u - q(x)u \quad \text{in} \quad \Omega \times [0,T] \\ \frac{\partial u}{\partial \nu} &= 0 \qquad\qquad\quad\ \text{in} \quad \partial\Omega \times [0,T]\ . \end{aligned} \tag{4.1}$$

The function q has to be reconstructed from the observations

$$g_{y,z}(t) = u^y(z,t) \quad , \quad y, z \in \partial\Omega\ , \qquad 0 \leq t \leq T$$

where u^y is the solution of (4.1) with $u^y = 0$, $u_t^y = \delta_y$ for $t = 0$ where δ_y is the δ-function on $\partial\Omega$ concentred at y. It is clear that this problem is a special case of (1.1), (1.2). It is not a difficult problem since it can be reduced to the Radon problem by the progressive wave expansion, see e.g. [1]. We use it only for the explanation of our method in higher dimensions.

With a slight abuse of notation, (1.3) reads in this case

$$\int\limits_{\Omega} u^z(x,s)u^y(x,t)dx = G_{z,y}(s,t)\ , \quad 0 \leq s \leq t\ , \quad z, y \in \partial\Omega\ . \tag{4.2}$$

We note in passing that (4.2) is essentially also used in [3], with the difference that in that paper z, y refer to arbitrary functions rather than δ functions on $\partial\Omega$. G is determined in that paper from the Dirichlet-to-Neuman map by solving $(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2})G = 0$, while we consider G as data.

To fix ideas let Ω be the disk with midpoint 0 and radius ρ in $\mathbb{R}^2$. Introduce the polar coordinate grid

$$x_{i\nu} = r_i \begin{pmatrix} \cos\theta_\nu \\ \sin\theta_\nu \end{pmatrix} \quad , \qquad r_i = (n-i)\Delta r \ \ , \ \ \theta_\nu = \nu\Delta\theta\ ,$$

$$\Delta r = \frac{\rho}{n}\ , \ \ \Delta\theta = \frac{2\pi}{q}\ , \quad i = 0, \ldots, n-1\ , \ \ \nu = 0, \ldots, q-1\ .$$

Define the $q \times q$ matrices $U_{i\ell}$ by

$$(U_{i\ell})_{\mu\nu} = u^{x_{0\nu}}(x_{i\mu}, t_\ell)\sqrt{r_i \Delta r \Delta\theta}$$

where $t_\ell = \ell\Delta t,\ \ \ell = 0, \dots, n-1$ and put $U = (U_{i\ell})_{i,\ell=0,\dots,n-1}$. U is a (nq, nq) matrix, arranged as a (n,n)-matrix of (q,q) blocks. The same applies to the matrix

$$G = (G_{k\ell})_{k,\ell=0,\dots,n-1}\ ,\quad (G_{k\ell})_{\nu\mu} = G_{x_{0\nu},x_{0\mu}}(t_\ell, t_k)\ .$$

Applying a quadrature rule to (4.2) we obtain

$$\Delta r \Delta\theta \sum_{i=0}^{n-1} \sum_{\lambda=0}^{q-1} r_i u^{x_{0\mu}}(x_{i\lambda}, t_\ell) u^{x_{0\nu}}(x_{i\lambda}, t_k) = G_{x_{0\mu},x_{0\nu}}(t_\ell, t_k)\ .$$

In terms of the quantities just introduced this reads

$$\sum_{i=0}^{n-1} \sum_{\lambda=0}^{q-1} (U_{i\ell})_{\lambda\mu} (U_{ik})_{\lambda\nu} = (G_{k\ell})_{\nu\mu}$$

or

$$\sum_{i=0}^{n-1}\ U_{i\ell}^* U_{ik} = G_{k\ell}$$

or

$$U^* U = G\ . \tag{4.3}$$

By causality, $u^y(x,t) = 0$ for $|y - x| > t$. Hence $U_{i\ell} = 0$ for $i > \ell$ provided $\Delta r = \Delta t$. In this case, $U_{\ell\ell}$ is a diagonal matrix, again by causality. Hence (4.3) is simply a block Cholesky decomposition of G which determines U uniquely.

The extension to differential equations of the type

$$u_{tt} = \operatorname{div}(c^2 \nabla u) - q(x) u$$

is done exactly in the same way as in the $1D$ case. Putting $\Delta r = \bar{c}\Delta t$ yields $U_{i\ell} = 0$ for $i > \ell$, hence U is block upper triangular. However it is not clear what the analogue of the lemma is. Before this problem is dealt with it seems advisable to get numerical experience with the $1D$ case first.

We did not discuss the determination of c, q from the differential equation once u is determined. The exact extent to which this is possible is not yet clear.

5 First order evolutions

Of course the same can be done for first order evolutions

$$u_t^{\varphi} = Lu^{\varphi} \quad , \quad u^{\varphi}(0) = \varphi \; . \tag{5.1}$$

With $g_{\varphi,\psi}$ as in (1.2) we readily see that the identity (1.3) now reads

$$(u^{\psi}(s), u^{\varphi}(t)) = g_{\varphi,\psi}(s+t) \; . \tag{5.2}$$

The identity (5.2) is now used exactly in the same way as (1.3) to solve inverse problems for (5.1). However, since inverse first order evolution problems are notoriously unstable, the practical usefulness of (5.2) is probably very limited.

References

[1] Rose, J.H. - Cheney, M. - DeFacio, B.: The connection between time - and frequency domain three-dimensional inverse scattering methods. *J. Math. Phys.* **25**, 2995-3000 (1984).

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