An error bound for the Born approximation

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Abstract *We derive an explicit error bound for the Born approximation for the inverse scattering problem of the Helmholtz equation at fixed frequency.*

1 Introduction

We consider the scattering problem

$$\begin{aligned} \Delta u + k^2(1+f)u = 0 \quad \text{in} \quad \mathbb{R}^n \\ u = u_i + u_s \end{aligned} \tag{1.1}$$

where $u_i = e^{ikx\cdot\theta}$ is the incoming plane wave with wave number $k > 0$ and direction of propagation $\theta \in S^{n-1}$ and f is a compactly supported function, $f(x) = 0$ for $|x| > \rho$. u_s is the scattered wave that fulfills the Sommerfeld radiation condition at infinity. It satisfies the equation

$$\Delta u_s + k^2 u_s = -k^2 f u \ . \tag{1.2}$$

The Born approximation u_B to u_s is obtained by replacing u on the right hand side of (1.2) by u_i, obtaining

$$\Delta u_B + k^2 u_B = -k^2 f u_i \ .$$

Making use of the Green's function, i.e.

$$G_k(x) = \frac{i}{4} H_0(k|x|) \qquad (n=2)$$
$$G_k(x) = \frac{e^{ik|x|}}{4\pi|x|} \qquad (n=3)$$

we find for u_s, u_B

$$u_s(x) = k^2 \int_{|y|<\rho} G_k(x-y) f(y) u(y) dy \;, \tag{1.3}$$

$$u_B(x) = k^2 \int_{|y|<\rho} G_k(x-y) f(y) u_i(y) dy \;. \tag{1.4}$$

We also consider the inverse problem for (1.1). This problem calls for the determination of f from the function u_s on the hyperplanes $x \cdot \theta = r$, $|r| > \rho$, for a sufficiently large class of directions θ. Again the Born approximation f_B to f is obtained by replacing u in (1.2) by u_i, i.e.

$$\Delta u_s + k^2 u_s = -k^2 f_B u_i \;,$$

and again we have an integral equation

$$u_s(x) = k^2 \int_{|y|<\rho} G_k(x-y) f_B(y) u_i(y) dy \tag{1.5}$$

on $x \cdot \theta = r$.

In the present paper we give explicit estimates for $\|u - u_B\|$ and $\|f - f_B\|$ in the norm of $L_2(|x| < \rho)$. The former estimate follows immediately from estimates of the norm of the operator $\mathcal{G}_k$ with kernel $G_k(x-y)$ in $L_2(|x| < \rho)$. The latter estimate follows by a well known explicit formula (see (3.2)) for f_B.

The literature on the Born approximation is enormous. We give only a few hints. For the fundamentals see [3], [10], [12], [14]. The accuracy of the Born approximation has been studied in many papers, starting with heuristics [9], analytically solvable cases [13], and the interpretation of the Born as single scattering approximation [6]. Numerical studies have been done in [8], [7] and [4]. All these papers find that for the Born approximation to be valid one needs a condition of the form

$$k\rho \sup_{|x|<\rho} |f(x)| < 2\pi c \tag{1.6}$$

where c is a “small” constant. For instance, $c = 0.16$ in formula (24) of [4], $c = 0.25$ in formula (24) of [15]. Condition (1.6) is made plausible in [8]: The left hand side of (1.6) is a rough estimate for the phase shift of the total field u generated by the object f. This phase shift has to be much smaller than the wavelength $\lambda = 2\pi/k$ for the assumption underlying the Born approximation, namely the similarity of u and u_i, to be valid. Our condition for the Born approximation to hold (see Theorem 4) is of the same type as (1.6), but with a constant c that is based on an exact mathematical theory and with an explicit estimate of the ensuing error.

2 Estimate of operator norms

The goal of this section is an estimate for the norm of the operator

$$(\mathcal{G}_k u)(x) = \int\limits_{|y|<\rho} G_k(x-y)u(y)dy$$

in $L_2(|x| < \rho)$.

Theorem 2.1 *There exists a function* $\gamma_n : (0,\infty) \to (0,\infty)$ *such that*

$$\|\mathcal{G}_k\|_{L_2(|x|<\rho)} = \gamma_n(k\rho)\ \frac{\rho}{k}$$

and $\overline{\gamma}_n = \sup\limits_{(0,\infty)} \gamma_n < \infty$.

Proof: It suffices to prove the theorem for $\rho = 1$. The general case follows from the relations

$$\begin{aligned} G_k(\rho x) &= \rho^{2-n} G_{k\rho}(x)\ , \\ \|\mathcal{G}_k\|_{L_2(|x|<\rho)} &= \rho^2 \|\mathcal{G}_{\rho k}\|_{L_2(|x|<1)}\ . \end{aligned}$$

For $\rho = 1$ the result is a special case of [2]. In that paper, the operator

$$(Tu)(x) = \int G_k(x-y)p(x,y)u(y)dy$$

is considered, with $p \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and it is shown that there exists a positive constant $c_n < \infty$ such that

$$\|T\|_{L_2(\mathrm{R}^n)} \leq c_n/k$$

(Theorem 2.1 of [2]).

□

In principle it is possible to get explicit upper bounds for γ_n by making the constructive proofs in [2] or [1] more explicit. However, it is much easier to compute γ_n numerically. We simply discretized $\mathcal{G}_k$ in $|x| < 1$ and computed the largest eigenvalue λ_k of $\mathcal{G}_k^* \mathcal{G}_k$. Obviously,

$$\|\mathcal{G}_k\|_{L_2(|x|<1)} = \sqrt{\lambda_k} = \gamma_n(k)/k \ .$$

For $n = 2$ we obtained by the power method the following values:

k	$\gamma_2(k)$
1	0.63
1.5	0.71
2	0.74
2.5	0.72
3	0.68
4	0.64
8	0.63
16	0.63

Thus a numerically safe value for the number $\overline{\gamma}_2$ is 0.8.

3 Error estimates

In the light of Theorem 2.1, the error bound for u_B is almost obvious.

Theorem 3.1 *Let*

$$q = \overline{\gamma}_n \rho k \sup_{|x|<\rho} |f(x)| < 1 \ .$$

Then,

$$\begin{aligned} \|u_B\| &\leq \overline{\gamma}_n \rho k \|f\| \ , \\ \|u_s\| &\leq \frac{\overline{\gamma}_n \rho k}{1-q} \|f\| \ , \\ \|u_s - u_B\| &\leq \frac{q}{1-q} \, \overline{\gamma}_n \ \rho k \|f\| \ . \end{aligned}$$

Proof: The estimates for u_B, u_s follow immediately from (1.4), (1.3). Subtracting (1.3), (1.4) yields

$$(u_s - u_B)(x) = k^2 \int\limits_{|y|<\rho} G_k(x-y) f(y) u_s(y) dy$$

hence

$$\begin{aligned} \|u_s - u_B\| &\leq k^2 \|\mathcal{G}_k\|_{L_2(|x|<\rho)} \sup_{|y|<\rho} |f(y)| \|u_s\| \\ &= q \|u_s\| \ . \end{aligned}$$

□

Note that the estimate on u_s implies

$$\|u_s - u_B\| \leq (V_n \rho^n)^{1/2} \ \frac{q^2}{1-q}$$

with V_n the volume of the unit ball in $\mathbb{R}^n$, provided that $q < 1$, and the estimate is of second order in q. This corroborates the results quoted in section 1, claiming that $k\rho \sup |f(x)|$ is the decisive quantity for the Born approximation to be valid.

Now let $|r| > \rho$ and let $U_\theta : L_2(|x| < \rho) \to L_2(\theta^\perp)$, $\theta^\perp$ the hyperplane perpendicular to θ, be the propagation operator

$$(U_\theta f)(z) = \int\limits_{|y|<\rho} G_k(r\theta + z - y) f(y) u_i(y) dy \ ;$$

see Devaney [5]. For the $(n-1)$-dimensional Fourier transform $(U_\theta f)^\wedge$ in $\theta^\perp$ we have with $\zeta \in \theta^\perp$

$$(U_\theta f)^\wedge(\zeta) = i\sqrt{\frac{\pi}{2}}\, e^{i|r|a(\zeta)}\ \frac{1}{a(\zeta)}\hat{f}((\varepsilon a(\zeta)-k)\theta+\zeta) \tag{3.1}$$

where $\varepsilon = sgn(r)$ and $a(\zeta) = \sqrt{k^2-\zeta^2}$. This is, with a slight change of notation, Theorem 3.1 in [11]. Combining (3.1) with (1.5) we see that f_B satisfies

$$\begin{aligned} \hat{f}_B((\varepsilon a(\zeta)-k)\theta+\zeta) &= -i\sqrt{\frac{2}{\pi}}\, e^{-i|r|a(\zeta)}a(\zeta)\hat{g}_\theta(\zeta)\ , \\ g_\theta(z) &= u_s(r\theta+z)\ . \end{aligned} \tag{3.2}$$

For $|\zeta| \leq k$ this determines $\hat{f}_3$ in the ball of radius $2k$ around 0.

Theorem 3.2 *For $\zeta \in \theta^\perp$, $\varepsilon = \pm 1$ we have*

$$(f_B - f)^\wedge((\varepsilon a(\zeta)-k)\theta+\zeta) = (u_s f)^\wedge(\varepsilon a(\zeta)\theta+\zeta)\ .$$

Proof: With the help of the propagation operator we get from (1.3), (1.5)

$$U_\theta f_{\mathrm{B}} = U_\theta\left(\frac{u}{u_i}f\right)\ .$$

Subtracting $U_\theta f$ yields

$$U_\theta(f_B - f) = U_\theta\left(\left(\frac{u}{u_i}-1\right)f\right) = U_\theta\left(\frac{u_s}{u_i}f\right)\ .$$

From (3.1) we get

$$(f_B - f)^\wedge((\varepsilon a(\zeta)-k)\theta+\zeta) = \left(\frac{u_s}{u_i}f\right)^\wedge((\varepsilon a(\zeta)-k)\theta+\zeta)\ .$$

Since

$$\left(\frac{u_s}{u_i}f\right)^\wedge(\xi) = \left(u_s f e^{-ik\theta\cdot x}\right)^\wedge(\xi) = (u_s f)^\wedge(\xi+k\theta)$$

we obtain the result.

□

As ζ varies in $\theta^\perp$ and θ in S^{n-1}, $(\varepsilon a(\zeta) - k)\theta + \zeta)$ runs through the ball of radius $2k$. Hence Theorem 3.2 gives information in Fourier space only up to spatial frequency $2k$. Therefore we introduce the function f_{2k} by

$$\hat{f}_{2k}(\xi) = \begin{cases} \hat{f}(\xi) & , \quad |\xi| \leq 2k \ , \\ 0 & , \quad \text{otherwise} \ , \end{cases}$$

i.e. f_{2k} is the low-pass filtered version of f with cut-off frequency $2k$.

Theorem 3.3 *Let* $q < 1$. *Then,*

$$\|f_B - f_{2k}\| \leq \pi^{-n/2} V_n^{1/2} \ \frac{\overline{\gamma}_n \rho}{1-q} \ k^{1+n/2} \|f\|^2 \ .$$

Proof: We have

$$\begin{aligned} (u_s f)^\wedge(\xi) &= (2\pi)^{-n/2} \int \hat{u}_s(\xi - \eta)\hat{f}(\eta) d\eta \\ &= (2\pi)^{-n/2} \int \left(e^{-i\xi \cdot x} u_s\right)^\sim (\eta)\hat{f}(\eta) d\eta \ . \end{aligned}$$

Using Parseval's relation we obtain

$$(u_s f)^\wedge(\xi) = (2\pi)^{-n/2} \int e^{-i\xi \cdot x} u_s(x) f(x) dx \ ,$$

hence

$$\|(u_s f)^\wedge(\xi)\| \leq (2\pi)^{-n/2} \|u_s\| \|f\| \ .$$

From Theorem 3.1 we obtain (remember that u_s depends on θ)

$$\begin{aligned} \sup_{|\xi| \leq 2k} |(f_B - f)^\wedge(\xi)| &\leq \sup_\theta \sup_{|\xi| = k} |(u_s f)^\wedge|(\xi) \\ &\leq (2\pi)^{-n/2} \sup_\theta \|u_s\| \|f\| \\ &\leq (2\pi)^{-n/2} \ \frac{\overline{\gamma}_n \rho^n k}{1-q} \ \|f\|^2 \ . \end{aligned}$$

It follows that

$$\|f_B - f_{2k}\| \leq \left(\int\limits_{|\xi|\leq 2k} |(f_B - f)^\wedge(\xi)|^2 d\xi \right)^{1/2} \leq V_n^{1/2}(2k)^{n/2}(2\pi)^{-n/2} \frac{\overline{\gamma}_n \rho^n k}{1-q} \|f\|^2 .$$

□

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