Range Conditions in Tomography

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Let $g = Rf$ be the 2D Radon transform of a function f supported in $|x| < 1$. The Helgason-Ludwig consistency conditions state that for $\theta \in S^1$

$$\int_{-1}^{+1} s^m g(\theta, s) ds = p_m(\theta) \ , \qquad m = 0, 1, \ldots \tag{1}$$

where p_m is a polynomial of degree m; see e.g. Natterer (1986). It is tempting to use (1) to force consistency on the data, in the hope of improving the reconstruction of f from g if g is corrupted by noise or modelling errors. The purpose of the note is to show that this is useless, at least for removing noise.

We decompose $L_2(S^1 \times [-1, +1]) = H_c \oplus H_i$,

$$\begin{aligned} H_c &= \text{span}\ \{wT_m(s)e^{ik\varphi} : |k| \le m\} \ , \\ H_i &= \text{span}\ \{wT_m(s)e^{ik\varphi} : |k| > m\} \end{aligned}$$

where T_m is the first kind Chebyshev polynomial of degree m, $w = (1-s^2)^{-1/2}$ and $\theta = (\cos\varphi, \sin\varphi)^T$. H_c is the space of consistent functions which satisfy (1). We apply the Radon inversion formula

$$f = \frac{1}{4\pi} R^* \ \frac{\partial}{\partial s} H g$$

to a function $g \in H_i$, e.g.

$$g = wT_m e^{ik\varphi} \ , \qquad |k| > m \ .$$

We have (see Abramowitz - Stegun [1], formula (22.13.3))

$$HwT_m(s) = U_{m-1}(s) = \frac{\sin ms}{m \sin s} \ , \qquad |s| \le 1$$

where T_m has been restricted to $[-1, +1]$. Thus we get

$$f(x) = \frac{1}{4\pi} \int_0^{2\pi} \frac{d}{ds} \ \frac{\sin ms}{m \sin s}\Big|_{s = x\cdot\theta} \ e^{ik\varphi} d\varphi$$

$$= \frac{1}{4\pi m} \int_0^{2\pi} \frac{d}{ds} \left(e^{i(m-1)s} + e^{i(m-3)s} + \cdots + e^{-i(m-1)s} \right)\Big|_{s=x\cdot\theta} e^{ik\varphi} d\varphi$$

$$= \frac{i}{4\pi m} \sum_{\ell=1-m}^{m-1}{}' \ \ell \int_0^{2\pi} e^{i\ell x\cdot\theta + ik\varphi} d\varphi$$

where the dash indicates that ℓ has the same parity as $m-1$. Using the integral representation of the Bessel function J_k of first order (see Abramowitz-Stegun [1], formula (9.1.21)),

$$J_k(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\cos\varphi - ik\varphi} d\varphi$$

we obtain

$$f(x) = \frac{i}{2m} \sum_{\ell=1-m}^{m-1}{}' \ \ell J_k(\ell|x|) e^{-i\varphi\psi} \tag{2}$$

where ψ is the argument of x. Debye's asymptotic relation for the Bessel functions states that $|J_k(z)|$ is negligibly small for $|z| < |k|$ and k large. Since $|\ell| < m < |k|$ in (2), $f(x)$ is negligible for $|x| \leq 1$ and k large. Thus inconsistent parts of g are mapped onto small functions by Radon's inversion formula.

We conclude that forcing consistency on the data by removing the inconsistent part in H_i prior to applying Radon's inversion formula has not much influence on the reconstruction. Only very low frequency inconsistencies (i.e. those for which $|k|$ is too small for Debye's asymptotic relation to be valid ($k \leq 5$, say)) can be removed in this way. Since noise is a high frequency phenomenon it can not be removed by forcing consistency on the data.

References

[1] M. Abramowitz and I.A. Stegun (1970), *Handbook of Mathematical Functions.* Dover.

[2] F. Natterer (1986), *The Mathematics of Computerized Tomography.* Wiley-Teubner.