The Empirical Cross Gramian for Parametrized Nonlinear Systems ⋆

Christian Himpe * **Mario Ohlberger** **

* *Westfälische Wilhelms Universtät, Münster, 48149 Germany (e-mail: christian.himpe@uni-muenster.de).*
** *Westfälische Wilhelms Universtät, Münster, 48149 Germany (e-mail: mario.ohlberger@uni-muenster.de)*

Abstract: The cross gramian matrix can be used for model order reduction as well as system identification of linear control systems, which are frequently used in the sciences. The empirical cross gramian is solely computed from trajectories and hence extends beyond linear state-space systems to nonlinear systems. In this work the applicability of the empirical cross gramian also for parametrized systems is demonstrated and assessed using a nonlinear benchmark problem.

Keywords: Model Reduction, Controllability, Observability, Nonlinear Systems, System Identification, Cross Gramian

1. INTRODUCTION

Large-scale parametrized state-space systems such as models of neuronal networks often require model order reduction to accelerate the evaluation, especially in a multi-query setting when varying parameters for example during optimization.

For a square linear control system, a linear control system with the same number of inputs and outputs,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned}$$

$\dim(u) = \dim(y)$, the cross gramian matrix, reviewed in Antoulas (2005), enables the computation of a reduced order model, based on the systems controllability and observability.

The system gramians, the controllability gramian and observability gramian, used in balanced truncation, and the cross gramian classically exploit the linear control system structure for computation. The empirical gramians Lall et al. (1999) extend this concept to nonlinear systems,

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)),\\ y(t) &= g(x(t), u(t)).\end{aligned}$$

In this work, for parametrized square nonlinear systems:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t), \theta),\\ y(t) &= g(x(t), u(t), \theta),\end{aligned}$$

the empirical variant of the cross gramian Himpe and Ohlberger (2014a) is demonstrated to compute a reduced order model that is valid over a given parameter space, around a steady state $\bar{x}$, exemplary for a benchmark model.

⋆ This work was supported by the Deutsche Forschungsgemeinschaft, DFG EXC 1003 Cells in Motion Cluster of Excellence, Münster, Germany, as well as the Center for Developing Mathematics in Interaction, DEMAIN, Münster, Germany.

2. CROSS GRAMIAN

The cross gramian W_X is defined as the product of controllability operator $\mathcal{C}(u) := \int_0^\infty e^{At} Bu(t)\,\mathrm{d}t$ and observability operator $\mathcal{O}(x_0) := Ce^{At}x_0$,

$$W_X := \mathcal{C} \circ \mathcal{O} = \int_0^\infty e^{At} BCe^{At}\,\mathrm{d}t,$$

which is classically computed as the solution of a Sylvester equation: $AW_X + W_X A = -BC$. If the underlying system is symmetric: $\mathcal{OC} = (\mathcal{OC})^*$, then the following property relates the cross gramian to the controllability gramian $W_C := \mathcal{CC}^*$ and observability gramian $W_O := \mathcal{O}^*\mathcal{O}$,

$$\begin{aligned}W_X^2 &= \mathcal{COCO}\\ &= \mathcal{CC}^*\mathcal{O}^*\mathcal{O} = W_C W_O \\ \Rightarrow |\lambda(W_X)| &= \sqrt{\lambda(W_C W_O)}.\end{aligned} \tag{1}$$

The singular value decomposition of the cross gramian then yields a projection which can be truncated to generate a reduced order model similar to balanced truncation.

3. EMPIRICAL CROSS GRAMIAN

The empirical cross gramian from Himpe and Ohlberger (2014a) is an empirical gramian based on computing trajectories for perturbed input $u \in Q_U$ and initial states $x_0 \in Q_X$. Additionally for parametrized systems, trajectories for perturbed parameters $\theta \in Q_\Theta$ can be computed:

$$\widehat{W}_X := \frac{1}{|Q_U||Q_X||Q_\Theta|} \sum_{h=1}^{|Q_U|} \sum_{i=1}^{|Q_X|} \sum_{j=1}^{|Q_\Theta|} \int_0^\infty \Psi^{hij}(t)\,\mathrm{d}t,$$

$$\Psi_{k,i}^{hij}(t) = (x_k^{hj}(t) - \bar{x}_k)(y_h^{ij}(t) - \bar{y}_h),$$

with $x_k^{hj}(t)$ being the k-th component of the state trajectory with perturbations from Q_U and Q_Θ around the steady state $\bar{x}$, and $y_h^{ij}(t)$ being the h-th component of the output trajectory with perturbations from Q_X and

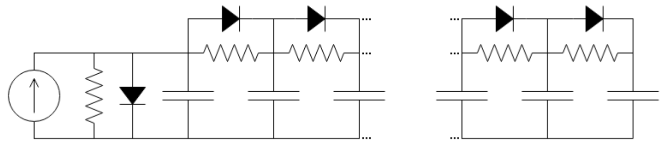

Fig. 1. Nonlinear resistor capacitor network [1].

Q_Θ with steady state output $\bar{y}$. The number of perturbations $|Q_U|, |Q_X|, |Q_\Theta|$ depends on the operating region of the system. For linear systems the (linear) cross gramian equals the empirical cross gramian as shown in Himpe and Ohlberger (2014a), hence we use $\widehat{W}_X = W_X$. But, since the computation of the empirical cross gramian requires only simulations of the underlying system, this approach also extends to nonlinear systems. The (empirical) cross gramian is only applicable to square systems and exhibits its core property (1) for symmetric systems. The nonlinear extension of linear symmetric systems are gradient systems Scherpen and van der Schaft (2011), and only for gradient systems this empirical cross gramian is expected to yield workable results.

4. NUMERICAL RESULTS

The model reduction capabilities of the empirical cross gramian for parametrized systems are tested on a nonlinear benchmark problem listed at (MORwiki community, 2015, Nonlinear RC Ladder); a resistor-capacitor cascade with nonlinear resistors (see also the circuit schematics in Figure 1):

$$\dot{x}(t) = \begin{pmatrix} -g(x_1(t)) - g(x_1(t) - x_2(t)) + u(t) \\ g(x_1(t) - x_2(t)) - g(x_2(t) - x_3(t)) \\ \vdots \\ g(x_{k-1}(t) - x_k(t)) - g(x_k(t) - x_{x+1}(t)) \\ \vdots \\ g(x_{N-1}(t) - x_N(t)) \end{pmatrix}, \qquad (2)$$

$$y(t) = x_1(t),$$

with $g_\theta : \mathbb{R} \to \mathbb{R}$ given by:

$$g_\theta(x) = \exp(10x) + \theta x - 1.$$

The linear part of the resistor network is parametrized, with θ_i describing the i-th resistor's linear resistance value. Since (2) is a SISO system, the empirical cross gramian applies here, because all SISO system are gradient systems as noted in Scherpen and van der Schaft (2011).

For the following experiments [2], the empirical cross gramian (WX) is computed using the `empirical gramian framework` from Himpe and Ohlberger (2013); Himpe (2015) and compared to balanced truncation (BT) of the empirical controllability gramian and empirical observability gramian introduced in Lall et al. (1999). The number of states is set to $\dim(x) = 1000$, the input signal is selected as $u(t) = e^{-t}$ and the test parameters are drawn from a uniform distribution $P(\theta) = U(\frac{1}{2}, \frac{3}{2})$. Figure 2 shows, while both methods require nearly the same offline time to assemble the reduced order model, that the output error of the empirical cross gramian's reduced order model, is about five orders of magnitude below the output error of the reduced model computed by balanced truncation.

[1] Adapted from "Nonlinear RC Ladder", MORwiki `http://modelreduction.org/index.php/Nonlinear_RC_Ladder`

[2] The companion code to reproduce these results can be found at `http://www.runmycode.org/companion/view/1084`

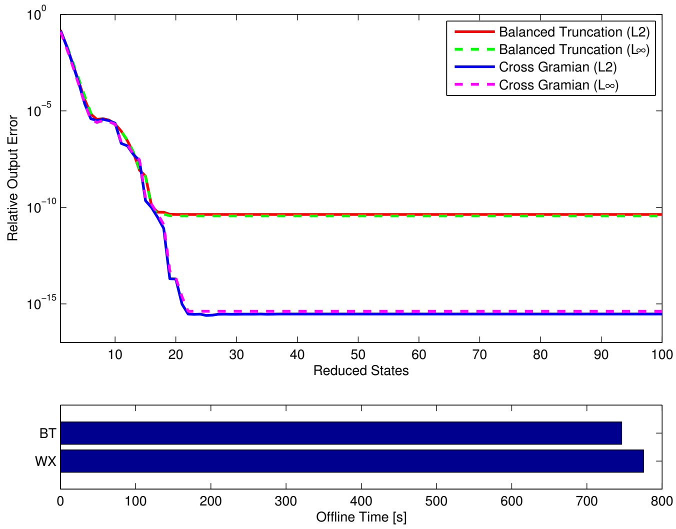

Fig. 2. Relative L_2 and L_∞ output errors for the first hundred reduced states and offline time required to compute the associated gramians.

5. REMARKS AND OUTLOOK

As demonstrated, the empirical cross gramian can be applied to parametrized nonlinear systems with an improved performance over balanced truncation. Future extensions will include expanding the scope of the empirical cross gramian beyond symmetric linear systems as in Himpe and Ohlberger (2014b), or gradient nonlinear systems to more general (non-symmetric) configurations Himpe and Ohlberger (2015).

REFERENCES

Antoulas, A. (2005). *Approximation of large-scale dynamical systems*, volume 6. SIAM.

Himpe, C. (2015). emgr - Empirical Gramian Framework. `http://gramian.de`.

Himpe, C. and Ohlberger, M. (2013). A Unified Software Framework for Empirical Gramians. *Journal of Mathematics*, 2013, 1–6.

Himpe, C. and Ohlberger, M. (2014a). Cross-Gramian Based Combined State and Parameter Reduction for Large-Scale Control Systems. *Mathematical Problems in Engineering*, 2014, 1–13.

Himpe, C. and Ohlberger, M. (2014b). Model Reduction for Complex Hyperbolic Networks. In *Proceedings of the European Control Conference*, 2739–2743.

Himpe, C. and Ohlberger, M. (2015). A Note on the Cross Gramian for Non-Symmetric Systems. *Submitted. Preprint at arXiv*, math.OC(1501.05519), 1–6.

Lall, S., Marsden, J., and Glavaski, S. (1999). Empirical model reduction of controlled nonlinear systems. In *Proceedings of the 14th IFAC Congress*, volume F, 473–478.

MORwiki community (2015). Model Order Reduction Wiki. `http://modelreduction.org`.

Scherpen, J. and van der Schaft, A. (2011). Balanced model reduction of gradient systems. *Information and Software Technology*, 12745–12750.