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The Versatile Cross Gramian

for System Theoretic Model Reduction and More

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Abstract:
For input-output systems, the cross gramian matrix encodes controllability and observability information into a single matrix, which are essential to system-theoretic applications. This system gramian can be used, in example, for model order reduction, sensitivity analysis, system identification, decentralized control and parameter identification. Beyond linear symmetric systems, the cross gramian is also available for parametric, non-symmetric, non-square and nonlinear systems.

Control System:
$\dot{x}(t) = f(x(t), u(t), \theta)$
$y(t) = g(x(t), u(t), \theta)$

Linear System:
$\dot{x}(t) = Ax(t) + Bu(t)$
$y(t) = Cx(t)$

Definition:
The cross gramian matrix W_X is defined as composition of the controllability operator $\mathcal{C}$ with the observability operator $\mathcal{O}$:

$$W_X := \mathcal{C}\mathcal{O} = \int_0^\infty e^{At} B C e^{At} \mathrm{d}t.$$

Cross Gramian

Classic Computation:
The cross gramian of a square linear system can be computed as the solution to a sylvester matrix equation:

$$AW_X + W_X A = -BC.$$

Empirical Computation:
The cross gramian of a general control system can be locally computed as product of state and adjoint or output impulse responses:

$$W_X = XY^{<2,3,1>}.$$

Model Reduction

Symmetric System:
A system is symmetric if its transfer function $G(s)$ is symmetric:
$$G(s) = G(s)^T.$$

Core Property:
For a symmetric system, the following relation between the system gramians holds:
$$W_X^2 = W_C W_O.$$

Hankel Singular Values:
The Hankel singular values relate to symmetric systems' cross gramian by:
$$|\lambda_i(W_X)| = \sqrt{\lambda_i(W_C W_O)}.$$

Model Reduction:
An approximate balancing projection is given by the left singular vectors of a TSVD of W_X:
$$U = \mathrm{pod}(W_X).$$

Sensitivity Analysis

Hankel Operator:
The Hankel operator H is defined as the composition of $\mathcal{O}$ with $\mathcal{C}$:
$$H := \mathcal{O}\mathcal{C}.$$

Trace:
The trace of the Hankel operator is equal to the trace of the cross gramian:
$$\mathrm{tr}(H) = \mathrm{tr}(W_X).$$

System Gain:
The system gain equals twice the transfer function at zero frequency:
$$\mathrm{tr}(W_X) = -\tfrac{1}{2}\mathrm{tr}(CA^{-1}B).$$

Sensitivity Analysis:
Treating parameters as inputs, the gain can be used sensitivity measure:
$$\tfrac{\mathrm{d}y}{\mathrm{d}u} = 2\,\mathrm{tr}(W_X).$$

System Identification

State-Space Symmteric:
For state-space symmetric systems all system gramians are equal:
$$W_C = W_O = W_X.$$

Minimality:
If the cross gramian has full rank, then the system is controllable, observable and thus of minimal order.

Cauchy-Index:
The signature of the cross gramian matrix is equal to the systems Cauchy-index:
$$\mathrm{sign}(W_X) = I(G(s))$$

Singularity-Index:
The rank of the controllability operator of the virtual system (W_X, B, C) equals the singularity index.

Decentralized Control

Decentralized Control:
Decomposition of a MIMO system into a set of SISO systems exposing the dominant input-output relations.

Super Position:
For a square MIMO system, W_X can be written as sum of SISO subsystem cross gramians:
$$W_X = \textstyle\sum_i \int_0^\infty e^{At} b_i c_i e^{At} \mathrm{d}t.$$

Interaction Measures:
The subsystem input-output coherence can be measured for example by the subsystem gain:
$$\Psi_{ij} := \mathrm{tr}(W_{X,ij}).$$

Participation Matrix:
A matrix of all input-output combinations of interaction measures yields the participation matrix Ψ.

Non-Symmetric Systems

Non-Symmetric System:
For a square non-symmetric system holds:
$$\textstyle\sum_i^k \sigma_i(H) \geq \sum_i^k \sigma_i(W_X) \wedge \sum_i^k \sigma_{N-i+1}(W_X) \geq \sum_i^k \sigma_{N-i+1}(H)$$

Orthogonal Symmetry:
The cross gramian extends to orthogonally symmetric systems:
$$W_X = \textstyle\int_0^\infty e^{At} BUC e^{At} \mathrm{d}t.$$

Symmetric Embedding:
An approximate cross gramian is given by embedding using symmetrizer J:
$$W_X \approx \textstyle\int_0^\infty e^{At} J C^T C B B^T J^{-1} e^{At} \mathrm{d}t.$$

Non-Symmetric Cross Gramian:
The non-symmetric cross gramian is defined as sum of all subsystem cross gramians:
$$W_Z := \textstyle\sum_i \sum_j \int_0^\infty e^{At} b_i c_j e^{At} \mathrm{d}t$$

Nonlinear Systems

Generalized Symmetry:
System symmetry can be extended to general and thus nonlinear systems by:
$$\mathrm{Im}\,\mathcal{C}^+ = \mathrm{Im}\,\mathcal{O}.$$

Gradient System:
A gradient system is symmetric and can be written using a real, smooth potential function V:
$$\dot{x} = \nabla V(x) - \langle u, \nabla g(x) \rangle.$$

Nonlinear Cross Gramian:
The general cross operator or cross map solves the nonlinear Sylvester equation:
$$\tfrac{\mathrm{d}W_X}{\mathrm{d}x} f(x) = -f(W_X(x)) - g(W_X(x)).$$

Symmetry Test:
A nonlinear system is symmetric if the cross gramian is invertible and:
$$\mathcal{O}(x) = \mathcal{C}^+(W_X(x)).$$

Parameter Identification

Augmented System:
Treating parameters as states yields an augmented system:
$$\begin{pmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} f(x(t), u(t), \theta) \\ 0 \end{pmatrix},$$
$$y(t) = g(x(t), u(t), \theta).$$

Joint Gramian:
The joint gramian is the cross gramian of an augmented system:
$$W_J = \begin{pmatrix} W_X & W_M \\ 0 & 0 \end{pmatrix}.$$

Cross-Identifiability:
The symmetric part of W_J leads to cross-gramian-based identifiability information:
$$W_{\tilde{I}} := W_M^T (W_X + W_X^T)^{-1} W_M.$$

Identification:
An SVD of the cross-identifiability gramian yields the dominant subspace:
$$U = pod(W_{\tilde{I}})$$

Read Me:
C. Himpe and M. Ohlberger. "A note on the non-symmetric cross gramian". *arXiv Preprint*, math.OC(1501.05519):1–6, 2015.
C. Himpe and M. Ohlberger. "Cross-gramian based combined state and parameter reduction for large-scale control systems". *Mathematical Problems in Engineering*, 2014:1–13, 2014.
L.A. Mironovskii and T.N. Solvéva. "Analysis of multiplicity of hankel singular values of control systems". *Automation and Remote Control*, 76(2):205–218, 2015.
T.C. Ionescu, K. Fujimoto, and J.M.A. Scherpen. "Singular value analysis of non-linear symmetric systems". *Transactions on Automatic Control of the IEEE*, 56(9):2073–2086, 2011.