A Dimensional Reduction Approach Based on the Application of Reduced Basis Methods in the Framework of Hierarchical Model Reduction

Mario Ohlberger

(joint work with Kathrin Smetana)

Many phenomena in nature have dominant spatial directions along which the essential dynamics occur. Examples are blood flow problems, fluid dynamics in pipes or river beds, and subsurface flow. Motivated by a project on adaptive hydrological modelling of coupled hydrological processes [2] we started to investigate a new dimensional reduction approach [5, 7] for problems with dominant direction which is based on the application of reduced basis (RB) techniques in the hierarchical model reduction (HMR) framework (cf. [8] and the refrences therein). In detail let $\Omega \subset \mathbb{R}^2$ be a computational domain. We define the solution space V such that $H^1_0(\Omega) \subseteq V \subseteq H^1(\Omega)$ and consider the following general elliptic problem:

$$\text{Find } \; p \in V: \quad a(p, v) = f(v) \quad \forall v \in V,$$

where $a(\cdot, \cdot)$ is a coercive and continuous bilinear form and f a linear form.

The idea of HMR, which goes back to the work of Vogelius and Babuska [10], is to perform a Galerkin projection onto a reduced space of rank m, i.e.

$$V_m = \left\{ v_m(x, y) = \sum_{k=1}^{m} \overline{v}_k(x)\, \phi_k(y), \;\; \overline{v}_k(x) \in X, \right\},$$

which combines the full solution space X in the dominant direction with a reduction space $Y := \text{span}(\phi_1, ..., \phi_m)$ in the transverse direction. The latter is spanned by modal orthonormal basis functions. While so far the basis functions in the HMR approach have been chosen a priori, for instance, as Legendre or trigonometric polynomials, in this work a highly nonlinear approximation is employed for the construction of the reduction space. To this end we first derive a lower dimensional parametrized problem in the transverse direction from the full problem where the parameters reflect the influence from the unknown solution in the dominant direction. For the derivation of a suitable 1D PDE in transverse direction, we first make the following tenso product ansatz

$$p(x, y) \approx U(x) \cdot \mathcal{P}(y).$$

Here, $U(x)$ represents the behavior of the full solution in the dominant direction, which is unknown at this stage. By choosing the test functions as $v(x, y) = U(x) \cdot \upsilon(y)$ for any $\upsilon \in Y$ we obtain a parameterized reduced problem: Given any $U \in X$, find $\mathcal{P} \in Y$ such that

$$a(U\mathcal{P}, U\upsilon) = f(U\upsilon) \;\; \forall \upsilon \in Y.$$

Exploiting the good approximation properties of RB methods, we then construct a reduction space by applying a proper orthogonal decomposition to a set of snapshots of the parametrized partial differential equation. For an efficient construction of the snapshot set we apply adaptive refinement in parameter space (cf. [4]) based on an a posteriori error estimate that is also derived in this article. We introduce our method for general elliptic problems such as advection-diffusion equations in two space dimensions. Numerical experiments demonstrate a fast convergence of the proposed dimensionally reduced approximation to the solution of the full dimensional problem and the computational efficiency of our new adaptive approach.

In a next step, we extend the reduced basis-hierarchical model reduction framework for the application to nonlinear partial differential equations [9]. The major new ingredient to accomplish this goal is the introduction of an adaptive Empirical Projection Method, which is an adaptive integration algorithm based on empirical interpolation [1, 3]. In detail, let $u(\mu,\cdot) \in L^2(\omega)$ be given, e.g. as the image of a nonlinear operator A, i.e. $u(\mu,\cdot) = A(v(\mu,\cdot))$. By $\mathcal{M}_\Xi := \{u(\mu,\cdot),\, \mu \in \Xi\}$ we deonte a snapshot set, where $\Xi \subset \mathcal{D}$ is a training set of size $|\Xi| = n$. The collateral space $W_k = \text{span}\{\kappa_1, ..., \kappa_k\}$ with $(\kappa_i, \kappa_j)_{L^2(\omega)} = \delta_{ij}$ is then defined through a POD of the snapshot set. By projection of u into W_k, we obtain

$$P_k[u](\mu, y) := \sum_{l=1}^{k} \int_\omega u(\mu, z)\, \kappa_l(z)\, dz\ \kappa_l(y)$$

For the usage of such a projection in our dimension reduction approach we need a separation of variables in $u(\mu, z)$ in order to be able to precompute the integral on ω independent of μ. To achieve this goal, the adaptive EPM subdivides ω into subintervals and applies locally a generalized empirical interpolation (GEIM), i.e.

$$\begin{aligned} P_k^L[u](\mu, y) &:= \sum_{l=1}^{k} \int_\omega \mathcal{I}_L[u](\mu, z)\, \kappa_l(z)\, dz\ \kappa_l(y) \\ \text{with} \qquad \mathcal{I}_L[u](\mu, z) &:= \sum_{I\in\mathfrak{I}} \mathcal{I}_{k_I}^I[u](\mu, z) = \sum_{I\in\mathfrak{I}} \sum_{j=1}^{k_I} \sigma_j^I(u(\mu,\cdot))\vartheta_j^I(z). \end{aligned}$$

Here $(\vartheta_l^j)_j$ is a basis of the localized spaces W_k^I, and $(\sigma_j^I)_j$ a corresponding nodal basis of the dual space, generated by the GEIM (cf. [1, 3]). Using the adaptive EPM, we project both the variational formulation and the range of the nonlinear operator onto reduced spaces. Those combine the full dimensional space in an identified dominant spatial direction and a reduction space or collateral basis space spanned by modal orthonormal basis functions in the transverse direction. Both the reduction and the collateral basis space are constructed in a highly nonlinear fashion by introducing a parametrized problem in the transverse direction and associated parametrized operator evaluations, and by applying reduced basis methods to select the bases from the corresponding snapshots as in the linear case. Rigorous a priori and a posteriori error estimators which do not require additional regularity of the nonlinear operator are proven for the Empirical Projection Method and then used to derive a rigorous a posteriori error estimator for the resulting hierarchical model reduction approach. Numerical experiments for an elliptic nonlinear diffusion equation demonstrate a fast convergence of the proposed dimensionally reduced approximation to the solution of the full-dimensional problem. Run-time experiments verify a linear scaling of the reduction method in the number of degrees of freedom used for the computations in the dominant direction.

Finally, we also investigate the application of our HMR-RB approach in the presence of interfaces or strong gradients in the solution which are skewed with respect to the coordinate axes [6]. Usually, tensor-based model reduction procedures show bad convergence rates for such situations. The key ideas to recover the good approximation properties are the detection of the interface and a subsequent removal of the interface from the solution by choosing the determined interface as the lifting function of the Dirichlet boundary conditions. For prescribed interfaces we demonstrate in numerical experiments that the proposed procedure yields a significantly improved convergence behavior even in the case when we only consider an approximation of the interface.

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