The Total Surgery Obstruction I

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These notes are the first in a set of three, corresponding to a series of three Obseminar talks given at the University of Münster in June 2009, aimed at showing the existence of the total surgery obstruction of a Poincaré complex X, that is zero if and only if X is homotopy equivalent to a topological manifold. This first part covers the basic definitions needed for algebraic surgery as well as very rough explanations of how to go from geometric objects to algebraic ones.

The main aim of this first part is to establish a long exact sequence of L-groups, consisting of the symmetric, quadratic and normal L-groups of a ring with involution R.

The book by Ranicki [Ran92] covers everything I do here, but is not always the best place to look first. A good introduction to algebraic surgery itself is the short paper [Ran01b]. Normal complexes are nicely dealt with in [Ran01a]. For the long exact sequence of L-groups, see [Ran92, Chapter 2].

Convention. All chain complexes are taken to be finitely generated and free.

1 Revision

There are some general definitions and results that I will need later, and so these are included here as a reminder for people who have seen them before and as a very quick explanation for anyone who has not.

Definition 1.1. A *ring with involution* is a ring R together with a map

$$\bar{\ }: R \to R \quad \text{s.t.}$$
$$\bar{\bar{a}} = a, \quad \overline{a+b} = \bar{a} + \bar{b}, \quad \overline{ab} = \bar{b}\bar{a}.$$

We like rings with involution because they allow us to turn a left module into a right module and vice versa, which is useful if you are interested in taking tensors, which we will be. To make this more precise, given a left R-module M, we turn it into a right R-module by

$$m \,.\, r := \bar{r} \,.\, m$$

So now we can think of left modules as right modules and vice versa, so we can also consider the object $M \otimes_R M$ for any module M.

Example 1.2. Let R be a commutative ring and G be a group. Then the group ring $R[G]$ is a ring with involution, where we define the involution by

$$\overline{rg} = rg^{-1}$$

We will often be interested in the case $R = \mathbb{Z}$ and $G = \mathbb{Z}[\pi_1(X)]$, for some space X.

Convention. A ring R will be always be a ring with involution.

Proposition 1.3. *Let R be a ring with involution and let M and N be finitely generated, projective, left R-modules. Then the* slant map

$$\begin{aligned} \phi : M \otimes_R N &\to \mathrm{Hom}_R(M^*, N) \\ x \otimes y &\to (f \to \overline{f(x)}.y) \end{aligned}$$

is an isomorphism, where $M^* = \mathrm{Hom}_R(M, R)$.

I won't prove this here, but I shall remind you of the fact that, for finitely generated vectors spaces V, W over a field k, we have the following facts

- $\mathrm{Hom}_k(V, W) \cong V^* \otimes_k W$
- $(V^*)^* \cong V$

and combining these facts proves the slant isomorphism if M and N are vector fields, i.e. when R is a field.

Definition 1.4. Let C,D be chain complexes over a ring R. Let $f : C \to D$ be a chain map. The *algebraic mapping cone* of f is the chain complex $\mathcal{C}(f)$ with

$$\begin{aligned} \mathcal{C}(f)_r &:= D_r \oplus C_{r-1} \\ d^{\mathcal{C}(f)} &:= \begin{pmatrix} d^D & (-1)^{r-1} f \\ 0 & d^C \end{pmatrix} \end{aligned}$$

2 Symmetric Complexes

Motivation. The objects of interest in algebraic surgery theory are Poincaré complexes. We have many ways of encoding a space X into algebra, for instance we could take the singular chain complex, but we also want to keep track of the Poincaré duality. We know the Poincaré duality chain equivalence is capping with some fundamental class $[X]$,

$$\_ \cap [X] : C(X)^{n-*} \to C(X)$$

where n is the dimension of the fundamental class $[X]$. Using the slant isomorphism, we can think of the Poincaré duality map as an element of $C(X) \otimes C(X)$.

To make this a little more precise, let R be a ring, let X be a CW-complex, and let $C(X)$ be the cellular chain complex of X with coefficients in R. For any element $\sigma \in C(X)^n$ there is a map

$$\_ \cap \sigma \colon C(X)^{n-*} \to C(X)$$

corresponding to an element of $(C(X) \otimes_R C(X))_n$. Thus we can define a map

$$\phi \colon C(X) \to C(X) \otimes C(X)$$

taking the element σ to the element corresponding to capping with σ.

We have defined ϕ purely algebraically and it would be nice to have some geometric idea of where it comes from. Recall that we can think of capping a chain with a cochain as letting the cochain 'eat' the first part of the chain and then leaving the rest (after it is full). Compare the definition of the cap product with the slant isomorphism and you may just become convinced that the diagonal map $X \to X \times X$ is a geometric map inducing the map ϕ on the cellular chain complexes, where we use the cell structure on X to obtain a cell structure on $X \times X$. The key property of the diagonal map is that it is symmetric.

To give a more rigorous idea of what I mean when I say "symmetric", let $\mathbb{Z}/2 = \{e, T\}$, and let T act on $X \times X$ by $T.(x, y) = (y, x)$. Then the diagonal map has image lying in the fixed point set of $X \times X$ under this action.

The problem is that this diagonal map is not a cellular map and thus does not induce a map on the cellular chain complexes.

Example 2.1. If $X = \Delta^1$ then we have an obvious CW-structure on X consisting of two 0-cells and one 1-cell. Then the cell structure on $X \times X$ consists of four 0-cells and four 1-cells forming a square and a 2-cell filling in the square. Now the diagonal map has image straight through the middle of the 2-cell, see figure 1, where the dotted diagonal line is the image of the diagonal map.

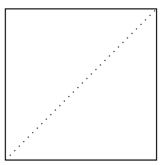


Figure 1: The image of the diagonal map is not cellular

So we need a map ϕ_0 homotopic to the diagonal map that is also symmetric upto homotopy, meaning that $T\phi_0 \simeq \phi_0$ and the homotopy itself is symmetric upto homotopy (so we get recursion) . We do have a cellular map $\phi_0 \colon X \to X \times X$ by going round the boundaries of the cells, and then $T\phi_0$ corresponds to going around to the other part of the boundary of the cell.

In example 2.1 above, we could take our ϕ_0 to go along the bottom and right edges of the square. Then $T\phi_0$ would be going along the left and top edges. These paths are clearly homotopic, by dragging the path through the interior of the square.

In general, we still get that $T\phi_0$ homotopic to ϕ_0 by dragging across the interiors of the cells. So let $\phi_1 \colon X \to X \times X$ be a homotopy from ϕ_0 to $T\phi_0$, going through the cell. Then We get that $T\phi_1$ is just dragging across cells in the other direction and so is homotopic to ϕ_1. We can repeat this process of dragging across cells to get higher homotopies of homotopies as far up as we want.

To keep track of all these homotopies, we introduce the space S^∞. Let S^0 be the CW-complex with two 0-cells, which we can think of these as corresponding to e and T. We add two 1-cells joining these two points to form S^1. Call these new 1-cells e_1 and Te_1. Then add two 2-cells to get S^2 and so on. We can from all the standard spheres this way, and set $S^\infty = \bigcup_{n\in\mathbb{N}} S^n$. We can then define a map

$$\phi \colon S^\infty \times X \to X \times X$$

such that $\phi|_{e_n \times X} = \phi_n$ and $\phi|_{Te_n \times X} = T\phi_n$.

Then ϕ is a cellular map that contains all information about the symmetry of ϕ_0. It is this map ϕ that we will use in our algebraic data. And so, after a rather long piece of motivation, we come to algebra.

Definition 2.2. The chain complex W is defined to be the cellular chain complex of the space S^∞. Explicitly, it is the chain complex

$$\cdots \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \longrightarrow 0$$

Definition 2.3. Let R be a ring and let C be a chain complex over R. Using the involution we can from the tensor product $C \otimes_R C$ and we turn this into a $\mathbb{Z}[\mathbb{Z}/2]$-module using the flipping $\mathbb{Z}/2$-action (upto sign), namely,

$$\begin{aligned} C_p \otimes C_q &\to C_q \otimes C_p \\ x \otimes y &\to (-1)^{pq} y \otimes x \end{aligned}$$

We define the chain complex $W^{\%}(C)$ by

$$W^{\%}(C) := \mathrm{Hom}_R(W, C \otimes C)$$

Then the *symmetric Q-groups of C* are defined by

$$Q^n(C) := H_n(W^{\%}(C))$$

Remark 2.4. Any element $\phi \in W^{\%}(C)_n$ can be though of as a collection of maps $\{\phi_s \colon C^{n+s-*} \to C | s \in \mathbb{N}\}$ by

$$\phi_s := \phi(1_s) \in (C \otimes C)_{n+s} \cong \mathrm{Hom}_R(C^{n+s-*}, C)_0$$

where $1_s \in W_s$ is the identity in $\mathbb{Z}[\mathbb{Z}/2]$.

Definition 2.5. Let R be a ring. An *n-dimensional symmetric chain complex over R* is a pair (C, ϕ), where C is a chain complex over R and ϕ is an n-cycle in $W^{\%}(C)$. We introduce the notation n-SAC to abbreviate the term n-dimensional symmetric chain complex.

Remark 2.6. The n refers only to the degree of the element ϕ and has nothing to do with the chain complex C.

Construction 2.7. Let X be a topological space. Let $\tilde{X}$ be the universal cover of X and set R to be the group ring $\mathbb{Z}[\pi_1(X)]$. Then $C(\tilde{X}; \mathbb{Z})$ is a chain complex over $\mathbb{Z}[\pi_1(X)]$, using the action of the fundamental group on the universal cover. The *symmetric construction of X* is a map

$$\phi_X \colon C(X; \mathbb{Z}) \to W^{\%}(C(\tilde{X}, \mathbb{Z}))$$

such that for every $\sigma \in C(\tilde{X}; R)^{-*}$, we have $\phi_X(\sigma)_0$ is the map corresponding to capping with the class σ.

Remark 2.8. If X is simply-connected, then the map ϕ_X is an algebraic version of the geometric map $\phi \colon S^\infty \times X \to X \times X$.

Remark 2.9. If X is a geometric Poincaré complex of formal dimension n, with fundamental class $[X]$, then $\phi_X([X])_0$ is the Poincaré duality chain equivalence. This motivates the following definition.

Definition 2.10. An n-SAC (C, ϕ) over a ring R is called *Poincaré* if the map $\phi_0 \colon C^{n-*} \to C$ is a chain equivalence. We use the notation n-SAPC for an n-dimensional symmetric algebraic Poincaré chain complex.

We want an algebraic analogue of the geometric surgery toolbox, and for this we will something data as input for our surgery. This input will come in the form of a *symmetric pair*.

Definition 2.11. Let C, D be chain complexes over a ring R. Let $f \colon C \to D$ be a chain map. Then the map $f \otimes f \colon C \otimes C \to D \otimes D$ induces a map $f^{\%} \colon W^{\%}(C) \to W^{\%}(D)$. Explicitly this map is given by

$$(f^{\%}(\phi))_s := f \phi_s f^*$$

Definition 2.12. An *(n+1)-dimensional symmetric pair* is a map, written

$$(f \colon C \to D, (\delta\phi, \phi))$$

where

– (C, ϕ) is an n-SAC over R

– D is a chain complex over R

– $f \colon C \to D$ is a chain map

– $\delta\phi \in (W^{\%}(D))_{n+1}$ s.t.

  – $d(\delta\phi) = f^{\%}(\phi)$ for all $s \in \mathbb{N}$.

Remark 2.13. We use the symbol $\delta\phi$ as a single object, choosing this notation to reflect the role played by $\delta\phi$ in the definition of a symmetric pair.

Remark 2.14. The conditions on the symmetric structures on C and D in definition 2.12 mean we have an $(n+1)$-cycle $(\delta\phi, \phi)$ of $\mathcal{C}(f^{\%})$. There is a map $\mathcal{C}(f^{\%}) \to W^{\%}(\mathcal{C}(f))$, which is not explained here, and thus from $(\delta\phi, \phi)$ we get an $(n+1)$-cycle of $W^{\%}(\mathcal{C}(f))$.

Definition 2.15. An $(n+1)$-dimensional symmetric pair $(f \colon C \to D, (\delta\phi, \phi))$ is called *Poincaré* if the map

$$\begin{pmatrix} \delta\phi_0 \\ \phi_0 f \end{pmatrix} : D^{n+1-*} \to \mathcal{C}(f)$$

is a chain equivalence.

Why this map is special will be clear when I define algebraic surgery. Finally, a techincal definition that will be needed later on.

Definition 2.16. Let C be a chain complex over some ring R. Let ϕ be an n-cycle of $W^{\%}(C)$. Recall that ϕ is determined by the maps $\phi_k \colon C^{n+k-*} \to C$, for $k \in \mathbb{N}$. We define the *suspension* map

$$S \colon W^{\%}(C) \to S^{-1}(W^{\%}(SC))$$

by

$$(S(\phi))_k := \phi_{k-1}$$

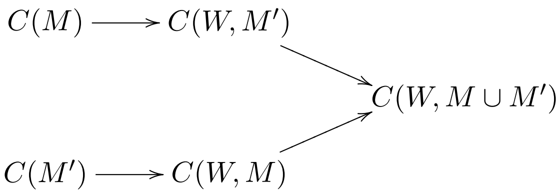
The idea behind this definition is

$$\begin{aligned} S^{-1}(W^{\%}(SC)) &= S^{-1} \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, (SC) \otimes (SC)) \\ &= S^{-1} \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, S^2(C \otimes C) \\ &= \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(S^{-1}W, C \otimes C) \end{aligned}$$

3 Algebraic Surgery

We use geometric surgery to motivate the definition of algebraic surgery, so first a quick recap of what happens with geometric surgery.

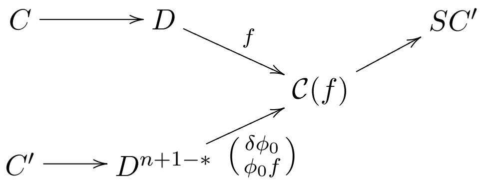
If we have a closed manifold M and perform surgery on it then we obtain a new closed manifold M' and a cobordism W from M to M' called the trace of the surgery. Thus we obtain a diagram



where the chain complexes $C(W, M')$ and $C(W, M)$ are dual. This is just a form of Poincaré-Lefschetz duality, since $\partial W = M \cup M'$. Geometrically, this corresponds to the fact that the cone of the inclusion map $M \hookrightarrow W/M'$ is homotopy equivalent to $W/(M \cup M')$.

We want to try to replicate this algebraically.

Let $(f : C \to D, (\delta\phi, \phi))$ be an $(n+1)$-dimensional symmetric pair. The map f corresponds to the inclusion $M \hookrightarrow W/M'$, so we consider the following diagram



The chain complex C' is defined to be the homotopy fibre of the would-be duality map $\begin{pmatrix} \delta\phi_0 \\ \phi_0 f \end{pmatrix}$ (see definition 2.15), which algebraically means we have

$$C' = S^{-1}\mathcal{C}(\begin{pmatrix} \delta\phi_0 \\ \phi_0 f \end{pmatrix})$$

In particular,

$$C'_k = D_{k+1} \oplus C_k \oplus D^{n+1-k}$$

So the symmetric pair being Poincaré is equivalent to saying the algebraic map f corresponds to the inclusion $\partial W \hookrightarrow W$.

We have a chain complex but we really want an n-SAC, so we need some symmetric structure ϕ' on C'. We have an $(n+1)$-cycle of $W^{\%}(\mathcal{C}(f))$ coming from $(\delta\phi, \phi)$, and then we can push this forward to an $(n+1)$-cycle of $W^{\%}(SC')$. It is a fact that this cycle is in the image of the suspension map and so can be desuspended to an n-cycle of $W^{\%}(C')$, which we denote ϕ', and this is the symmetric structure we take on C'.

Definition 3.1. We call the n-SAC (C', ϕ'), obtained using the process above, the *effect* of surgery on the symmetric pair.

Remark 3.2. Algebraic surgery preserves the homotopy type of the boundary of (C, ϕ). A definition of the boundary of an n-SAC will be given later, but for now it is enough to know that it is contractible if and only if (C, ϕ) is Poincaré. In this special case, we get that

$$(C, \phi) \text{ is Poincaré} \Leftrightarrow (C', \phi') \text{ is Poincaré}$$

In the geometric case, the surgery gave rise to a cobordism, the trace of the surgery. A similar thing happens in the algebraic case.

Definition 3.3. A *cobordism* of n-SAPCs $(C, \phi), (C', \phi')$ is an $(n+1)$-symmetric Poincaré pair

$$((f\, f') \colon C \oplus C' \to E, (\delta\phi, \phi \oplus -\phi'))$$

Remark 3.4. Any algebraic surgery gives rise to such a cobordism.

Remark 3.5. The equivalence relation generated by surgery and homotopy equivalence is the same as the equivalence relation generated by being cobordant.

4 Quadratic Complexes

Definition 4.1. Let C be a chain complex over a ring R. We define the chain complex $W_{\%}(C)$ by

$$W_{\%}(C) := W \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (C \otimes_R C)$$

Then the *quadratic Q-groups of C* are

$$Q_n(C) := H_n(W_{\%}(C))$$

Now that we have defined the quadratic Q-groups, we can also define an *n-dimensional quadratic chain complex* (shortened to n-QAC) and an $(n+1)$*-dimensional quadratic pair* analogously to the symmetric case, which was done in definitions 2.5 and 2.12

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It would be nice if we could relate symmetric and quadratic chain complexes in some way, and for this we introduce *hyperquadratic Q-groups.*

Definition 4.2. We define the chain complex $\hat{W}$ to be $\mathbb{Z}[\mathbb{Z}/2]$ in every dimension, with differentials alternating between $1-T$ and $1+T$, so it looks like

$$\cdots \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \cdots$$

Hence the chain complex $\widehat{W}$ is an extension of the chain complex W into negative dimensions.

We define the chain complex $\widehat{W}^{\%}(C)$ by

$$\widehat{W}^{\%}(C) := \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\widehat{W}, C \otimes_R C)$$

And then the *hyperquadratic Q-groups of C* are

$$\widehat{Q}^n(C) := H_n(\widehat{W}^{\%}(C))$$

Which is very similar to the symmetric case, see definition 2.3.

Fact. The hyperquadratic Q-groups are the stabilisation of the symmetric Q-groups.

$$\widehat{Q}^n(C) = \operatorname*{colim}_{k\to\infty} Q^{n+k}(S^k C)$$

Proposition 4.3. *Let C be a chain complex over a ring R. Then there is a long exact sequence of Q-groups*

$$\cdots \longrightarrow Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{J} \widehat{Q}^n(C) \xrightarrow{H} Q_{n-1}(C) \longrightarrow \cdots$$

Proof. Define a new chain complex W_- to be $\mathbb{Z}[\mathbb{Z}/2]$ in negative dimensions and zero everywhere else, such that there is a short exact sequence of chain complexes

$$0 \longrightarrow W_- \longrightarrow \widehat{W} \longrightarrow W \longrightarrow 0$$

Apply the functor $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\_, C \otimes_R C)$ to this SES to get another SES

$$0 \longrightarrow \mathrm{Hom}(W, C \otimes C) \longrightarrow \mathrm{Hom}(\widehat{W}, C \otimes C) \longrightarrow \mathrm{Hom}(W_-, C \otimes C) \longrightarrow 0$$

We recognise the first and second term in this SES as $W^{\%}(C)$ and $\widehat{W}^{\%}(C)$, but what is the third term?

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W_-, C \otimes_R C) &= (W_-)^{-*} \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (C \otimes_R C)\\ &= (SW) \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (C \otimes_R C)\\ &= SW_{\%}(C)\end{aligned}$$

Thus the LES in homology induced by the SES is exactly the LES sequence from the statement of the proposition. ☐

Definition 4.4. Let (C, ψ) be an n-QAC. Using the map $1+T$ we get an n-SAC $(C, (1+T)\psi)$. We call (C, ψ) a n-dimensional quadratic algebraic *Poincaré* chain complex (or a n-QAPC for short), if the n-SAC $(C, (1+T)\psi)$ is an n-SAPC.

Construction 4.5. We saw earlier why symmetric chain complexes are interesting, but what about quadratic complexes? To try to motivate them I introduce the *quadratic construction.*

Example 4.6. Let X be a geometric Poincaré complex and let $f\colon M \to X$ be a degree one normal map. We can use Spanier-Whitehead-duality (shortened hereafter to S-duality) to get a map $F\colon \Sigma X_+^k \to \Sigma^k M_+$ for some $k \in \mathbb{N}$. We hope that this map F is a homotopy inverse to f, but for this we obviously need $k = 0$. The quadratic construction will be an obstruction to $k = 0$.

In general, let X, Y be any simply-connected pointed spaces. Given a map $F\colon \Sigma^k X \to \Sigma^k Y$ there is a map $\Psi_F\colon H_*(X) \to Q_*(C(Y))$ such that

$$k = 0 \;\Rightarrow\; \Psi_F \equiv 0$$

We don't need X, Y to be simply-connected, we just need $\pi_1(X) = \pi_1(Y)$, but if we consider non-simply-connected spaces then we need to pass to the universal covers and use $\mathbb{Z}[\pi_1(X)]$ coefficients, so sticking to the simply-connected case keeps the notation much more simple.

I won't give an explicit construction of the map Ψ_F, merely try to convince you that such a thing should exist. We use the naturality property of the symmetric construction, namely for any map $g\colon U \to V$, we get

$$\phi_V g_* = g^{\%}\phi_U$$

Set $f\colon C(X) \to S^{-k}C(\Sigma^k X) \xrightarrow{F} S^{-k}C(\Sigma^k Y) \to C(Y)$. Then the following diagram does not necessarily commute, since f does not come from a geometric map

$$\begin{array}{ccc} H_n(X) & \xrightarrow{\phi_X} & Q^n(C(X)) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f^{\%}} \\ H_n(Y) & \xrightarrow{\phi_Y} & Q^n(C(Y)) \end{array}$$

So we look at the difference $f^{\%}\phi_X - \phi_Y f_*$, and use the LES of Q-groups to get a commutative diagram

$$\begin{array}{ccccccccc} & & & \swarrow^{\Psi_F}\!\!\!\!\! & H_n(X) & \searrow^{\equiv 0} & & & \\ & & & & \big\downarrow{\scriptstyle f^{\%}\phi_X - \phi_Y f_*} & & & & \\ \cdots & \longrightarrow & Q_n(C(Y)) & \longrightarrow & Q^n(C(Y)) & \longrightarrow & \widehat{Q}^n(C(Y)) & \longrightarrow & \cdots \end{array}$$

The map $H_n(X) \to \widehat{Q}^n(C(Y))$ is the stablisation of the map $f^{\%}\phi_X - \phi_Y f_*$, but when we stabilise f we recover the map $F\colon C(\Sigma^k X) \to C(\Sigma^k Y)$, which does come from a geometric map, and so, by the naturality of the symmetric construction, the map $H_n(X) \to \widehat{Q}^n(C(Y))$ is zero.

Then exactness tells us there is a pre-image. If we look on the chain level, $C(X)$ is free so there should be a map $C(X) \to W_{\%}(C(Y))$ and this can be chosen in some sort of canonical way, and hence we obtain our map Ψ_F with some nice naturality properities, which will not be explained here. For details about the quadratic construction see Andrew Ranicki's original paper [Ran80].

5 Normal Complexes

Definition 5.1. An *n-dimensional geometric normal complex* (or n-GNC for short) is a triple (X, ν, ρ) consisting of a space X with a k-spherical fibration ν and a map $\rho\colon S^{n+k} \to \mathrm{Th}(\nu)$.

Remark 5.2. There are no conditions on the map ρ. It may be surprising that we ask for these three objects, but allow the choice to be arbitrary. The notion is a more general thing than a Poincaré complex, meaning any geometric Poincaré complex gives rise to a geometric normal complex, as shown in the next example.

Example 5.3. Let X be a geometric Poincaré complex of formal dimension n (abbreviated to n-GPC) with fundamental class $[X]$. Then X has a Spivak normal fibration (SNF) $\nu_X\colon X \to BG(k)$, which is unique up to stabilisation, such that there is a map $\rho_X\colon S^{n+k} \to \mathrm{Th}(\nu_X)$ with

$$[X] = u_{\nu_X} \cap h(\rho_X)$$

where u_{ν_X} is the Thom class and h is the Hurewicz homomorphism. Thus we get an n-dimensional geometric normal complex (X, ν_X, ρ_X).

Remark 5.4. For an arbitrary n-GNC complex (X, ν, ρ) we can define a class $[X] \in C(X)_n$ by

$$[X] := u_\nu \cap h(\rho)$$

So the dimension of the geometric normal complex is the degree of the associated class, namely the dimension of the source sphere of ρ minus the dimension of the spherical fibration.

Definition 5.5. Let X, Y be pointed spaces. A map $\alpha\colon S^N \to X \wedge Y$ is an *S-duality map* if

$$\alpha_*([S^N])\backslash\_\colon \tilde{C}(X)^{N-*} \to \tilde{C}(Y)$$

is a chain equivalence. We say the spaces X, Y are *S-dual.*

Fact. Every finite CW-complex has an S-dual.

We can formulate geometric Poincaré complexes in terms of normal complexes using S-duality. We have already shown in example 5.3 how to obtain an n-GNC from an n-GPC. We need some condition on when a normal complex is Poincaré.

For any n-GNC (X, ν, ρ) we can use the map $\rho\colon S^{n+k} \to \mathrm{Th}(\nu)$ to define a "would-be" S-duality map $\alpha\colon S^{n+k} \to \mathrm{Th}(\nu) \wedge X_+$ by precomposing with some sort of diagonal map. To make this explicit, let W be a neighbourhood of X embedded in the total space of the spherical fibration ν. We define the map $\tilde{\Delta}$ by

$$\tilde{\Delta}\colon \mathrm{Th}(\nu) \simeq \frac{W}{\partial W} \xrightarrow{\Delta} \frac{W \times W}{W \times \partial W} \simeq \frac{W}{\partial W} \times W_+ \simeq \mathrm{Th}(\nu) \wedge X_+$$

where Δ is the actual diagonal map. Then define α to be the composite

$$\alpha\colon S^{n+k} \xrightarrow{\rho} \mathrm{Th}(\nu) \xrightarrow{\tilde{\Delta}} \mathrm{Th}(\nu) \wedge X_+$$

Example 5.6. Let (X, ν_X, ρ_X) be the n-GNC coming from a geometric Poincaré complex with fundamental class $[X]$ (see example 5.3). Then the map α is an S-duality map, which can be seen by observing the that fact that

$$\alpha_*([S^{n+k}]) = \tilde{\Delta}_*(h(\rho))$$

Moreover, the following diagram commutes upto homotopy

$$\begin{array}{ccc} C(X)^{n-*} & \xrightarrow{\text{Thom}} & \tilde{C}(\mathrm{Th}(\nu))^{n+k-*} \\ & \text{Poincaré} \searrow \quad \circlearrowleft \quad \swarrow \text{S-duality} & \\ & C(X) = \tilde{C}(X_+) & \end{array}$$

Fact. An n-GNC (X, ν, ρ) is an n-GPC with fundamental class $u_\nu \cap h(\rho)$ if and only if the map $\alpha\colon S^{n+k} \to \mathrm{Th}(\nu) \wedge X_+$ is an S-duality map.

If we are going to be able to encode such a geometric normal complex into algebra, we first need some idea of what a spherical fibration over a chain complex might look like.

Definition 5.7. Let C be a chain complex over a ring R. A *chain bundle* over C is a 0-cycle γ of $\widehat{W}^{\%}(C^{-*})$.

Construction 5.8. Let X be a finite CW-complex. Let ν be a k-spherical fibration over X. Then $\mathrm{Th}(\nu)$ is also a finite CW-complex and hence has an S-dual. We use this to construct a chain bundle γ_ν over $C(X)$. We call the process *hyperquadratic construction* and although it is a chain level construction, I only give it in homology here because it makes the notation much easier and it is clearer what is going on. We start with the Thom class $u_\nu \in \tilde{H}^k(\mathrm{Th}(\nu))$, and apply the following composition to it:

$$\begin{aligned}
\tilde{H}^k(\mathrm{Th}(\nu)) &\xrightarrow{\text{S-duality}} \tilde{H}_{N-k}(Y) \\
&\xrightarrow{\phi_Y} Q^{N-k}(\tilde{C}(Y)) \\
&\xrightarrow{\text{S-duality}} Q^{N-k}(\tilde{C}(\mathrm{Th}(\nu))^{N-*}) \\
&\xrightarrow{\text{Thom}} Q^{N-k}(C(X)^{N-k-*}) \\
&\xrightarrow{J} \widehat{Q}^{N-k}(C(X)^{N-k-*}) \\
&\xrightarrow{S^{-n}} \widehat{Q}^0(C(X)^{-*})
\end{aligned}$$

The end result of this is our chain bundle γ_ν over $C(X)$.

Now we can define an algebraic analogue of a geometric normal complex.

Definition 5.9. An *n-dimensional normal algebraic chain complex* (n-NAC for short) is a 4-tuple (C, ϕ, γ, χ) such that

– (C, ϕ) is an n-SAC

– γ is a chain bundle over C

– $\chi \in (\widehat{W}^{\%}(C))_{n+1}$ satisfies $d\chi = J(\phi) - (\widehat{\phi_0})^{\%}(S^n\gamma)$

where $(\widehat{\phi_0})^{\%} \colon \widehat{W}^{\%}(C^{n-*}) \to \widehat{W}^{\%}(C)$ is given by $((\widehat{\phi_0})^{\%}(\alpha))_s := \phi_0\alpha_s\phi_0^*$ for all $s \in \mathbb{Z}$ (cf definition 2.11).

The third condition may look a little strange, but it should be thought of as a way of relating the symmetric structure to the chain bundle. I hope the following example will make it a little less mysterious.

Example 5.10. Let X be a finite CW-complex and a geometric Poincaré complex of formal dimension n with fundamental class $[X]$. Let ν_X be the SNF and $\rho_X \colon S^{n+k} \to \mathrm{Th}(\nu_X)$ a map such that

$$[X] = u_{\nu_X} \cap h(\rho_X)$$

Then (X, ν_X, ρ_X) is a geometric normal complex and we get an n-NAC (C, ϕ, γ, χ) with

– $C = C(X)$

– $\phi = \phi_X([X])$

– $\gamma = \gamma_{\nu_X}$

We still have to show where the element χ comes from. Consider the hyperquadratic construction in this special case. Since X is Poincaré, we get that X_+ is a space S-dual to $\mathrm{Th}(\nu_X)$. We also know that the composition of the Thom map and the S-duality map is the Poincaré duality map. Thus we obtain

$$\begin{aligned}\tilde{H}^k(\mathrm{Th}(\nu_X)) &\xrightarrow{\text{S-duality}} \tilde{H}_n(X_+) = H_n(X)\\ &\xrightarrow{\phi_X} Q^n(C(X))\\ &\xrightarrow{\text{Poincaré}} Q^n(C(X)^{n-*})\\ &\xrightarrow{J} \widehat{Q}^n(C(X)^{n-*})\\ &\xrightarrow{S^{-n}} \widehat{Q}^0(C(X)^{-*})\end{aligned}$$

The S-duality map sends the Thom class $u_{\nu_X} \in \tilde{H}(\mathrm{Th}(\nu_X))$ to the fundamental class $[X] \in H_n(X)$. Also, the map J commutes with the Poincaré map. Therefore we get

$$J(\phi) = (\widehat{\phi_0})^{\%}(S^n\gamma) \in \widehat{Q}^n(C)$$

which means that such a $\chi \in (\widehat{W}^{\%}(C))_{n+1}$ exists to make them equivalent on the chain level.

In fact, we only need a geometric normal complex to get an algebraic normal complex, but in this more general case we don't have the relationship between S-duality and Poincaré duality to show the existence of the element χ. This makes it more complicated to construct the algebraic normal complex, and it is something that will not be covered here, although this fact will be used in the notes of the later talks. For those who are interested in how it works, see Michael Weiss's PhD thesis ([Wei85a, §3] and [Wei85b, §7]), and Ranicki [Ran81, §7.4] could also be useful.

Definition 5.11. The *symmetric L-groups* of a ring R are

$$L^n(R) := \{\text{cobordism classes of } n\text{-SAPCs}\}$$

The *quadratic L-groups* are

$$L_n(R) := \{\text{cobordism classes of } n\text{-QAPCs}\}$$

And the *normal L-groups* are

$$NL^n(R) := \{\text{cobordism classes of } n\text{-NACs}\}$$

Proposition 5.12. *Let R be a ring with involution. Then there is a LES*

$$\cdots \longrightarrow L_n(R) \xrightarrow{1+T} L^n(R) \xrightarrow{J} NL^n(R) \xrightarrow{\partial} L_{n-1}(R) \longrightarrow \cdots$$

The maps in the proposition need to be explained. The map $1+T$ is straightforward,

$$1+T\colon (C,\psi)\mapsto (C,(1+T)\psi)$$

For the other two maps, I will need a definition and then a lemma.

Definition 5.13. Let (C,ϕ) be an n-SAC. The *boundary* of (C,ϕ) is the $(n-1)$-SAC obtained from surgery on the symmetric pair $(0\to C,(\phi,0))$.

We denote the boundary by $\partial(C,\phi)$ or $(\partial C,\partial\phi)$.

Remark 5.14. An n-SAC is Poincaré if and only if its boundary is contractible. This comes from the observation that $(\partial C)_k=\mathcal{C}(\phi_0)_{k+1}$.

Lemma 5.15. *Let (C,ϕ) be an n-SAC. Then (C,ϕ) can be extended to a normal complex (C,ϕ,γ,χ) if and only if the boundary $(\partial C,\partial\phi)$ has a quadratic refinement.*

Proof. We compare the LES of Q-groups of ∂C and the LES arising from the map $(\widehat{\phi_0})^{\%}$.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & Q_{n-1}(\partial C) & \xrightarrow{1+T} & Q^{n-1}(\partial C) & \xrightarrow{J} & \widehat{Q}^{n-1}(\partial C) & \longrightarrow \cdots \\
 & & & & & \| & \\
\cdots \longrightarrow & \widehat{Q}^{n}(C^{n-*}) & \xrightarrow{(\widehat{\phi_0})^{\%}} & \widehat{Q}^{n}(C) & \longrightarrow & \widehat{Q}^{n}(\mathcal{C}(\phi_0)) & \longrightarrow \cdots
\end{array}$$

The element $\partial\phi\in Q^{n-1}(\partial C)$ corresponds to the element $J(\phi)\in\widehat{Q}^n(C)$. Thus if (C,ϕ) has a normal structure, then there is a pre-image of $J(\phi)$ in $\widehat{Q}^n(C^{n-*})$, namely $S^n\gamma$, and so $J(\phi)$ maps to zero in $\widehat{Q}^n(\mathcal{C}(\phi_0))$. Hence $\partial\phi$ also maps to zero in $\widehat{Q}^{n-1}(\partial C)$ and by exactness it has a pre-image, which is our quadratic refinement.

Conversely, if $(\partial C,\partial\phi)$ has a quadratic refinement then there is a $\partial\psi\in Q_{n-1}(\partial C)$ that maps to $\partial\phi$ and so $\partial\phi$ maps to zero in $\widehat{Q}^{n-1}(\partial C)$. Hence $J(\phi)$ maps to zero in $\widehat{Q}^n(\mathcal{C}(\phi_0))$ and by exactness we obtain a pre-image of $J(\phi)$ lying in $\widehat{Q}^n(C^{n-*})$, say μ. Set $\gamma=S^{-n}\mu$, and then we get

$$(\widehat{\phi_0})^{\%}(S^n\gamma)=(\widehat{\phi_0})^{\%}(\mu)=J(\phi)\in\widehat{Q}^n(\mathcal{C}(\phi_0))$$

□

Now we can define the remaining two maps J and ∂. An element of $L^n(R)$ is an n-SAPC, and so $\partial C\simeq 0$. Therefore any symmetric structure on the boundary is trivial so automatically has a quadratic refinement, and then we use the lemma to get a normal structure on our n-SAPC. The map J takes an n-SAPC and adds the additional information of this normal structure to get an n-NAC.

Finally, we need to define ∂. Given an n-NAC (C,ϕ,γ,χ), we can take the symmetric boundary $(\partial C,\partial\phi)$, and this will have a quadratic refinement, say $\partial\psi$, using the lemma. So we define the *quadratic boundary* of (C, ϕ, γ, χ) to be

$$\partial(C, \phi, \gamma, \chi) := (\partial C, \partial\psi)$$

The quadratic boundary of a normal complex will be important later on, as will the long exact sequence of L-groups. The notes from the next talk will construct a braid of L-groups, by using the long exact sequence given here to relate the different types of L-groups.

For further reading, I suggest the notes to the remaining two talks, which, together wtih my notes, should not be taken as three independent objects but instead thought of as one continuous set in order to get a more complete idea of the total surgery obstruction.

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