THE TOTAL SURGERY OBSTRUCTION II

PHILIPP KÜHL

Abstract. The total surgery obstruction $s(X)$ of a finite n-dimensional Poincaré complex X is an element of a certain abelian group $\mathbb{S}_n(X)$ with the property that $s(X) = 0$ if and only if X is homotopy equivalent to a closed n-dimensional topological manifold for $n \geq 5$. These are the notes of the second of three talks which want to give a brief overview of the techniques used from algebraic surgery theory to define and proove this statement. Following the first part of [Ran92] particularly chapters 3-5 and 11-14 we concentrate in this part on the question of how algebraic bordism categories capture local Poincaré duality, which leads us to a braid of exact sequences using L-theory spectra.

1. Algebraic bordism categories

We denote the algebraic mapping cone of a chain map f by $\mathcal{C}(f)$.

Definition 1.1. An algebraic bordism category $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C}, (T, e))$ consists of the following data:

$\mathbb{A}$ is an additive category.
$\mathbb{B}$ is a subcategory of the additive category of bounded chain complexes over $\mathbb{A}$.
$\mathbb{C}$ is a full subcategory of $\mathbb{B}$ closed under taking cones, i.e. $\mathcal{C}(f) \in \mathbb{C}$ for all morphisms $f \in \mathbb{C}$. This category will be used to define different types of (Poincaré) duality.
T is a contravariant functor $T : \mathbb{A} \to \mathbb{B}$ which builds together with e the *chain duality* of Λ.
e is a natural transformation $e : T^2 \to (\mathrm{id} : \mathbb{A} \to \mathbb{B})$ such that
- $e_M : T^2(M) \to M$ is a chain equivalence.
- $e_{T(M)} \circ T(e_M) = \mathrm{id}$.

(by $\mathrm{id} : \mathbb{A} \to \mathbb{B}$ we mean an object $A \in \mathbb{A}$ goes to the 0-dimensional chain complex $C \in \mathbb{B}$ with $C_0 = A$)

and finally for any object $B \in \mathbb{B}$

- $\mathcal{C}(\mathrm{id} : B \to B) \in \mathbb{C}$ and
- $\mathcal{C}(e(B) : T^2(B) \xrightarrow{\simeq} B) \in \mathbb{C}$

has to be satisfied.

Remark 1.2. It will be important that the dual $T(M)$ of a chain module is a chain complex. We can extend $T : \mathbb{A} \to \mathbb{B}$ in a natural way to a duality $T_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$ on chain complexes by using the total chain complex as follows. Let

$$C : \ldots \to M_r \to M_{r-1} \to M_{r-2} \to \ldots \in \mathbb{C}.$$

T is contravariant, hence we get the following picture

$$
\begin{array}{ccccc}
T(M_r)_s & \longleftarrow & T(M_{r-1})_s & \longleftarrow & T(M_{r-2})_s \\
\downarrow & & \downarrow & & \downarrow \\
T(M_r)_{s-1} & \longleftarrow & T(M_{r-1})_{s-1} & \longleftarrow & T(M_{r-2})_{s-1} \\
\downarrow & & \downarrow & & \downarrow \\
T(M_r)_{s-2} & \longleftarrow & T(M_{r-1})_{s-2} & \longleftarrow & T(M_{r-2})_{s-2}
\end{array}
$$

$$\mathbf{T}_{\mathbb{B}}(\mathbf{C})_{\mathbf{p}} = \sum_{\mathbf{r}-\mathbf{s}=\mathbf{p}} (\mathbf{T}(\mathbf{M}_{\mathbf{r}})_{\mathbf{s}})$$

Example 1.3. Let R be a ring with involution. We denote by

$$\Lambda(R) = (\mathbb{A}(R), \mathbb{B}(R), \mathbb{C}(R), (T, e))$$

the algebraic bordism category with

- $\mathbb{A}(R)$ the additive category of finitely generated free R-modules,
- $\mathbb{B}(R)$ the bounded chain complexes in $\mathbb{A}(R)$,
- $\mathbb{C}(R)$ the contractible chain complexes of $\mathbb{B}(R)$ and
- T is defined by $T(M) \coloneqq \mathrm{Hom}_R(M, R)$, so we get
- e as the inverse of the isomorphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$, $x \mapsto (f \mapsto f(x))$.

Before we introduce the main example of an algebraic bordism category which we will use for the total surgery obstruction we want to transfer the concept of L-groups of the previous talk to this setting.

1.1. **L-groups of algebraic bordism categories.** For objects M and N in $\mathbb{A}$ define the chain complex of abelian groups

$$M \otimes_{\mathbb{A}} N \coloneqq \mathrm{Hom}_{\mathbb{A}}(T(M), N)$$

and the duality isomorphism

$$T_{M,N} : M \otimes_{\mathbb{A}} N \to N \otimes_{\mathbb{A}} M,\ (f : TM \to N) \mapsto (e_M \circ T(f) : TN \to T^2M \to M).$$

The properties of e yield

$$T_{M,N} \circ T_{N,M}(f) = e_M \circ T(e_N \circ T(f)) = e_M \circ T^2(f)T(e_N) = f$$

which gives us $(T_{M,N})^{-1} = T_{N,M}$. We can extend this to finite chain complexes C and D in $\mathbb{A}$:

$$C \otimes_{\mathbb{A}} D \coloneqq \mathrm{Hom}_{\mathbb{A}}(T(C), D)$$

and

$$T_{C,D} = \sum (-1)^{pq} T_{C_p,D_q} : C \otimes_{\mathbb{A}} D \to D \otimes_{\mathbb{A}} C,$$
$$\sum_{p+q+r=n} (C_p \otimes_{\mathbb{A}} D_q)_r \to \sum_{p+q+r=n} (D_q \otimes_{\mathbb{A}} C_p)_r$$

Using $T^2_{C,C} = 1$ we can consider $C \otimes_{\mathbb{A}} C$ as a $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex and we are able to use the constructions of the previous talk to get symmetric, quadratic and normal complexes in this setting of additive categories.

$$W_{\%}C \coloneqq W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_{\mathbb{A}} C) \qquad W^{\%}C \coloneqq \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C)$$

In the first talk Q-groups were introduced as homology groups of $W^{\%}C$ and $W_{\%}C$. But in this setting it is enough to work with cycles.

Definition 1.4. An *n-dim. symmetric algebraic complex* (C,φ) in $\Lambda = (\mathbb{A},\mathbb{B},\mathbb{C},(T,e))$ is a chain complex $C \in \mathbb{B}$ together with an n-cycle $\varphi \in (W^{\%}C)_n$ such that $\partial C = \Sigma^{-1}\mathcal{C}(\varphi_0 : \Sigma^n TC \to C) \in \mathbb{C}$

Definition 1.5. An $(n+1)$-*dimensional symmetric pair* $(f : C \to D, (\delta\varphi,\varphi))$ in Λ is a chain map $f : C \to D$ with $C, D \in \mathbb{B}, (\delta\varphi,\varphi) \in \mathcal{C}(f^{\%})$ $(n+1)$-cycle and $\mathcal{C}(\delta\varphi_0, \varphi_0 f^*) \in \mathbb{C}$.

Definition 1.6. A *cobordism* between two n-dimensional symmetric complexes (C,φ) and (C',φ') is an $(n+1)$-dim. symmetric pair $(C \oplus C' \to D, (\delta\psi, \varphi \oplus -\varphi')$ in Λ.

Analogously to the first talk we can also state all these definitions in quadratic and normal flavour. We define the symmetric/quadratic/normal *L-groups*

$$L^n(\Lambda) \;/\; L_n(\Lambda) \;/\; NL^n(\Lambda)$$

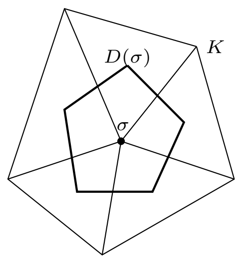
as the cobordism groups of n-dimensional symmetric/quadratic/normal algebraic complexes in Λ.

We use QAC as an abbreviation for 'quadratic algebraic complex respectively a SAC for symmetric one.

Main example 1.7. Let $\Lambda = (\mathbb{A},\mathbb{B},\mathbb{C},(T,e))$ be an algebraic bordism category and K a locally finite simplicial complex. By $\Delta_*(K)$ we denote the simplicial chain complex and by $\Delta^*(K)$ the simplicial cochain complex of K. Local Poincaré duality means for each simplex there is a duality. This local duality comes in two flavours.

(1) We can think of each simplex as a manifold with boundary and so there is a duality between $\Delta_*(\sigma,\partial\sigma)$ and $\Delta^*(\sigma)$

(2) Instead of the simplex itself we can consider its dual cell and in the case that K is a homology manifold we get a duality between $\Delta_*(D(\sigma,K),\partial D(\sigma,K))$ and $\Delta^*(D(\sigma,K))$



We want to encode these two phenomena in two so called *K-based algebraic bordism categories* which differ only in their morphism sets.

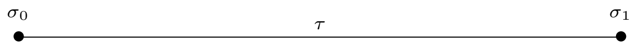
Definition 1.8. The addivitive categories of K-based objects $\mathbb{A}^*(K)$ and $\mathbb{A}_*(K)$ are defined by

$$\mathrm{Obj}(\mathbb{A}^*(K)) = \mathrm{Obj}(\mathbb{A}_*(K)) = \{\textstyle\sum_{\sigma\in K} M_\sigma \mid M_\sigma \in \mathbb{A}\},$$

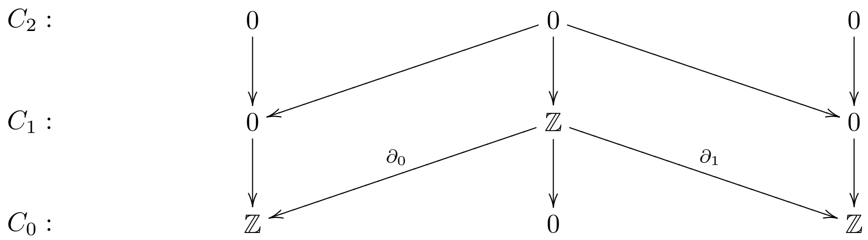
(1) $\mathrm{Mor}(\mathbb{A}^*(K)) =$
$\{\sum\limits_{\sigma\leq\tau} f_{\tau,\sigma} : \sum\limits_{\sigma\in K} M_\sigma \to \sum\limits_{\tau\in K} N_\tau \mid (f_{\tau,\sigma} : M_\sigma \to N_\tau) \in \mathrm{Mor}(\mathbb{A})\}$

(2) $\mathrm{Mor}(\mathbb{A}_*(K)) =$
$\{\sum\limits_{\sigma\geq\tau} f_{\tau,\sigma} : \sum\limits_{\sigma\in K} M_\sigma \to \sum\limits_{\tau\in K} N_\tau \mid (f_{\tau,\sigma} : M_\sigma \to N_\tau) \in \mathrm{Mor}(\mathbb{A})\}$

So in $\mathbb{A}^*$ the morphisms only go from bigger to smaller simplices and in $\mathbb{A}_*$ the other way round. We call a map $f_{\tau,\sigma}$ a skew map if $\sigma \neq \tau$ and a straight map if $\sigma = \tau$. Let's have a look at an example. We can consider the simplicial chain complex of the standard 1-simplex $\Delta_*(\Delta^1)$ as a chain complex in $\mathbb{A}(\mathbb{Z})^*(\Delta^1)$ using the following construction:



$$C(\sigma_0) = \Delta_*(\sigma_0, \partial\sigma_0) \qquad C(\tau) = \Delta_*(\tau, \partial\tau) \qquad C(\sigma_1) = \Delta_*(\sigma_1, \partial\sigma_1)$$



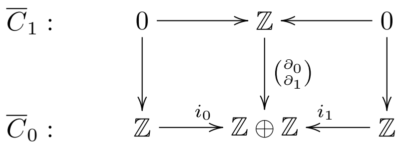
Let C be a chain complex in $\mathbb{A}^*(K)$. For each simplex σ we get a chain complex $C(\sigma)$ by restriction to the corresponding column.

We want to encode in the chain duality functor T^* of $\mathbb{A}^*(K)$ the local Poincaré duality of all simplices of K such that we recover the local Poincaré duality in each column chain complex. But the dimension of these local Poincaré dualities varies with the dimension of the simplices and we have to deal with the boundaries. We need three steps for the construction of T^*:

(1) We assemble in each column chain complex $C(\sigma)$ the column chain complexes of the boundary of σ:

$$\overline{C}(\sigma)_r := C(\sigma)_r \oplus \bigoplus_{\tau \in \partial\sigma} C(\tau)_r.$$

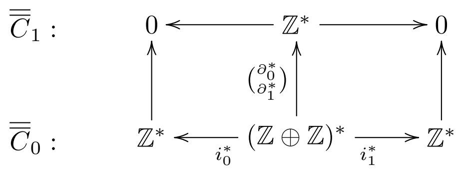
We integrate the the skew maps of C into the straight maps of $\overline{C}$. The skew maps of $\overline{C}$ are all defined to be 0 but we introduce horizontal inclusions $\overline{C}_r(\tau) \to \overline{C}_r(\sigma)$ for $\tau \in \partial\sigma$ which in the next step will become the skew maps in the dual chain complex.



(2) To each column chain complex $\overline{C}(\sigma)$ we apply the chain duality T of $\mathbb{A}$. T gives us a chain complex for each chain module so we have to use the total chain complex to get the dual chain complex

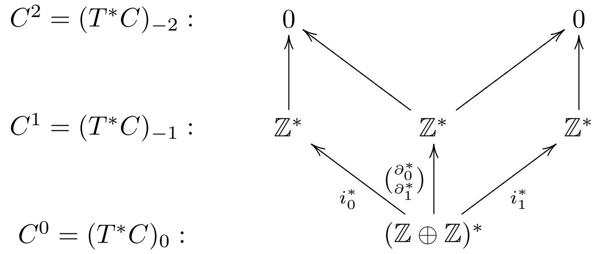
$$\overline{\overline{C}}_r(\sigma) = \sum_{p+q=r} (T_{\mathbb{A}(\mathbb{Z})}(\overline{C}(\sigma)_{-p}))_q.$$

In the example it is rather simple because the dual of each module is a 0-dimensional chain complex.



(3) In the whole dual complex we shift each column $\overline{\overline{C}}(\sigma)$ by the dimension of σ:

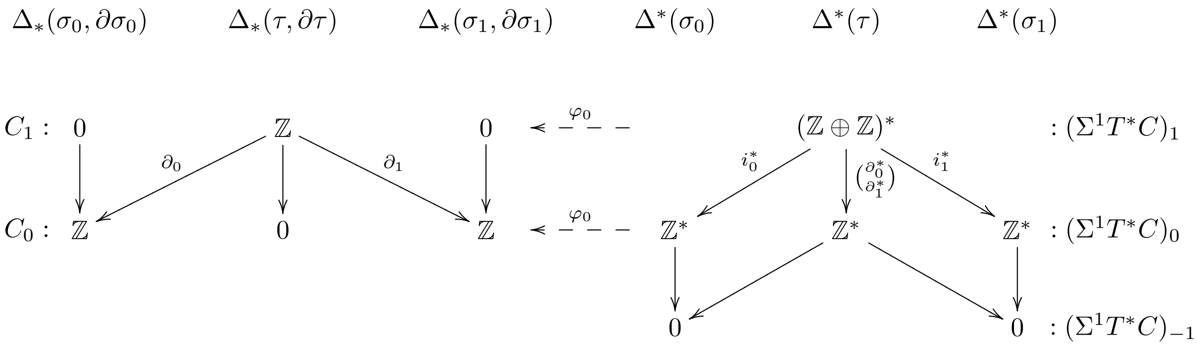
$$T^*(C)(\sigma)_r = \overline{\overline{C}}_{r-|\sigma|}(\sigma).$$



So we end up with

$$T^* : \mathbb{A}^*(K) \to \mathbb{B}^*(K),\ T^*(\sum_{\sigma\in K} M_\sigma))_r(\tau) = (T(\bigoplus_{\tau\geq\tilde{\tau}} M_{\tilde{\tau}}))_{r+|\tau|}.$$

for which $H_{n-k}(C(\sigma)) \cong H^k(C(\sigma))$ holds where n is independent of σ. C in the example is supposed to be an 1-dimensional Poincaré chain complex and in fact a 1-dimensional structure map $\varphi_0 : \Sigma^1 T^*C \to C$ induces isomorphisms on homology $H^{1-n}(T^*C(\sigma)) \cong H_n(C(\sigma))$



In $\mathbb{A}_*(K)$ we want to encode in each column chain complex $C(\sigma)$ a chain complex coming from the dual cell of σ. There are two crucial observations for the definition of T_*:

(1) The higher the dimension of a simplex σ the lower the dimension of the corresponding dual cell, so we have to shift the dual complex in the other direction as for T^* to encode the local Poincaré dualities in the right way.
(2) The boundary of a dual cell $D(\sigma, K)$ consists of the dual cells of bigger simplices by which we mean the simplices which contain σ. So we have to assemble in each column chain complex all column chain complexes coming from bigger simplices. But that's fine because in A_* the skew maps go only to bigger simplices.

This leads us to the definition

$$T_* : \mathbb{A}^*(K) \to \mathbb{B}^*(K),\ T_*(\sum_{\sigma\in K} M_\sigma))_r(\tau) = (T(\bigoplus_{\tau\leq\tilde{\tau}} M_{\tilde{\tau}}))_{r-|\tau|}.$$

An algebraic bordism category induces a K-based algebraic bordism category by the following construction

Definition 1.9. Let $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C}, (T, e))$ be an algebraic bordism category and K a locally finite simplicial complex. Then $\Lambda^*(K)$ is defined by

$\mathbb{A}^*$	$= \mathbb{A}^*(K)$ the additive category of K-based objects in $\mathbb{A}$.
$\mathbb{B}^*$	the bounded chain complexes in $\mathbb{A}^*$
$\mathbb{C}^*$	$= \{C \in \mathbb{B}^* \mid C(\sigma) \in \mathbb{C}$ for all $\sigma \in K\}$.
(T^*, e^*)	the K-based chain duality as constructed above.

Definition 1.10. Let K be a simplicial complex, $L \subset K$ a subcomplex of K and Λ an algebraic bordism category. The induced K-based algebraic bordism category relative L $\Lambda^*(K, L)$ is defined as subcategory of $\Lambda^*(K)$ with $C(\sigma) = 0$ for all $\sigma \in L$ and $C \in \mathbb{B}^*$ chain complex in $\Lambda^*(K, L)$.

1.2. **Constructions.** We have seen in the previous talk how to get out of a "geometric situation" symmetric and quadratic algebraic complexes in $\Lambda(\mathbb{Z})$. There is a way to extend these constructions to $\Lambda(\mathbb{Z})_*(X)$. We start with a triangulated Poincaré complex X for the symmetric construction. For each simplex $\sigma \in X$ we get a map

$$\phi_\sigma : (X, \emptyset) \to (X, X\backslash D(\nu_\sigma)) \simeq (D(\nu_\sigma), \partial D(\nu_\sigma)) \simeq S^{|\sigma|} \wedge (D(\sigma, \partial D(\sigma))$$

using excision. This gives us for each $\sigma \in X$ a class

$$\phi_\sigma([X]) = [D(\sigma), \partial D(\sigma)] \in C_{n-|\sigma|}(D(\sigma), \partial D(\sigma))$$

which fit together to create an n-dimensional symmetric algebraic complex in $\Lambda(\mathbb{Z})_*(X)$.

The input for the quadratic construction was a normal degree one map $f : M \to X$. We used Spanier-Whitehead duality to get a map $\Sigma^k X \to \Sigma^k M$ and ended up with a quadratic complex in $\Lambda(\mathbb{Z})$. Now let X be a triangulated manifold. Making f transverse to $D(\sigma)$ we get a map

$$f_\sigma : f^{-1}(D(\sigma), \partial D(\sigma)) \to (D(\sigma), \partial D(\sigma))$$

for each $\sigma \in X$. We apply to these maps the quadratic construction to get a quadratic complex in $\Lambda(\mathbb{Z})^*(X)$.

2. Surgery sequences

Definition 2.1. A functor of algebraic bordism categories

$$F : \Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C}) \to \Lambda' = (\mathbb{A}', \mathbb{B}', \mathbb{C}')$$

is a covariant functor of additive categories, such that

- $F(B) \in \mathbb{B}'$ for all $B \in \mathbb{B}$,
- $F(C) \in \mathbb{C}'$ for all $C \in \mathbb{C}$,
- for every $A \in \mathbb{A}$ there is a chain map

$$G_A : T'F(A) \to FT(A)$$

with $\mathcal{C}(G_A) \in \mathbb{C}'$ and a commutative diagram

$$\begin{array}{ccc} T'FT(A) & \xrightarrow{G_{T(A)}} & FT^2(A) \\ {\scriptstyle T'G_A}\downarrow & & \downarrow{\scriptstyle Fe_A} \\ T'^2F(A) & \xrightarrow{e'_{F(A)}} & F(A) \end{array} .$$

Proposition 2.2 ([Ran92, Prop. 3.8])**.** *For a functor $F : \Lambda \to \Lambda'$ of algebraic bordism categories there are relative L-groups $L_n(F)$, $L^n(F)$ and $NL^n(F)$ which fit into the long exact sequences*

$$\ldots \to L_n(\Lambda) \to L_n(\Lambda') \to L_n(F) \to L_{n-1}(\Lambda) \to \ldots,$$

$$\ldots \to L^n(\Lambda) \to L^n(\Lambda') \to L^n(F) \to L^{n-1}(\Lambda) \to \ldots,$$

$$\ldots \to NL^n(\Lambda) \to NL^n(\Lambda') \to NL^n(F) \to NL^{n-1}(\Lambda) \to \ldots.$$

There will be no change of $\mathbb{A}$ in the following so it will be omitted from the notation of algebraic bordism categories and we will write $L_n(\mathbb{B}, \mathbb{C})$ for the L-groups of an algebraic bordism category $(\mathbb{A}, \mathbb{B}, \mathbb{C})$. We need the statement of proposition 2.2 only in the special case where F is an inclusion of algebraic bordism categories. In this situation we get a non-relative description for the relative L-groups.

Proposition 2.3 ([Ran92, Prop. 3.9])**.** *Let $(\mathbb{B}, \mathbb{C}, \mathbb{D})$ be a triple of categories of chain complexes over $\mathbb{A}$ with $\mathbb{D} \subset \mathbb{C} \subset \mathbb{B}$. The relative symmetric L-groups for the inclusion $F : (\mathbb{B}, \mathbb{D}) \to (\mathbb{B}, \mathbb{C})$ are given by*

(i) $L^n(F) = L^{n-1}(\mathbb{C}, \mathbb{D})$

and in the quadratic and normal case by

(ii) $L_n(F) = L_{n-1}(\mathbb{C}, \mathbb{D}) = NL^n(F)$.

We are particularly interested in case (ii) where we get by Proposition 2.2 the long exact sequences

(1) $\ldots \to L_n(\mathbb{C}, \mathbb{D}) \to L_n(\mathbb{B}, \mathbb{D}) \to L_n(\mathbb{B}, \mathbb{C}) \to L_{n-1}(\mathbb{C}, \mathbb{D}) \to \ldots$

and

(2) $\ldots \to L_n(\mathbb{C}, \mathbb{D}) \to NL^n(\mathbb{B}, \mathbb{D}) \to NL^n(\mathbb{B}, \mathbb{C}) \to L_{n-1}(\mathbb{C}, \mathbb{D}) \to \ldots.$

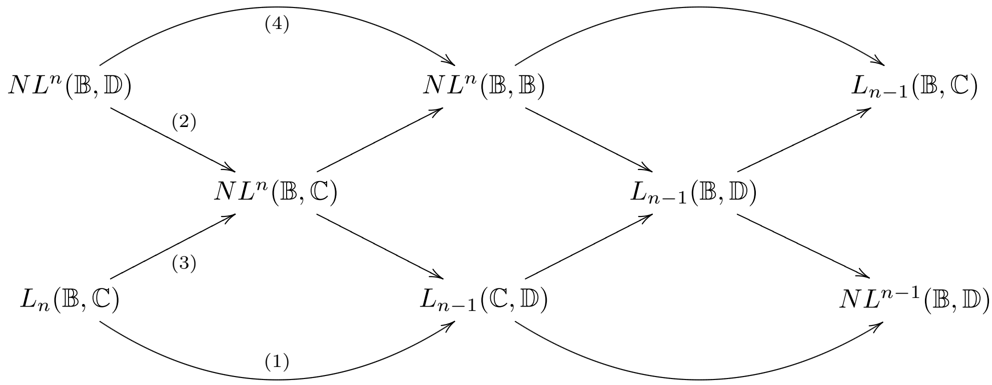
We can complete these two sequences to a whole braid of sequences by considering the inclusions $(\mathbb{B}, \mathbb{C}) \to (\mathbb{B}, \mathbb{B})$ and $(\mathbb{B}, \mathbb{D}) \to (\mathbb{B}, \mathbb{B})$. They induce the long exact sequences

(3) $\ldots \to L_n(\mathbb{B}, \mathbb{C}) \to NL^n(\mathbb{B}, \mathbb{C}) \to NL^n(\mathbb{B}, \mathbb{B}) \to L_{n-1}(\mathbb{B}, \mathbb{C}) \to \ldots$

and

(4) $\ldots \to L_n(\mathbb{B}, \mathbb{D}) \to NL^n(\mathbb{B}, \mathbb{D}) \to NL^n(\mathbb{B}, \mathbb{B}) \to L_{n-1}(\mathbb{B}, \mathbb{D}) \to \ldots.$

We get the following braid.



Comments on the proofs of 2.2 and 2.3. Let $F : \Lambda \to \Lambda'$ be a functor of algebraic bordism categories. An element in $L_n(F)$ is an $(n-1)$-dimensional QAC (C, ψ) in $L_n(\Lambda)$ together with a quadratic pair $(F(C) \to D, (\delta\psi, F(\psi)))$. The maps in the sequence are given by

$$\cdots \to L_n(\Lambda) \xrightarrow{F} L_n(\Lambda') \longrightarrow L_n(F) \longrightarrow L_{n-1}(\Lambda) \to \cdots$$

$$(C, \psi) \longmapsto (F(C), F(\psi))$$

$$(C', \psi') \longmapsto ((0,0), F(0) \to C', (\psi', F(0)))$$

$$((C, \psi), F(C) \to D, (\delta\psi, F(\psi))) \mapsto (C, \psi)$$

Essentially this exact sequence is an analogue of the long exact sequence of cobordism groups $\Omega_n(X)$. For the non-trivial parts of the proof of exactness a definition of cobordism of pairs is necessary which is given in [Ran81].

The isomorphism $L_n(F) \cong L_{n-1}(\mathbb{C}, \mathbb{D})$ is given by

$$((C, \psi), C \to D, (\delta\psi, \psi)) \mapsto (C', \psi')$$

where (C', ψ') is the effect of algebraic surgery on (C, ψ) by the pair $(C \to D, (\delta\psi, \psi))$. The inverse is given by

$$(C, \psi) \mapsto ((C, \psi), C \to 0, (0, \psi)) \,.$$

Similar for $NL_n(F) \cong L_{n-1}(\mathbb{C}, \mathbb{D})$. We perform algebraic surgery on an $(n-1)$-dimensional algebraic complex (C, ψ) using the normal pair $(C \to D, (\delta\psi, \psi))$ where $((C, \psi), C \to D, (\delta\psi, \psi)) \in NL_n(F)$ and get an $(n-1)$-dimensional quadratic complex (see [Ran92] 2.8(ii)). □

Now we define for a locally finite simplicial complex K a special algebraic bordism category that we put into the braid to get the exact sequences we need for the total surgery obstruction.

(1) $\mathbf{A} = \mathbb{A}(R)_*(K)$ is the additive category of finitely generated free R-modules based over K.
(2) $\mathbf{B} = \mathbb{B}(\mathbb{A}(R)_*(K))$ as usual the bounded chain complexes in $\mathbb{A}$
(3) $\mathbf{C} = \{C \in \mathbb{B} \mid A(C) \simeq *\}$ are the *globally contractible* chain complexes where $A : \mathbb{A}_*(K) \to \mathbb{A}[\pi_1(K)]$ is the *assembly* functor defined by

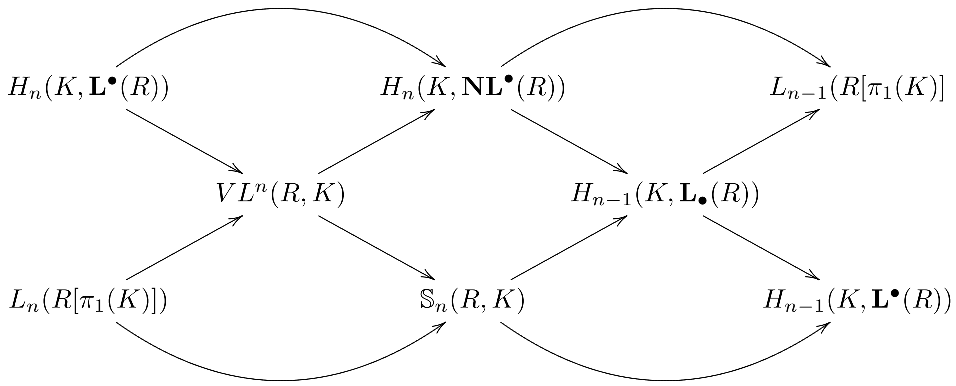
$$M \mapsto \sum_{\tilde{\sigma} \in \tilde{K}} M(p(\tilde{\sigma}))$$

(4) $\mathbf{D} = \{C \in \mathbb{B} \mid C(\sigma)\ simeq * \forall \sigma \in K\}$ is the category of *locally contractible* chain complexes.

We use the equations

$$L_{n-1}(\mathbf{C}, \mathbf{D}) = \mathbb{S}_n(R, K) \tag{2.1}$$
$$NL^n(\mathbf{B}, \mathbf{C}) = VL^n(R, K) \tag{2.2}$$
$$L_n(\mathbf{B}, \mathbf{C}) = L_n(R[\pi_1(K)]) \tag{2.3}$$
$$L_n(\mathbf{B}, \mathbf{D}) = H_n(K, \mathbf{L}_\bullet(R)) \tag{2.4}$$
$$NL^n(\mathbf{B}, \mathbf{D}) = H_n(K, \mathbf{L}^\bullet(R)) \tag{2.5}$$
$$NL^n(\mathbf{B}, \mathbf{B}) = H_n(K, \mathbf{NL}^\bullet(R)) \tag{2.6}$$

to get the following version of the braid.



(2.1) is just a definition. The structure group $\mathbb{S}_n(R, K)$ is where the total surgery obstruction lives. Strictly speaking not exactly in this group but a connective version. See chapter 15 and 17 of [Ran92] for details or the notes of the third talk for a brief overview.

(2.2) is also just a definition. The groups $VL^n(R, K)$ are called visible L-groups. They coincide with the visible L-groups of [Wei92] in some special cases. See Remark 9.8 in [Ran92].

(2.3) is the result of the algebraic π-π-theorem ([Ran92, chapter 10]).

(2.4)-(2.6) The algebraic bordism categories $(\mathbf{A}, \mathbf{B}, \mathbf{D})$ and $(\mathbf{A}, \mathbf{B}, \mathbf{B})$ are induced by $(\mathbb{A}(R), \mathbb{B}(R), \mathbb{C}(R))$ and $(\mathbb{A}(R), \mathbb{B}(R), \mathbb{B}(R))$ where $\mathbb{C}(R)$ is the category of contractible chain complexes of finitely generated free R-modules. That is the formal reason why we can describe these groups in terms of homology groups with coefficients in L-theory spectra. For (2.5) we also use that every symmetric algebraic complex in $(\mathbf{B}, \mathbf{D})$ has a normal structure ([Ran92, Prop. 3.5]).

3. L-Spectra

Let's have a closer look at L-theory spectra to get an idea why we get a description for some terms in the braid as homology groups. We restrict in the following to the quadratic case. Everything works in exactly the same way for the symmetric and normal cases just by replacing the word "quadratic" in the definitions. The standard n-simplex is denoted by Δ^n.

We briefly recall the definitions and most important facts we will need about (Kan) Δ-sets. We consider Δ-sets only without degeneracy maps so a Δ-set K comes with a sequence of sets K^n for each dimension $n \geq 0$ and with face maps $\partial_i^n : K^n \to K^{n-1}$ such that $\partial_i^n \partial_j^{n-1} = \partial_{j-1}^n \partial_i^{n-1}$ for $j > i$. A Δ-map $f : K \to L$ sends k-simplices to k-simplices: $f(K^{(k)}) \subset L^{(k)}$. For Δ-sets K, L we define the *function* Δ*-set* L^K to be the Δ-set whose n-simplices are the Δ-maps $K \times \Delta^n \to L$ and face maps induced by the inclusions $\partial_i : \Delta^{n-1} \to \Delta^n$.

Let $\Lambda_i^n = \partial\Delta^n \setminus \partial_i^n \Delta^n$ be the boundary of the standard n-simplex with the i-th face removed. We say a Δ-set K has the Kan extension property if any Δ-map $f : \Lambda_i^n \to K$ extends to a Δ-map $\tilde{f} : \Delta^n \to K$ not necessarily in an unique way. For instance in the simplicial world the standard simplices are Kan, in the singular world everything is Kan. The Kan property gives us that homotopy is a equivalence relation where a *homotopy* of Δ-maps $f_0, f_1 : K \to L$ is defined to be a Δ-map $h : K \oplus \Delta^1 \to L$ with the expected conditions. A Δ-set K is *pointed* if there is a base simplex $\emptyset \in K^{(n)}$ for each $n \geq 0$. We denote by K_+ the Δ-set K with an added base point in each dimension. For a pointed Kan Δ-set K the *homotopy groups* $\pi_n(K)$ are given by

$$\pi_n(K) \coloneqq [\partial\Delta^{n+1}, K] = \{x \in K^{(n)} \mid \partial_i x = \emptyset \in K^{(n-1)}, 0 \leq i \leq n\}/\sim$$

with $x \sim y$ if there is a $z \in K^{(n+1)}$ such that

$$\partial_i z = \begin{cases} x & \text{if } i = 0 \\ y & \text{if } i = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

The *loop* Δ*-set* ΩK is defined by

$$(\Omega K)^{(n)} = \left(K^{S^1}\right)^{(n)} = \{x \in K^{(n+1)} \mid \partial_0\partial_1 \ldots \partial_n x = \emptyset \in K^{(0)}, \partial_{n+1} x = \emptyset \in K^n\}$$

where S^1 is the pointed Δ-set defined by

$$(S^1)^{(n)} = \begin{cases} \{s, \emptyset\} & \text{if } n = 1 \\ \{\emptyset\} & \text{if } n \neq 1. \end{cases}$$

Let Λ be an algebraic bordism category. We assume K to be a finite Δ-set. Everything works fine with even locally finite Δ-sets using certain colimit constructions.

Proposition 3.1 (Shift principle)**.** *Let (C, φ) be an n-dimensional quadratic complex in $\Lambda^*(K)$ and $\sigma \in K$. Then (C, φ) restricted to the column $(C(\sigma), \varphi(\sigma))$ in $\Lambda^*(K)$ is an $n + |\sigma|$-dimensional quadratic complex in Λ.*

Proof. φ is an n-dimensional quadratic structure on the 0-simplex column chain complexes $C(\tau)$, $|\tau| = 0$. Because of the shift in the chain duality

$$T^* : \mathbb{A}^*(K) \to \mathbb{B}^*(K),\ T^*(\sum_{\sigma \in K} M_\sigma))_{r(\tau)} = (T_{\mathbb{A}(\mathbb{Z})}(\bigoplus_{\tau \geq \tilde{\tau}} M_{\tilde{\tau}}))_{r+|\tau|}.$$

φ is an $n + |\sigma|$-dimensional quadratic structure restricted to $C(\sigma)$. □

Definition 3.2. Let $\mathbf{L}_n(\Lambda)$ be the pointed Δ-set with k-simplices

$$\mathbf{L}_n(\Lambda)^{(k)} = \{n\text{-dim. quadratic complexes in } \Lambda^*(\Delta^k)\}.$$

The face maps

$$\mathbf{L}_n(\Lambda)^{(k)} \to \mathbf{L}_n(\Lambda)^{(k-1)}, \quad (C, \varphi) \in \Lambda^*(\Delta^k) \mapsto (C', \varphi') \in \Lambda^*(\Delta^{k-1})$$

are induced by the face inclusions $\partial_i : \Delta^{k-1} \to \Delta^k$ and the base point is the 0-chain complex.

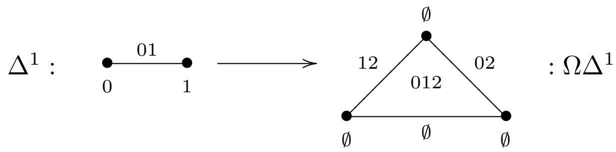
Proposition 3.3. $\mathbf{L}_\bullet(\Lambda) \coloneqq \{\mathbf{L}_n(\Lambda) \mid n \in \mathbb{Z}\}$ *is an Ω-spectrum of pointed Kan Δ-sets with $\pi_n(\mathbf{L}_\bullet(\Lambda)) \cong L_n(\Lambda)$.*

Proof. For the proof of the Kan extension condition we refer to [Ran92, p. 137].

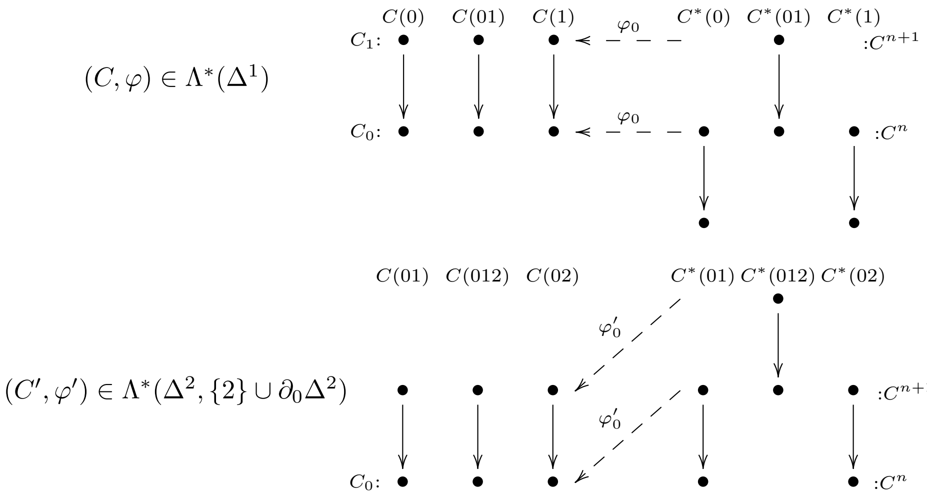
For the Ω-spectrum property we prove that in fact $(\mathbf{L}_{n+1}(\Lambda))^{(k)}$ and $(\Omega\mathbf{L}_n(\Lambda))^{(k)}$ are different descriptions of the same Δ-set. Be aware that the index goes in the opposite direction than usual. Let

(C,φ) be an $(n+1)$-dimensional QAC in $\Lambda^*(\Delta^k)$ which is a k-simplex in $\mathbf{L}_{n+1}(\Lambda)$ and
(C',φ') an n-dimensional QAC in $\Lambda^*(\Delta^{k+1},\{k+1\}\cup\partial_0\Delta^{k+1}) = \Lambda^*(\Omega\Delta(k))$ which is a k-simplex in $\Omega\mathbf{L}_n(\Lambda)$

There is a one-to-one correspondence between n-dimensional simplices in Δ^k and $(n+1)$-dimensional simplices in $\Omega\Delta(k)$.



On the chain complex level we have $C'(\{k+1\}) = 0$ and $C'(\sigma) = 0$ for every $\sigma \in \partial_0\Delta^{k+1}$. So the shift principle yields a one-to-one correspondence between $(n+1)$-dimensional QACs in $\Lambda^*(\Delta^k)$ and n-dimensional QACs in $\Lambda^*(\Delta^{k+1},\{k+1\}\cup\partial_0\Delta^k)$ The following picture visualizes what happens for $k=1$.



The proof of $\pi_n(\mathbf{L}_\bullet(\Lambda)) = L_n(\Lambda)$ is essentially the same story. A k-simplex in $\pi_n(\mathbf{L}_m(\Lambda))$ is an m-dimensional QAC (C,φ) in $\Lambda^*(K)$ with $C(\sigma)=0$ for all $\sigma\in K$ with $|\sigma|<n$. There is only one non-trivial column chain complex in C namely the biggest simplex τ with $|\tau|=n$. Using the shift principle we see that (C,φ) is an $(n+m)$-dimensional QAC in Λ. The homotopy relation corresponds to the cobordism relation hence $\pi_n(\mathbf{L}_\bullet(\Lambda)) = \pi_{n+m}(\mathbf{L}_{-m}(\Lambda)) = L_n(\Lambda)$. □

Proposition 3.4. *Let K be a finite simplicial complex and Λ an algebraic bordism category. Then*

(i) $\mathbf{L}_\bullet(\Lambda)^{K_+} \cong \mathbf{L}_\bullet(\Lambda^*(K))$
(ii) $K_+ \wedge \mathbf{L}_\bullet(\Lambda) \simeq \mathbf{L}_\bullet(\Lambda_*(K))$

Remark 3.5. In particular we get

$$L_n(\Lambda) = \pi_n(\mathbf{L}_\bullet(\Lambda_*(K))) = \pi_n(K_+\wedge\mathbf{L}_\bullet(\Lambda)) = H_n(K,\mathbf{L}_\bullet(\Lambda))$$

and so for our special choice we made for the braid

$$L_n(\mathbf{B},\mathbf{D}) = L_n(\Lambda(R)_*(K)) = H_n(K,\mathbf{L}_\bullet(R)).$$

Proof of (i). We have to show $\mathbf{L}_\bullet(\Lambda^*(K)) = \mathbf{L}_\bullet(\Lambda)^{K_+}$. In the upper star category the skew maps go only from bigger simplices to smaller simplices. Hence we can split an n-dimensional QAC $(C,\varphi)\in\Lambda^*(K)$ into a collection of n-dimensional QAC

$\{(C_\sigma, \varphi_\sigma) \in \Lambda^*(\Delta^{|\sigma|})\}$ such that the $(C_\sigma, \varphi_\sigma)$ are related to each oter in the same way the corresponding simplices are related to each other. the chain complexes $(C_\sigma, \varphi_\sigma)$ fit together. $(C_\sigma, \varphi_\sigma)$ is an $|\sigma|$-simplex in $\mathbf{L}_n(\Lambda)$ and the compatibility conditions are contained in the notion of Δ-maps. Hence we get

$$\begin{aligned}(C, \varphi) &= \{n\text{-dim. QAC } (C_\sigma, \varphi_\sigma) \in \Lambda^*(\Delta^{|\sigma|}) \mid \\ &\qquad\qquad \sigma \in K \text{ and } C_\sigma(\partial_i \sigma) = C_{\partial_i \sigma}(\partial_i \sigma)\} \\ &= \Delta\text{-map } f_C : K_+ \to \mathbf{L}_n(\Lambda) \text{ with } f(\sigma) = C_\sigma\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{L}_n(\Lambda^*(K))^{(k)} &= \{n\text{-dim. QAC } (C, \varphi) \in \Lambda^*(K)^*(\Delta^k) = \Lambda^*(K \times \Delta^k)\} \\ &= \{f : (K \times \Delta^k)_+ \to \mathbf{L}_n(\Lambda) \mid f \text{ is a pointed } \Delta\text{-map}\} \\ &= (\mathbf{L}_n(\Lambda)^{K_+})^{(k)}\end{aligned}$$

□

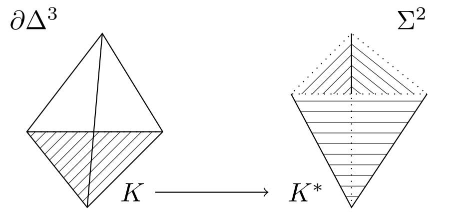
Proof of (ii). For $m \in \mathbb{N}$ large enough there is an embedding $i : K \to \partial\Delta^{m+1}$. Let Σ^m be the simplicial complex with one k-simplex σ^* for each $(m-k)$-simplex $\sigma \in \partial\Delta^{m+1}$ and σ^* is a face of τ^* in Σ^m if and only if τ is a face of σ in $\partial\Delta^{m+1}$. The *supplement* of K is the the subcomplex

$$\overline{K} \coloneqq \{\sigma^* \in \Sigma^m \mid \sigma \in \partial\Delta^{m+1} \setminus i(K)\} \subset \Sigma^m.$$

There are two ways in which we map between Σ^m and $\partial\Delta^{m+1}$. First there is a map

$$* : \partial\Delta^{m+1} \to \Sigma^m,\ \sigma \mapsto \sigma^*$$

which is a bijection between $(\partial\Delta^{m+1})^{(m-k)}$ and $(\Sigma^m)^{(k)}$ and sends face maps $\partial_i : (\partial\Delta^{m+1})^{(m-k)} \to (\partial\Delta^{m+1})^{(m-k-1)}$ to inclusions $\partial_i^* : (\Sigma^m)^{(k)}) \to (\Sigma^m)^{(k+1)}$.



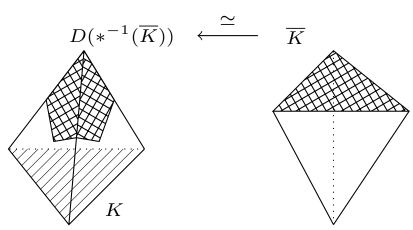
This map yield

$$\mathbf{L}_n(\Lambda^*(\Sigma, \overline{K})) = \mathbf{L}_n(\Lambda_*(K)). \tag{3.1}$$

On the other hand we get for each $\sigma^* \in \Sigma^m$ a homotopy equivalence between σ^* and the dual cell $D(\sigma, \partial\Delta^{m+1})$. We apply this homotopy equivalence to the supplement $\overline{K}$ and we get

$$\Sigma^m / \overline{K} \simeq N(K)/\partial N(K)$$

where $N(K)$ is a neighbourhood of K in $\partial\Delta^{m+1}$. For $m = 2$ it looks like this:



Using Spanier-Whitehead duality (see [Whi62], 7.5) we get

$$\begin{aligned} K_+ \wedge \mathbf{L}_n(\Lambda) &\overset{SW}{=} \mathbf{L}_n(\Lambda)^{(\Sigma^m,\overline{K})} \\ &\overset{(i)}{=} \mathbf{L}_n(\Lambda^*(\Sigma^m,\overline{K})) \\ &\overset{(3.1)}{=} \mathbf{L}_n(\Lambda_*(K)) \end{aligned}$$

□

Remark 3.6. There is also a proof without these simplex constructions by Weiss in [Wei92] where he shows that the functor $K \to \mathbf{L}_\bullet(\Lambda_*(K))$ from Δ-sets to spectra is homotopy invariant and excisive.

References

[Ran81] A. Ranicki. Exact sequences in the algebraic theory of surgery, volume 26 of Mathematical Notes, 1981.

[Ran92] A. Ranicki. *Algebraic L-theory and topological manifolds.* Cambridge Univ Pr, 1992.

[Wei92] M. Weiss. Visible L-theory. In *Forum Math*, volume 4, pages 465–498, 1992.

[Whi62] G.W. Whitehead. Generalized homology theories. *Transactions of the American Mathematical Society*, pages 227–283, 1962.