MODULAR TENSOR CATEGORIES FROM QUANTUM DOUBLES OF FINITE GROUPS

These notes grew out of the question for the easiest non-trivial examples of modular tensor categories. These are given by the representation categories of the Drinfeld double of finite groups. There is a very short account of this in the book by Bakalov and Kirillov [1, section 3.2]. We try to be as self-contained as possible in these notes.

Some important examples of modular tensor categories arise from representation categories of Hopf algebras, others from certain representation categories in conformal field theory [3]. The representation category of the quantum double is a simple yet important toy example of the first construction. Since this is the special case of the quantum double of a Hopf algebra, we will need the basics about the latter first.

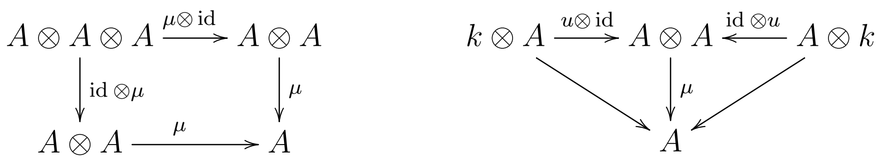
1. Basics about Hopf algebras

As we have seen in the example of 2-dimensional topological quantum field theories, the complex group ring $A = \mathbb{C}G$ of a finite group G has a lot of algebraic structure. First of all, it is an algebra over $\mathbb{C}$ with respect to the multiplication

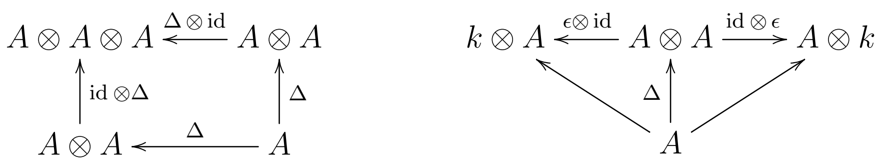
$$\mu\colon A \otimes A \to A \quad ; \quad \left(\sum_{g \in G} \lambda_g g\right) \otimes \left(\sum_{h \in G} \rho_h\, h\right) \mapsto \sum_{g,h \in G} \lambda_g\, \rho_h\, gh\ .$$

with unit $e \in G \subset \mathbb{C}G$. Dual to this notion $\mathbb{C}G$ also carries a coalgebra structure, which turns it into a bialgebra in the sense of the following definition [5].

Definition 1.1. A *bialgebra* A over a field k is a tuple $(A, \mu, u, \Delta, \epsilon)$, such that (A, μ, u) is an algebra over k with multiplication $\mu\colon A \otimes A \to A$ and unit $u\colon k \to A$ together with a *comultiplication* $\Delta\colon A \to A \otimes A$ and a *counit* $\epsilon\colon A \to k$, such that Δ and ϵ are algebra homomorphisms. The algebra axioms (associativity and unit) are encoded in the commutativity of the following diagrams



in which the unlabeled arrows mark the canonical isomorphisms. The axioms of a coalgebra yield the commutativity of



and the compatibility of both structures forces the following diagrams to commute

(1)
$$\begin{CD} A\otimes A @>{\mu}>> A \\ @V{\Delta\otimes\Delta}VV @VV{\Delta}V \\ A\otimes A\otimes A\otimes A @>{\tau}>> A\otimes A\otimes A\otimes A \xrightarrow{\mu\otimes\mu} A\otimes A \end{CD} \qquad \begin{CD} A @>{\Delta}>> A\otimes A \\ @A{u}AA @AA{u\otimes u}A \\ k @<<< k\otimes k \end{CD}$$

(2)
$$\begin{CD} A\otimes A @>{\mu}>> A \\ @V{\epsilon\otimes\epsilon}VV @VV{\epsilon}V \\ k\otimes k @>>> k \end{CD} \qquad \begin{array}{ccc} A & \xrightarrow{\epsilon} & k \\ {\scriptstyle u}\uparrow & \nearrow_{=} & \\ k & & \end{array}$$

where the unlabeled arrows are again canonical and $\tau\colon A^{\otimes 4} \to A^{\otimes 4}$ is given by $\tau(a\otimes b\otimes c\otimes d) = a\otimes c\otimes b\otimes d$.

Example 1.2. The complex group algebra $A = \mathbb{C}G$ with the above multiplication and unit together with the comultiplication

$$\Delta\colon A \to A\otimes A \quad ; \quad \sum_{g\in G} \lambda_g\, g \mapsto \sum_{g\in G} \lambda_g\,(g\otimes g)$$

and counit $\epsilon(\sum_{g\in G} \lambda_g g) = \sum_{g\in G} \lambda_g$ is a bialgebra. Checking this is a nice exercise.

Warning! We have seen that $\mathbb{C}G$ also carries a trace

$$\mathrm{tr}\colon A \to \mathbb{C} \quad ; \quad \sum_{g\in G} \lambda_g\, g \mapsto \lambda_e\ ,$$

which endows it with the structure of a *Frobenius* algebra. The coalgebra structure obtained from this differs from the one in the above definition in that the conditions (1) and (2) are not satisfied for it. The comultiplication $\delta\colon A \to A\otimes A$ of a Frobenius algebra only has to be an A-*module homomorphism*, whereas for a bialgebra we demand $\Delta\colon A \to A\otimes A$ to be an *algebra homomorphism* with respect to the algebra structure on A. Therefore Frobenius algebras and bialgebras should be considered as two different kind of structures.

The group algebra $A = \mathbb{C}G$ has yet another feature not encoded in the bialgebra structure: Each group element has an *inverse*. This leads to the following definition

Definition 1.3. Let A be a bialgebra. A k-linear homomorphism $S\colon A \to A$ is called an *antipode* if it fits into the following commutative diagram

$$\begin{array}{ccccccc} & & A\otimes A & \xrightarrow{S\otimes \mathrm{id}_A} & A\otimes A & & \\ & {\scriptstyle\Delta}\nearrow & & & & \searrow{\scriptstyle\mu} & \\ A & & \xrightarrow{\epsilon} & k & \xrightarrow{u} & & A \\ & {\scriptstyle\Delta}\searrow & & & & \nearrow{\scriptstyle\mu} & \\ & & A\otimes A & \xrightarrow{\mathrm{id}_A\otimes S} & A\otimes A & & \end{array}$$

A bialgebra A together with an antipode $S\colon A \to A$ is called a *Hopf algebra*.

Example 1.4. The complex group algebra $A = \mathbb{C}G$ is a Hopf algebra with the antipode

$$S\left(\sum_{g\in G}\lambda_g g\right) = \sum_{g\in G}\lambda_g g^{-1} \ .$$

Remark 1.5. Using Δ and μ we can turn $\mathrm{Hom}(A,A)$ into an algebra: Let $f,g \in \mathrm{Hom}(A,A)$. Then we define

$$f * g\colon\ A \xrightarrow{\Delta} A\otimes A \xrightarrow{f\otimes g} A\otimes A \xrightarrow{\mu} A$$

This multiplication, which is *different* from the canonical one given by composition, turns $\mathrm{Hom}(A,A)$ into an algebra. The unit with respect to $*$ is given by

$$u\circ\epsilon\colon A \to k \to A \ .$$

Therefore the condition that $S\colon A \to A$ is an antipode means that it is an inverse of id_A with respect to $*$, i.e. $\mathrm{id}_A * S = S * \mathrm{id}_A = u\circ\epsilon$. Unravelling the definition of $*$ yields the commutative diagram in definition 1.3.

Lemma 1.6. *Let A be a Hopf algebra over a field k, which is finite dimensional as a k-vector space. Then its dual space $A^* = \mathrm{Hom}(A,k)$ is in a canonical way described in the proof again a Hopf algebra.*

Proof. Since $(A\otimes A)^* = A^*\otimes A^*$ in the finite dimensional case, the space A^* is a bialgebra with multiplication $\mu' = \Delta^*\colon A^*\otimes A^* \to A^*$, unit $u' = \epsilon^*\colon k \to A^*$, comultiplication $\Delta' = \mu^*\colon A^* \to A^*\otimes A^*$ and counit $\epsilon' = u^*\colon A^* \to k$. Since the dualizing functor is contravariant, it turns the first two diagrams of definition 1.1 into the second two diagrams and vice versa. The compatibility condition of the comultiplication Δ and the counit ϵ was that they are algebra homomorphisms. Rotating the diagrams (1) and (2) by 90 degrees, we see that this condition is equivalent to the fact that μ and u are *coalgebra homomorphisms*. Therefore, the corresponding diagrams for Δ', μ', ϵ' and u' still commute.

If we apply the dualizing functor to the defining diagram of the antipode given in definition 1.3, it follows from the symmetry of that diagram that $S^*\colon A^* \to A^*$ is an antipode for the bialgebra A^*. □

Example 1.7. Let $A = \mathbb{C}G$ be the complex group algebra of a finite group G. As we have seen above, A is a Hopf algebra. A basis for this algebra is given by the elements $g \in G \subset A$. The dual space A^* also has a canonical basis, given by the linear functionals $\delta_g \in \mathrm{Hom}(A,\mathbb{C})$ with

$$\delta_g(h) = \delta_{g,h} = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{else} \end{cases} \ .$$

Let $g,h,x \in G$. Then the multiplication $\mu' = \Delta^*$ in A^* is defined on the basis elements δ_g and δ_h by

$$\begin{aligned}(\delta_g\cdot\delta_h)(x) &:= \mu'(\delta_g\otimes\delta_h)(x) = \Delta^*(\delta_g\otimes\delta_h)(x) \\ &= (\delta_g\otimes\delta_h)(\Delta(x)) = (\delta_g\otimes\delta_h)(x\otimes x) = \delta_{g,x}\,\delta_{h,x} = \delta_{g,h}\,\delta_g(x) \ .\end{aligned}$$

Therefore we get $\delta_g \cdot \delta_h = \delta_h \cdot \delta_g = \delta_{g,h}\, \delta_g$. This means that A^* is a commutative algebra. Its unit is given by $u' \colon \mathbb{C} \to A^*$ with

$$u'(1)(x) = 1 \text{ for all } x \in G \quad \Leftrightarrow \quad u'(1) = \sum_{g \in g} \delta_g \ .$$

For the comultiplication of a basis element $\delta_g \in A^*$ we get with $x, y \in G$:

$$\begin{aligned}\Delta'(\delta_g)(x \otimes y) &:= \mu^*(\delta_g)(x \otimes y) = \delta_g(\mu(x \otimes y)) = \delta_g(xy) = \delta_{g,xy} \\ &= \sum_{h \in G} \delta_{g,xh}\, \delta_{h,y} = \sum_{h \in G} \delta_{gh^{-1},x}\, \delta_{h,y} = \sum_{h \in G} (\delta_{gh^{-1}} \otimes \delta_h)(x \otimes y) \ .\end{aligned}$$

Therefore $\Delta'(\delta_g) = \sum_{h \in G} \delta_{gh^{-1}} \otimes \delta_h$ with counit $\epsilon' \colon A^* \to \mathbb{C}$ given by $\epsilon'(\delta_g) = \delta_{g,e}$. The antipode $S' \colon A^* \to A^*$ is obtained from

$$S'(\delta_g)(x) = \delta_g(S(x)) = \delta_g(x^{-1}) = \delta_{g,x^{-1}} = \delta_{g^{-1}}(x)$$

for all $x \in G$, i.e. $S'(\delta_g) = \delta_{g^{-1}}$.

2. Representation categories of Hopf algebras

Finite dimensional representations of a finite group G yield an easy example of a rigid monoidal category with a symmetry. In this section we will generalize this concept to the case of Hopf algebras.

Definition 2.1. Let A be an algebra over the field k. A *representation* of A is a k-vector space V together with an algebra homomorphism $\pi \colon A \to \mathrm{End}(V)$. Two representations π on V and π' on V' are called *equivalent* if there exists an isomorphism $T \colon V \to V'$ such that for all $a \in A$ and $v \in V$

$$T\,\pi(a)\,v = \pi'(a)\,T\,v \ . \tag{3}$$

A k-linear map $T \colon V \to V'$ that satisfies (3) for all $a \in A$ and $v \in V$ will be called *equivariant homomorphism* or *intertwiner*. A representation V is called *finite-dimensional*, if $\dim(V) < \infty$. Let (V, π) be a representation. A subspace $W \subset V$ is called *invariant*, if $\pi(A)W \subset W$. A representation V is called *irreducible*, if 0 and V are its only invariant subspaces. Denote by $\mathrm{Rep}(A)$ the category with objects the finite dimensional representations of A and with intertwiners as morphisms. Equivalence classes of representations correspond to isomorphism classes of objects in this category.

Remark 2.2. Most people would say (left) A-module instead of representation. We stick to our notation, due to the close connection between Hopf algebras and group theory.

In case A is a bialgebra, the category $\mathrm{Rep}(A)$ can be endowed with a tensor product as follows. Let $(V, \pi_V), (W, \pi_W) \in \mathrm{obj}(\mathrm{Rep}(A))$, then the tensor product $V \otimes W$ over k is again a representation with the action of A given by

$$\pi_V * \pi_W \colon A \xrightarrow{\Delta} A \otimes A \xrightarrow{\pi_V \otimes \pi_W} \mathrm{End}(V) \otimes \mathrm{End}(W) \longrightarrow \mathrm{End}(V \otimes W) \ .$$

The ground field k also provides a representation via the counit $\pi_k = \epsilon \colon A \to k = \mathrm{Hom}(k, k)$, i.e. $a \cdot \lambda = \epsilon(a) \cdot \lambda$ for $a \in A$ and $\lambda \in k$. It is easy to check that $(V \otimes k, \pi_V * \pi_k)$ and $(k \otimes V, \pi_k * \pi_V)$

are naturally isomorphic to (V, π_V) via the obvious isomorphism. This yields the tensor unit in the monoidal category.

Let $\tau_{V,W} : V \otimes W \to W \otimes V$ be the canonical twist map, let $\tau = \tau_{A,A}$. Observe that

$$\tau_{V,W} \circ (\pi_V * \pi_W)(a) \circ \tau_{V,W}^{-1} = (\pi_W *^{\mathrm{op}} \pi_V)(a) \ ,$$

where $\pi_W *^{\mathrm{op}} \pi_V$ is defined via

$$\pi_W *^{\mathrm{op}} \pi_V : A \xrightarrow{\Delta} A \otimes A \xrightarrow{\tau} A \otimes A \xrightarrow{\pi_W \otimes \pi_V} \mathrm{End}(W \otimes V) \ .$$

Since $\tau \circ \Delta$ may not coincide with Δ, the objects $(V \otimes W, \pi_V * \pi_W)$ and $(W \otimes V, \pi_W * \pi_V)$ may be non-isomorphic in $\mathrm{Rep}(A)$. To obtain a *braided* monoidal category, we need further structure on the bialgebra A.

Definition 2.3. Let A be a bialgebra. An element $R \in A \otimes A$ is called a *universal R-matrix*, if it is invertible and satisfies the following axioms:

a) $(\tau \circ \Delta)(a)\, R = R\, \Delta(a)$,
b) $(\mathrm{id}_A \otimes \Delta)\, R = R_{13}\, R_{12}$,
c) $(\Delta \otimes \mathrm{id}_A)\, R = R_{13}\, R_{23}$.

Here we used the following notation: If $R = \sum_i a_i \otimes b_i \in A \otimes A$, then

$$R_{13} = \sum_i a_i \otimes 1 \otimes b_i \quad \text{and} \quad R_{23} = \sum_i 1 \otimes a_i \otimes b_i$$

and so on. A bialgebra A together with a universal R-matrix is called *quasi-triangular.*

Lemma 2.4. *Let A be a quasi-triangular bialgebra with universal R-matrix R. Then*

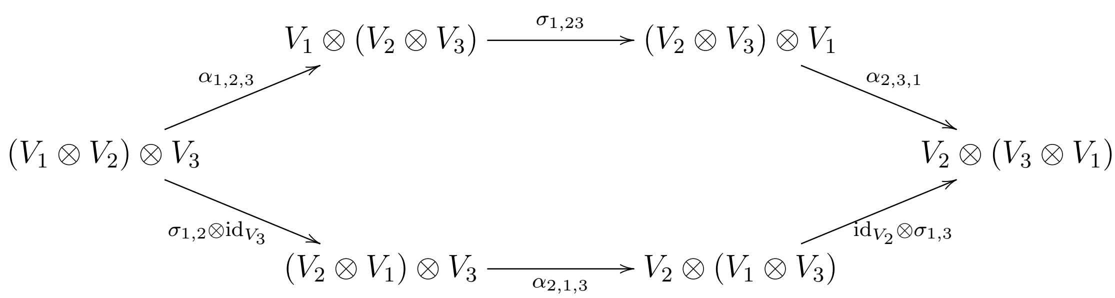
$$\sigma_{V,W} : \ V \otimes W \xrightarrow{(\pi_V \otimes \pi_W)(R)} V \otimes W \xrightarrow{\tau_{V,W}} W \otimes V$$

is a braiding on the monoidal category $\mathrm{Rep}(A)$.

Proof. We first have to show that $\sigma_{V,W}$ is equivariant with respect to the actions $\pi_V * \pi_W$ and $\pi_W * \pi_V$ on $V \otimes W$ and $W \otimes V$ respectively. This follows from the small calculation

$$\begin{aligned} &\sigma_{V,W}\, (\pi_V \otimes \pi_W)(\Delta(a)) = \tau_{V,W}\, (\pi_V \otimes \pi_W)(R)\, (\pi_V \otimes \pi_W)(\Delta(a)) \\ &= \tau_{V,W}\, (\pi_V \otimes \pi_W)(R\, \Delta(a)) = \tau_{V,W}\, (\pi_V \otimes \pi_W)(\tau\, \Delta(a))\, (\pi_V \otimes \pi_W)(R) \\ &= (\pi_W \otimes \pi_V)(\Delta(a))\, \tau_{V,W} (\pi_V \otimes \pi_W)(R) = (\pi_W \otimes \pi_V)(\Delta(a))\, \sigma_{V,W} \end{aligned}$$

where we used point a) of definition 2.3. Now let V_1, V_2 and V_3 be three representations of A and denote by α_{ijk} the corresponding associators of the tensor product. The commutativity of the braiding diagram



follows from the following observation, in which we neglect all associators:

$$\begin{aligned}\sigma_{1,23} &= \tau_{1,23}\, \pi_1 \otimes (\pi_2 * \pi_3)(R) = \tau_{1,23}\, (\pi_1 \otimes \pi_2 \otimes \pi_3)(\mathrm{id}_A \otimes \Delta)(R) \\ &= \tau_{1,23}\, (\pi_1 \otimes \pi_2 \otimes \pi_3)(R_{13}\, R_{12}) = (\mathrm{id}_2 \otimes \tau_{1,3})\, (\tau_{1,2} \otimes \mathrm{id}_3)\, (\pi_1 \otimes \pi_2 \otimes \pi_3)(R_{13}\, R_{12}) \\ &= (\mathrm{id}_2 \otimes \tau_{1,3})\, (\tau_{1,2} \otimes \mathrm{id}_3)\, (\pi_1 \otimes \pi_2 \otimes \pi_3)(R_{13})\, (\pi_1 \otimes \pi_2 \otimes \pi_3)(R_{12}) \\ &= (\mathrm{id}_2 \otimes \tau_{1,3})\, \mathrm{id}_2 \otimes (\pi_1 \otimes \pi_3)(R)\, (\tau_{1,2} \otimes \mathrm{id}_3)\, (\pi_1 \otimes \pi_2)(R) \otimes \mathrm{id}_3 \\ &= (\mathrm{id}_2 \otimes \sigma_{1,3}) \circ (\sigma_{1,2} \otimes \mathrm{id}_3)\ .\end{aligned}$$

Here we used property b) of definition 2.3. The corresponding diagram for σ^{-1} finally uses property c). □

Remark 2.5. The converse of the above statement is also true in case A is a *finite-dimensional* bialgebra, since then $A \in \mathrm{obj}(\mathrm{Rep}(A))$. Then we have: If $\mathrm{Rep}(A)$ is braided, then A is quasi-triangular. The universal R-matrix in this case is given by $R = \tau(\sigma_{A,A}(1 \otimes 1)) \in A \otimes A$.

2.1. **Hopf algebras and duals.** As we have seen in the last section the representation category of a bialgebra A is monoidal. If A is quasi-triangular, then $\mathrm{Rep}(A)$ is even braided. Using the category of finite-dimensional representations of a finite group as a guideline the next structure to consider is that of dual representations: If V is a representation of G then G also acts on the dual space V^* via $\xi \mapsto g \cdot \xi$ with $(g \cdot \xi)(v) = \xi(g^{-1} \cdot v)$ for $\xi \in V^*$, $v \in V$ and $g \in G$. We will generalize this to the case of Hopf algebras. To achieve this, we need the following properties of the antipode:

Lemma 2.6. *Let A be a Hopf algebra with antipode S, then we have*

a) $S(ab) = S(b)\, S(a)$ for all $a, b \in A$,
b) $S(1) = 1$.

In short: S is an anti-homomorphism *of algebras.*

Proof. Consider the space $\mathrm{Hom}(A \otimes A, A)$. This is an algebra with respect to the following operation: Let $f, g \in \mathrm{Hom}(A \otimes A, A)$ and define

$$f * g\colon\ A \otimes A \xrightarrow{\Delta\otimes\Delta} A \otimes A \otimes A \otimes A \xrightarrow{\mathrm{id}_A\otimes\tau\otimes\mathrm{id}_A} A \otimes A \otimes A \otimes A \xrightarrow{f\otimes g} A \otimes A \xrightarrow{\mu} A\ .$$

Associativity of this operation follows from the associativity of μ and the coassociativity of Δ. A unit for $*$ is given by $a \otimes b \mapsto u(\epsilon(a)\epsilon(b))$, which we will denote by $u(\epsilon \cdot \epsilon)$. Define $P\colon A \otimes A \to A$ by $P = S \circ \mu$, i.e. $P(a \otimes b) = S(ab)$ and observe that

$$\begin{aligned}P * \mu = (S \circ \mu) * \mu &= \mu \circ ((S \circ \mu) \otimes \mu) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta) \\ &= \mu \circ (S \otimes \mathrm{id}_A) \circ (\mu \otimes \mu) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta) \\ &= \mu \circ (S \otimes \mathrm{id}_A) \circ \Delta \circ \mu = (S * \mathrm{id}_A) \circ \mu = u\epsilon \circ \mu = u(\epsilon \cdot \epsilon)\end{aligned}$$

where we have used the diagrams (1) and (2) of definition 1.1. Likewise let $N\colon A \otimes A \to A$ be given by $N(a \otimes b) = S(b)\, S(a)$. Let $a, b \in A$ and $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$ and $\Delta(b) =$

$\sum_i b_i^{(1)} \otimes b_i^{(2)}$. Note that

$$(\mu * N)(a \otimes b) = \sum_{i,j} a_i^{(1)} b_j^{(1)} \, S\left(b_j^{(2)}\right) S\left(a_i^{(2)}\right) = \sum_i a_i^{(1)} \, (\mathrm{id}_A * S)(b) \, S\left(a_i^{(2)}\right)$$
$$= u\epsilon(b) \sum_i a_i^{(1)} \, S\left(a_i^{(2)}\right) = u\epsilon(b)(\mathrm{id}_A * S)(a) = u(\epsilon(b)\epsilon(a))$$
$$= u(\epsilon \cdot \epsilon)(a \otimes b) \ .$$

Now we have

$$P = P * u(\epsilon \cdot \epsilon) = P * (\mu * N) = (P * \mu) * N = u(\epsilon \cdot \epsilon) * N = N \ ,$$

i.e. $S(ab) = S(b)\, S(a)$. The second claim follows from

$$S(1) = (S * \mathrm{id}_A)(1) = (u\epsilon)(1) = 1 \ . \qquad \square$$

As we see from this lemma, the antipode S of a Hopf algebra A allows us to define an action of A on the dual space $V^* = \mathrm{Hom}(V, k)$ of a representation $V \in \mathrm{obj}(\mathrm{Rep}(A))$ via

$$(a \cdot \xi)(v) = \xi(\pi_V(S(a))v)$$

for $\xi \in V^*$, $v \in V$ and $a \in A$. More generally for two representations $V, W \in \mathrm{obj}(\mathrm{Rep}(A))$, then k-linear homomorphisms $\mathrm{Hom}(V, W)$ carry an action of A as follows: Let $a \in A$ and $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$. Then for $f \in \mathrm{Hom}(V, W)$

$$(a \cdot f)(v) = \sum_i \pi_W(a_i^{(1)}) \, f(\pi_V(S(a_i^{(2)}))v)$$

defines an action of A, where we used that Δ is an algebra homomorphism. This is consistent with the previous definition, since for $a \in A$ and $f \in \mathrm{Hom}(V, k)$ we have

$$(a \cdot f)(v) = \sum_i \epsilon(a_i^{(1)}) f(S(a_i^{(2)})v) = f\left(\sum_i u\epsilon(a_i^{(1)}) S(a_i^{(2)}) v\right)$$
$$= f((u\epsilon * S)(a)v) = f(S(a)v) \ .$$

Lemma 2.7. *Let $V, W \in \mathrm{obj}(\mathrm{Rep}(A))$. The canonical map*

$$\kappa \colon W \otimes V^* \to \mathrm{Hom}(V, W) \quad ; \quad w \otimes \xi \mapsto (v \mapsto \xi(v)\, w)$$

is an equivalence of representations of A.

Proof. Let $a \in A$ with $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$ and $w \otimes \xi \in W \otimes V^*$. Then we have for $v \in V$

$$\kappa(a\,(w \otimes \xi))(v) = \sum_i \kappa(a_i^{(1)} w \otimes a_i^{(2)} \xi)(v) = \sum_i a_i^{(1)} w \, \xi(S(a_i^{(2)})v) = (a\, \kappa(w \otimes \xi))(v)$$

where we have dropped the endomorphisms π_V, π_W etc. from our notation. This proves that κ is an intertwiner in the sense of equation (3). The fact that κ is an isomorphism is well-known from linear algebra. $\square$

Lemma 2.8. *Let $V \in \mathrm{obj}(\mathrm{Rep}(A))$. The unit map $u \colon k \to \mathrm{End}(V)$; $\lambda \mapsto \lambda\, \mathrm{id}_V$ is an intertwiner.*

Proof. Let $a \in A$ with $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$ and $v \in V$. Then the action of a on id_V is given by

$$(a \cdot \mathrm{id}_V)(v) = \sum_i a_i^{(1)}\, S(a_i^{(2)})\, v = (\mathrm{id}_A * S)(a)\, v = \epsilon(a)\, v\ .$$

The claim follows from this. □

Lemma 2.9. *Let $V \in \mathrm{obj}(\mathrm{Rep}(A))$. The pairing $e_V : V^* \otimes V \to k$ is an intertwiner.*

Proof. Let $a \in A$ with $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$ and $\xi \otimes v \in V^* \otimes V$. We have

$$e_V(a \cdot (\xi \otimes v)) = \sum_i e_V(a_i^{(1)}\xi \otimes a_i^{(2)} v) = \sum_i \xi(S(a_i^{(1)}) a_i^{(2)} v) = \xi((S * \mathrm{id}_A)(a)\, v) = \epsilon(a)\, \xi(v)\ .$$

Thus, e_V is equivariant with respect to the action of A. □

Combining the last two lemmata, we get that $V^* \in \mathrm{obj}(\mathrm{Rep}(A))$ together with the intertwiners

$$e_V : V^* \otimes V \to k \quad \text{and} \quad i_V :\ k \xrightarrow{u} \mathrm{End}(V) \xrightarrow{\kappa^{-1}} V \otimes V^*$$

is a right dual of V in the sense of rigid monoidal categories. To see this, let $e_i \in V$ be a basis of the finite-dimensional vector space V and denote by $e^i \in V^*$ the dual basis, then $i_V(1) = \sum_i e_i \otimes e^i$. Therefore for $v \in V$

$$(\mathrm{id}_V \otimes e_V) \circ (i_V \otimes \mathrm{id}_V)(v) = \sum_i e_i\, e^i(v) = v \tag{4}$$

and for $\xi \in V^*$

$$(e_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes i_V)(\xi) = \sum_i \xi(e_i) e^i = \xi\ . \tag{5}$$

Thus, the snake relations hold. It is tempting to think that there should be no difference between left and right duals in the category $\mathrm{Rep}(A)$, since (4) and (5) are just a matter of linear algebra. However, due to the asymmetry of the tensor product, lemma 2.7 and lemma 2.9 do not hold for these left duals. To obtain the right left duals[1], we therefore need to change the action of A on $\mathrm{Hom}(V, k)$.

Let A be a Hopf algebra over k with *invertible antipode* S and define ${}^*V = \mathrm{Hom}(V, k)$ with the action of A given by

$$(a \cdot \xi)(v) = \xi(\pi_V(S^{-1}(a))\, v)\ .$$

Lemma 2.10. *Let $V \in \mathrm{obj}(\mathrm{Rep}(A))$, let e_i be a basis of V and denote by e^i the dual basis. The maps*

$$i_V : k \to {}^*V \otimes V \quad ; \quad \lambda \mapsto \lambda \sum_i e^i \otimes e_i$$
$$e_V : V \otimes {}^*V \to k \quad ; \quad v \otimes \xi \mapsto \xi(v)$$

are intertwiners.

[1] pun not intended

Proof. Let $a \in A$ and let $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$. Since $S^{-1}(1) = 1$ we have

$$\begin{aligned} e_V(a \cdot (v \otimes \xi)) &= \sum_i e_V(a_i^{(1)} v \otimes a_i^{(2)} \xi) = \sum_i \xi(S^{-1}(a_i^{(2)})\, a_i^{(1)}\, v) \\ &= \sum_i \xi(S^{-1}(S(a_i^{(1)})\, a_i^{(2)})\ v) = \xi(S^{-1}((S * \mathrm{id}_A)(a))\, v) \\ &= \xi(S^{-1}(\epsilon(a)1)v) = \epsilon(a)\, \xi(v) = a \cdot e_V(v \otimes \xi)\ . \end{aligned}$$

The equivariance of the linear map i_V only needs to be checked at $1 \in k$, i.e. $a \cdot i_V(1) = i_V(a \cdot 1)$. This follows from the following small calculation:

$$\begin{aligned} a \cdot i_V(1) &= \sum_{i,j} a_i^{(1)} e^j \otimes a_i^{(2)} e_j = \sum_{i,j,k} a_i^{(1)} e^j(e_k) e^k \otimes a_i^{(2)} e_j \\ &= \sum_{i,j,k} e^j(S^{-1}(a_i^{(1)}) e_k)\, e^k \otimes a_i^{(2)} e_j = \sum_{i,j,k} e^k \otimes a_i^{(2)} e_j\, e^j(S^{-1}(a_i^{(1)}) e_k) \\ &= \sum_{i,k} e^k \otimes a_i^{(2)}\, S^{-1}(a_i^{(1)}) e_k = \sum_{i,k} e^k \otimes S^{-1}(a_i^{(1)}\, S(a_i^{(2)}))\, e_k \\ &= \sum_k e^k \otimes S^{-1}(\epsilon(a)\, 1)\, e_k = \epsilon(a) \sum_k e^k \otimes e_k = i_V(a \cdot 1)\ . \end{aligned}$$ □

The analogous equations of (4) and (5) hold for ${}^*V$ showing that it provides a left dual to V. Therefore we have shown

Theorem 2.11. *Let A be a* quasi-triangular *Hopf algebra with* invertible antipode S. *Then* $\mathrm{Rep}(A)$ *is a rigid braided monoidal category.*

2.2. **Ribbon Hopf algebras.** A ribbon category possesses a natural identification $\delta_V : V \to V^{**}$ that is compatible with the tensor product. This is equivalent to the following definition

Definition 2.12. A rigid braided monoidal category $(\mathcal{C}, \otimes, \dots)$ is said to be a *ribbon category* if it comes equipped with a family $\theta_V : V \to V$ of isomorphisms indexed by $V \in \mathrm{obj}(\mathcal{C})$ that satisfy

a) $\theta_{V \otimes W} = (\theta_V \otimes \theta_W)\, \sigma_{W,V} \circ \sigma_{V,W}$
b) $\theta_{V^*} = (\theta_V)^*$

where f^* denotes the adjoint morphism to f in $\mathcal{C}$. The morphisms θ_V are called the *twists* of $\mathcal{C}$.

This definition has a mirror image in quasi-triangular Hopf algebras, which ensures that the corresponding representation category is ribbon.

Definition 2.13. Let $(A, \mu, u, \Delta, \epsilon, R, S)$ be a quasi-triangular Hopf algebra with invertible antipode. An invertible element $\theta \in Z(A)$ (where $Z(A)$ denotes the center of A as an algebra) is called a *twist*, if it satisfies

a) $\Delta(\theta) = (\tau(R)R)^{-1}(\theta \otimes \theta)$,
b) $S(\theta) = \theta$.

A quasi-triangular Hopf algebra A with invertible antipode together with a twist θ is called a *ribbon Hopf algebra*.

This definition is taylored to make the following theorem work.

Theorem 2.14. *Let A be a ribbon Hopf algebra with twist $\theta \in Z(A)$. Then* $\mathrm{Rep}(A)$ *together with the natural morphism*

$$\theta_V : V \to V \quad ; \quad v \mapsto \pi_V(\theta^{-1})(v)$$

is a ribbon category.

Proof. Since $\theta \in Z(A)$, multiplication by θ^{-1} commutes with the multiplication by any other element $a \in A$. Therefore θ_V is indeed equivariant and defines an isomorphism in $\mathrm{Rep}(A)$.

To prove that θ_V defines a twist, let $V, W \in \mathrm{obj}(\mathrm{Rep}(A))$. Then we have for every $v \in V$ and $w \in W$:

$$\begin{aligned}(\theta_V \otimes \theta_W)\, \sigma_{W,V} \circ \sigma_{V,W}(v \otimes w) &= (\pi_V \otimes \pi_W)((\theta^{-1} \otimes \theta^{-1})\, \tau(R)\, R)(v \otimes w) \\ &= (\pi_V \otimes \pi_W)(\Delta(\theta^{-1}))(v \otimes w) = (\pi_V * \pi_W)(\theta^{-1})(v \otimes w) \\ &= \theta_{V \otimes W}(v \otimes w)\end{aligned}$$

by condition a) of definition 2.13. Moreover, for $\xi \in V^*$ and $v \in V$

$$\begin{aligned}(\theta_V)^*(\xi)(v) &= \xi(\theta_V(v)) = \xi(\pi_V(\theta^{-1})(v)) = \xi(\pi_V(S(\theta^{-1}))(v)) \\ &= (\pi_{V^*}(\theta^{-1})(\xi))(v) = \theta_{V^*}(\xi)(v)\ ,\end{aligned}$$

where we have used property b). □

Remark 2.15. For *finite dimensional* quasi-triangular Hopf algebras, the converse of the above statement is also true, i.e. if $\mathrm{Rep}(A)$ is a ribbon category, then A is a ribbon Hopf algebra. In this case $A \in \mathrm{obj}(\mathrm{Rep}(A))$ and similar to the above considerations it can be checked that $\theta = (\theta_A(1))^{-1}$ provides a twist for A.

3. The Quantum double of a finite group

Let G be a finite group. As we have seen in our introductory example $A = \mathbb{C}G$ is a Hopf algebra. Since A is finite dimensional with canonical basis $g \in G \subset A$, it follows from lemma 1.6 that A^* is a Hopf algebra as well and we refer to example 1.7 for a detailed explanation of the structure. A construction of Drinfeld endows the tensor product $A \otimes A^*$ with the structure of a Hopf algebra quite similar to the semidirect product of groups. This is called the *quantum double* of A. We will only need it in the case $A = \mathbb{C}G$, even though it works for all finite-dimensional Hopf algebras – in fact even the condition $\dim(A) < \infty$ may be relaxed: Let $D(G) = A \otimes A^*$ as a vector space and equip it with a new multiplication according to the rule

$$(g_1\, \delta_{h_1}) \cdot (g_2\, \delta_{h_2}) = g_1 g_2\, \delta_{g_2^{-1} h_1 g_2} \delta_{h_2}\ ,$$

where we have dropped the $\otimes$-symbol on both sides. On the right hand side we use the multiplications in A and A^* respectively. It is easy to check that this operation is associative. Let $1^* = \sum_{x \in G} \delta_x$ be the unit in A^*, $1 \in A$ be the one in A then $1 \otimes 1^*$ is a unit in $D(G)$, which we will write as $1 \cdot 1^*$. To obtain the coproduct on $D(G)$, we need to use the opposite comultiplication on the dual space defined by $\Delta_{A^*}^{\mathrm{op}} = \tau \circ \Delta_{A^*}$ and set

$$\Delta(g\,\delta_h) = \Delta_A(g)\,\Delta_{A^*}^{\mathrm{op}}(\delta_h) = \sum_{x\in G} g\,\delta_x \otimes g\,\delta_{hx^{-1}} \ .$$

Again, the coassociativity of Δ is straightforward to check. A counit for Δ is given by the linear extension of

$$\epsilon(g\,\delta_h) = \epsilon_A(g)\,\epsilon_{A^*}(\delta_h) = \delta_{e,h} \ .$$

Observe that there are algebra homomorphisms $\mathbb{C}G \to D(G)$ and $(\mathbb{C}G)^* \to D(G)$ given by $g \mapsto g\,1^*$ and $\delta_g \mapsto 1\,\delta_g$. We will sometimes drop the units from our notation and consider $g \in D(G)$ and $\delta_g \in D(G)$ via these inclusions. We define our candidate for the universal R-matrix in $D(G)$ to be

$$R = \sum_{g\in G} g\,1^* \otimes 1\,\delta_g = \sum_{g\in G} g \otimes \delta_g \in D(G)\otimes D(G) \ .$$

Furthermore, we define $S\colon D(G) \to D(G)$ to be the linear map

$$S(g\,\delta_h) = S_{A^*}(\delta_h)\,S_A(g) = \delta_{h^{-1}}\,g^{-1} = g^{-1}\,\delta_{gh^{-1}g^{-1}} \ .$$

For the twist we set

$$\theta = \sum_{g\in G} g^{-1}\,\delta_g \in D(G) \ . \tag{6}$$

Theorem 3.1. *With the above definitions, $D(G)$ is a ribbon Hopf algebra.*

Proof. What remains to be checked to prove that $D(G)$ is a bialgebra, is that Δ is an algebra homomorphism. This follows from

$$\begin{aligned}\Delta(g_1\,\delta_{h_1}) \cdot \Delta(g_2\,\delta_{h_2}) &= \Big(\sum_x g_1\,\delta_x \otimes g_1\,\delta_{h_1x^{-1}}\Big) \cdot \Big(\sum_y g_2\,\delta_y \otimes g_2\,\delta_{h_2y^{-1}}\Big)\\ &= \sum_{x,y} g_1g_2\,\delta_{g_2^{-1}xg_2}\delta_y \otimes g_1g_2\,\delta_{g_2^{-1}h_1x^{-1}g_2}\delta_{h_2y^{-1}}\\ &= \delta_{h_2,g_2^{-1}h_1g_2} \sum_x g_1g_2\,\delta_x \otimes g_1g_2\,\delta_{h_2x^{-1}}\\ &= \Delta(g_1g_2\,\delta_{h_2,g_2^{-1}h_1g_2}\,\delta_{h_2}) = \Delta(g_1\,\delta_{h_1}\cdot g_2\,\delta_{h_2})\end{aligned}$$

Likewise we have for the counit:

$$\begin{aligned}\epsilon(g_1\,\delta_{h_1}\cdot g_2\,\delta_{h_2}) &= \delta_{g_2^{-1}h_1g_2,\,h_2}\,\epsilon(g_1g_2\,\delta_{h_2}) = \delta_{g_2^{-1}h_1g_2,\,h_2}\,\delta_{h_2,\,e} = \delta_{g_2^{-1}h_1g_2,\,e}\,\delta_{h_2,\,e}\\ &= \delta_{h_1,\,e}\,\delta_{h_2,\,e} = \epsilon(g_1\,\delta_{h_1})\cdot\epsilon(g_2\,\delta_{h_2})\end{aligned}$$

It is easily checked that $\Delta(1\cdot 1^*) = 1\cdot 1^* \otimes 1\cdot 1^*$ and $\epsilon(1\cdot 1^*) = 1$. Therefore the diagrams (1) and (2) commute.

Next we will check that the map S defined above provides an antipode for the bialgebra $D(G)$. This follows from

$$(\mu\circ(S\otimes \mathrm{id}_A)\circ\Delta)(g\,\delta_h)=\sum_{x\in G}\delta_{x^{-1}}\,g^{-1}\,g\,\delta_{hx^{-1}}=\delta_{h,e}\sum_{x\in G}\delta_{x^{-1}}=\delta_{h,e}\,1^*=u\epsilon(g\,\delta_h)\;,$$

$$(\mu\circ(\mathrm{id}_A\otimes S)\circ\Delta)(g\,\delta_h)=\sum_{x\in G}g\,\delta_x\,\delta_{xh^{-1}}g^{-1}=\delta_{h,e}\sum_{x\in G}g\,\delta_x\,g^{-1}=\delta_{h,e}\sum_{x\in G}\delta_{gxg^{-1}}=u\epsilon(g\,\delta_h)\;.$$

Note that S is invertible, in fact we have $S^2=\mathrm{id}_{D(G)}$.

To see that $D(G)$ is in fact quasi-triangular with universal R-matrix R as above, we need to check the conditions a) to c) in definition 2.3. Regarding b) and c) note that

$$\begin{aligned}R_{13}\,R_{12}&=\sum_{x,y\in G}(x\otimes(1\cdot 1^*)\otimes\delta_x)\,(y\otimes\delta_y\otimes(1\cdot 1^*))=\sum_{x,y\in G}(xy\otimes\delta_y\otimes\delta_x)\\&=\sum_{g\in G}g\otimes\left(\sum_{y\in G}\delta_y\otimes\delta_{gy^{-1}}\right)=\sum_{g\in G}g\otimes\Delta(\delta_g)=(\mathrm{id}_{D(G)}\otimes\Delta)(R)\;,\\R_{13}\,R_{23}&=\sum_{x,y\in G}(x\otimes(1\cdot 1^*)\otimes\delta_x)((1\cdot 1^*)\otimes y\otimes\delta_y)=\sum_{x,y\in G}x\otimes y\otimes\delta_{x,y}\,\delta_y\\&=\sum_{x\in G}x\otimes x\otimes\delta_x=(\Delta\otimes\mathrm{id}_{D(G)})(R)\;.\end{aligned}$$

Regarding point a) we calculate

$$\begin{aligned}R\,\Delta(g\,\delta_h)&=\sum_{x,y\in G}xg\,\delta_y\otimes\delta_x\,g\,\delta_{hy^{-1}}=\sum_{x,y\in G}xg\,\delta_y\otimes g\,\delta_{g^{-1}xg}\delta_{hy^{-1}}\\&=\sum_{\tilde{x},y\in G}g\tilde{x}\,\delta_y\otimes g\,\delta_{\tilde{x}}\delta_{hy^{-1}}=\sum_{\tilde{x},y\in G}g\delta_{\tilde{x}y\tilde{x}^{-1}}\,\tilde{x}\otimes g\,\delta_{hy^{-1}}\delta_{\tilde{x}}\\&=\sum_{\tilde{x},y\in G}g\delta_{hyh^{-1}}\,\tilde{x}\otimes g\,\delta_{hy^{-1}}\delta_{\tilde{x}}=\sum_{\tilde{x},\tilde{y}\in G}g\delta_{h\tilde{y}^{-1}}\,\tilde{x}\otimes g\,\delta_{\tilde{y}}\delta_{\tilde{x}}=(\tau\circ\Delta)(g\,\delta_h)\,R\;,\end{aligned}$$

where we have used $\delta_{hy^{-1}}\delta_{\tilde{x}}=\delta_{\tilde{x},hy^{-1}}\,\delta_{\tilde{x}}$ from the second line to the third. An inverse for R is given by $R^{-1}=\sum_{x\in G}x^{-1}\otimes\delta_x\in D(G)\otimes D(G)$, since

$$R\cdot R^{-1}=\sum_{g,x\in G}gx^{-1}\otimes\delta_g\delta_x=\sum_{x\in G}(1\cdot 1^*)\otimes\delta_x=(1\cdot 1^*)\otimes(1\cdot 1^*)=R^{-1}\cdot R\;.$$

The remaining part of the proof concerns the twist $\theta\in D(G)$. First we need to see that θ is in fact a central element:

$$\theta\cdot g\,\delta_h=\sum_{x\in G}x^{-1}\,\delta_x\,g\,\delta_h=\sum_{x\in G}x^{-1}g\,\delta_{g^{-1}xg}\delta_h=\sum_{x\in G}x^{-1}g\,\delta_{x,ghg^{-1}}\delta_h=gh^{-1}\,\delta_h\;,$$

$$g\,\delta_h\cdot\theta=\sum_{x\in G}g\,\delta_h\,x^{-1}\,\delta_x=\sum_{x\in G}gx^{-1}\,\delta_{xhx^{-1}}\,\delta_x=gh^{-1}\,\delta_h\;.$$

We also obtain the invertibility of θ from this calculation with $\theta^{-1} = \sum_{x\in G} x\,\delta_x$. Regarding the conditions a) and b) from definition 2.13 we have that

$$\begin{aligned}
\tau(R)\,R\,\Delta(\theta) &= \sum_{g,h,x,y\in G} \delta_g\,hx^{-1}\,\delta_y \otimes g\,\delta_h\,x^{-1}\,\delta_{xy^{-1}} = \sum_{g,h,x,y\in G} \delta_g\,hx^{-1}\,\delta_y \otimes gx^{-1}\,\delta_{xhx^{-1}}\,\delta_{xy^{-1}} \\
&= \sum_{g,x,y\in G} \delta_g\,y^{-1}\,\delta_y \otimes gx^{-1}\,\delta_{xy^{-1}} = \sum_{g,x,y\in G} y^{-1}\,\delta_{ygy^{-1}}\delta_y \otimes gx^{-1}\,\delta_{xy^{-1}} \\
&= \sum_{x,y\in G} y^{-1}\,\delta_y \otimes yx^{-1}\,\delta_{xy^{-1}} = \sum_{x,y\in G} y^{-1}\,\delta_y \otimes x^{-1}\,\delta_x = \theta\otimes\theta\ ,
\end{aligned}$$

where from the first to the second row we have carried out the sum over h and from the second to the third row we summed over g. Since g commutes with $\delta_{g^{-1}}$, it is also clear that $S(\theta) = \theta$. □

Corollary 3.2. *The representation category* $\mathrm{Rep}(D(G))$ *is a ribbon category.*

3.1. **Semisimplicity of** $\mathrm{Rep}(D(G))$**.** In this section we will see that any object in $\mathrm{Rep}(D(G))$ decomposes into a direct sum of irreducible ones and that the number of irreducible representations is finite, i.e. we show that $\mathrm{Rep}(D(G))$ is semisimple.

Lemma 3.3. *Let* $V\in\mathrm{Rep}(D(G))$ *be a finite-dimensional representation of* $D(G)$ *and* $W\subset V$ *a subrepresentation (i.e. an invariant subspace). Then* V *has a decomposition*

$$V \cong W \oplus W'$$

as $D(G)$*-representation.*

Proof. Choose a $\mathbb{C}$-linear map $E\colon V\to W$, such that $E(w) = w$ for all $w\in W$, e.g. by completing a basis of W to a basis of V and sending the basis vectors of W to themselves and any other basis vector to 0. Now define $P\colon V\to W$ by

$$P(v) = \frac{1}{|G|}\sum_{x,y\in G} \pi_W(x\,\delta_y)\,E(\pi_V(\delta_y\,x^{-1})\,v)\ .$$

Since the restriction of π_V to W agrees with π_W by definition of subrepresentation, we will drop π_V and π_W from our notation. Now observe that for $w\in W$ we can drop the E and have

$$P(w) = \frac{1}{|G|}\sum_{x,y\in G} x\,\delta_y\,x^{-1}\,w = \frac{1}{|G|}\sum_{x,y\in G}\delta_{xyx^{-1}}\,w = \sum_{y\in G}\delta_y\,w = (1\cdot 1^*)\,w = w\ .$$

Thus, P is an idempotent projecting onto W. It is also an intertwiner as follows from the following calculation for $v\in V$, $g\,\delta_h\in D(G)$:

$$\begin{aligned}
g\,\delta_h\,P(v) &= \frac{1}{|G|}\sum_{x,y\in G} gx\,\delta_{x^{-1}hx}\,\delta_y\,E(\delta_y\,x^{-1}\,v) = \frac{1}{|G|}\sum_{x,y\in G} gx\,\delta_y\,E(\delta_y\,\delta_{x^{-1}hx}\,x^{-1}\,v) \\
&= \frac{1}{|G|}\sum_{x,y\in G} gx\,\delta_y\,E(\delta_y\,x^{-1}\,\delta_h\,v) = \frac{1}{|G|}\sum_{\tilde{x},y\in G}\tilde{x}\,\delta_y\,E(\delta_y\,\tilde{x}^{-1}\,g\,\delta_h\,v) \\
&= P(g\,\delta_h\,v)\ .
\end{aligned}$$

Thus, V decomposes as $W \oplus \ker P$ as a representation of $D(G)$. □

Remark 3.4. The argument given above is just a special case of a far more general statement: The element $s = \frac{1}{|G|}\left(\sum_{g\in G} g\right)\delta_e$ satisfies $\epsilon(s) \neq 0$ and

$$s\,a = \epsilon(a)\,s$$

and therefore constitutes what is called a *left integral* in the theory of Hopf algebras. The general statement now says that a finite dimensional Hopf algebra A is semisimple if and only if there exists left integral for A [5, Theorem 5.1.8].

Corollary 3.5. *Each finite-dimensional representation decomposes into a finite direct sum of irreducible ones.*

Proof. This follows from the above lemma by induction over the dimension n. Since the trivial representation is certainly irreducible, the statement is true for $n = 1$. Suppose it is true for all $k < n$ and let V be a representation of dimension n. If it is irreducible, there is nothing to prove, if not, there is a proper subrepresentation $W \subset V$ of dimension $k < n$. By the lemma $V = W \oplus W'$ as $D(G)$-modules and $\dim(W') < n$. Thus, W and W' both decompose into a finite direct sum of irreducible representations. □

An algebra A such that any (left) A-module M has the property described in lemma 3.3 is called semisimple itself. If we assume A to be finite-dimensional, then $A \in \mathrm{obj}(\mathrm{Rep}(A))$. The following structure theorem about semi-simple algebras follows from a theorem that was proven by Wedderburn and Artin (see [4, Theorem 3.5]).

Theorem 3.6. *Let A be a finite-dimensional semisimple algebra. Then the number of isomorphism classes of irreducible representations of A is finite. Let V_i with $i \in I$ and $|I| < \infty$ be a complete set of representatives, then*

$$A \;\cong\; \bigoplus_{i\in I} V_i^{n_i}\ ,$$

as representations of A, where $n_i = \dim(V_i)$. In particular, $\dim(A) = \sum_{i\in I} \dim(V_i)^2$.

In the case $A = D(G)$ we can give a concrete description of the irreducible representations. The following observations are blatantly copied from [2, Section 6]: Let $g \in G$. Denote by C_g the conjugacy class of g and by $Z(g)$ its centralizer. Then we have

$$G = \bigcup_{k=1}^{n} C_{g_k}$$

for some chosen representatives $g_k \in G$, where k runs through the finite list of all conjugacy classes of G . We choose $g_1 = e$. Let $Z_k = Z(g_k)$, $C_k = C_{g_k}$ and let $A_k = \mathbb{C}Z_k$ be the complex group ring of Z_k. For each $s \in C_k$ choose a fixed $\tau_s \in G$ such that $s = \tau_s\, g_k\, \tau_s^{-1}$. Choose $\tau_{g_k} = e \in G$.

Lemma 3.7. *a) $G = \bigcup_{s\in C_k} \tau_s Z_k$ and the union is disjoint.*
b) Let $g \in G$, $s \in C_k$, then for $t = gsg^{-1}$ we have

$$\tau_t^{-1}\, g\, \tau_s \in Z_k\ .$$

Proof. Suppose we have $h, h' \in Z_k$ with $\tau_t h = \tau_s h'$. Then $\tau_s^{-1}\tau_t = h'h^{-1} \in Z_k$ and

$$t = \tau_t\, g_k\, \tau_t^{-1} = \tau_s\, \tau_s^{-1}\tau_t\, g_k\, \tau_t^{-1}\tau_s\, \tau_s^{-1} = \tau_s\, g_k\, \tau_s^{-1} = s\ .$$

Therefore $\tau_s Z_k \cap \tau_t Z_k = \emptyset$ unless $s = t$ and

$$\left| \bigcup_{s\in C_k} \tau_s Z_k \right| = \sum_{s\in C_k} |Z_k| = |C_k| \cdot |Z_k| = |G|\ .$$

This proves a). As to b) note that for $g \in G$ and $s \in C_k$ there exists a unique $t \in C_k$ with $h := \tau_t^{-1}\, g\, \tau_s \in Z_k$. But

$$gsg^{-1} = g\, \tau_s\, g_k\, \tau_s^{-1}\, g^{-1} = \tau_t\, h\, g_k\, h^{-1}\, \tau_t^{-1} = \tau_t\, g_k\, \tau_t^{-1} = t\ . \qquad \square$$

Let V_α^k with $\alpha \in J$ be a complete set of representatives of the isomorphism classes of irreducible representations of Z_k and define

$$V_{k,\alpha} = (\mathbb{C}G) \otimes_{A_k} V_\alpha^k\ .$$

By lemma 3.7 a) it is spanned by vectors of the form $v_s = \tau_s \otimes v$ with $v \in V_\alpha^k$ and $s \in C_k$. Thus, $\dim(V_{k,\alpha}) = |C_k| \cdot \dim(V_\alpha^k)$. The action of $\mathbb{C}G$ on $V_{k,\alpha}$ in this notation looks like

$$g \cdot v_s = \left(\left(\tau_{gsg^{-1}}^{-1}\, g\, \tau_s \right) \cdot v \right)_{gsg^{-1}}\ ,$$

which makes sense since $\tau_{gsg^{-1}}^{-1}\, g\, \tau_s \in Z_k$ by lemma 3.7 b). The important point is that we can *extend* this to an action of $D(G)$ as follows

$$\delta_h \cdot v_s = \delta_{h,s}\, v_s\ , \tag{7}$$

i.e. δ_h projects to the subspace $V_{k,\alpha}(h) \subset V_{k,\alpha}$ spanned by the vectors $v_h \in V_{k,\alpha}$. As a vector space $V_{k,\alpha}$ decomposes into a direct sum:

$$V_{k,\alpha} = \bigoplus_{s\in C_k} V_{k,\alpha}(s)\ . \tag{8}$$

Each summand $V_{k,\alpha}(s)$ is an irreducible representation of $Z(s) = \tau_s\, Z_k\, \tau_s^{-1}$. That (7) indeed defines an action follows from

$$\begin{aligned} g_1\, \delta_{h_1} \cdot (g_2\, \delta_{h_2} \cdot v_s) &= \delta_{h_2,s}\, g_1\, \delta_{h_1} \cdot \left(\tau_{g_2sg_2^{-1}}^{-1} g \tau_s\, v \right)_{g_2sg_2^{-1}} \\ &= \delta_{h_2,s}\, \delta_{h_1,g_2sg_2^{-1}} \left(\tau_{g_1g_2s(g_1g_2)^{-1}}^{-1} g_1g_2\tau_s\, v \right)_{g_1g_2s(g_1g_2)^{-1}} \\ &= g_1g_2\, \delta_{g_2^{-1}h_1g_2}\delta_{h_2} \cdot v_s = (g_1\, \delta_{h_1} \cdot g_2\, \delta_{h_2}) \cdot v_s\ . \end{aligned}$$

Lemma 3.8. *The $D(G)$-representations $V_{k,\alpha}$ are irreducible.*

Proof. Let $c_k = \sum_{h\in C_k} \delta_h \in D(G)$ and choose $w \in V_{k,\alpha}$ with $w \neq 0$. Then $c_k \cdot w = w$. Therefore there exists an $s \in C_k$ with $\delta_s w \neq 0$. Note that $\delta_s w \in V_{k,\alpha}(s)$, which is an irreducible $Z(s)$-representation. Therefore

$$V_{k,\alpha}(s) = \mathbb{C}Z(s)\, \delta_s\, w \subseteq D(G)\, w\ .$$

The subspace $g\,V_{k,\alpha}(s) \subseteq V_{k,\alpha}(gsg^{-1})$ forms a non-zero $Z(gsg^{-1})$-subrepresentation. By the irreducibility of $V_{k,\alpha}(gsg^{-1})$ the inclusion is an equality. Therefore

$$V_{k,\alpha}(gsg^{-1}) = g\,V_{k,\alpha}(s) \subseteq D(G)\,w\ .$$

The decomposition (8) now yields

$$V_{k,\alpha} = \bigoplus_{s\in C_k} V_{k,\alpha}(s) \subseteq D(G)\,w \subseteq V_{k,\alpha}\ .$$

This shows that any non-zero subrepresentation of $V_{k,\alpha}$ already coincides with all of $V_{k,\alpha}$,i.e. the latter is irreducible. ☐

Lemma 3.9. *The irreducible $D(G)$-representations $V_{k,\alpha}$ and $V_{l,\beta}$ are isomorphic if and only if $k = l$ and V^k_α and V^l_β are isomorphic as Z_k-representations (i.e. $\alpha = \beta$).*

Proof. Suppose $k = l$ and $\varphi\colon V^k_\alpha \to V^k_\beta$ is an equivalence of Z_k-representations. Then $\psi(v_s) = (\varphi(v))_s$ provides an isomorphism of $D(G)$-representations.

For the other direction let $\psi\colon V_{k,\alpha} \to V_{l,\beta}$ be a $D(G)$-equivariant isomorphism. Because ψ commutes with δ_s, it has to map $V_{k,\alpha}(s)$ to $V_{l,\beta}(s)$, which is only possible if C_k and C_l agree, i.e. $l = k$. In case $s = g_k$ we have $V_{k,\alpha}(s) = e \otimes V^k_\alpha$ and $\psi(e \otimes V^k_\alpha) \subseteq e \otimes V^k_\beta$ as Z_k-representations. By the irreducibility of V^k_α and V^k_β this has to be an isomorphism. ☐

Lemma 3.10. *Every irreducible $D(G)$-representation is isomorphic to one of the $V_{k,\alpha}$.*

Proof. Observe that

$$\sum_{k,\alpha} \dim(V_{k,\alpha})^2 = \sum_k |C_k|^2 \sum_\alpha \dim(V^k_\alpha)^2\ .$$

Since $\mathbb{C}Z_k$ is a semisimple algebra, it follows from lemma 3.3 that $\sum_\alpha \dim(V^k_\alpha)^2 = |Z_k|$ and the above sum evaluates to

$$\sum_k |C_k|^2 \cdot |Z_k| = |G| \sum_k |C_k| = |G|^2\ .$$

Again by lemma 3.3, it follows that the sum over the dimensions of inequivalent irreducible representations equals $\dim(D(G)) = |G|^2$. Therefore the representations $V_{k,\alpha}$ exhaust all irreducibles. ☐

To summarize our efforts in this section: We have proven

Theorem 3.11. *Let G be a finite group. Then* $\mathrm{Rep}(D(G))$ *is a semisimple abelian ribbon category with simple objects $V_{k,\alpha}$.*

3.2. **Modularity of** $\mathrm{Rep}(D(G))$**.** The last missing piece is the proof of modularity of $\mathrm{Rep}(D(G))$. To compute the S-matrix, we first need to know how the twists act in the irreducible representations $V_{\alpha,k}$. By theorem 2.14 and (6) we have for $v_s \in V_{\alpha,k}$

$$\theta_{V_{\alpha,k}}(v_s) = \sum_{g\in G} \pi_{V_{\alpha,k}}(g\,\delta_g)(v_s) = \left(\pi_{V^k_\alpha}(\tau_s^{-1} s \tau_s)(v)\right)_s = \left(\pi_{V^k_\alpha}(g_k)(v)\right)_s = \frac{\mathrm{tr}(\pi_{V^k_\alpha}(g_k))}{\dim(V^k_\alpha)}\, v_s\ .$$

We will abbreviate $\mathrm{tr}(\pi_{V^k_\alpha}(x))$ by $\mathrm{tr}_{\alpha,k}(x)$ and set $\theta_{\alpha,k} = \theta_{V_{\alpha,k}}$. The S-matrix is then given by

$$S_{(\alpha,k),(\beta,l)} = \theta^{-1}_{\alpha,k}\,\theta^{-1}_{\beta,l}\,\mathrm{tr}\left(\theta_{V^*_{\alpha,k}\otimes V_{\beta,l}}\right) \tag{9}$$

Note that $(V_{\alpha,k})^*$ is isomorphic to V_{α^*,k^*} for some new indices α^* and k^*. Since $\delta_h \in D(G)$ acts on $V^*_{\alpha,k}$ via pullback with $S(\delta_h) = \delta_{h^{-1}}$, the direct sum decomposition of $V^*_{\alpha,k}$ is indexed by $C_{g_k^{-1}}$. Therefore g_{k^*} is the representative that is conjugate to g_k^{-1}. Note that $Z_{k^*} = h\,Z(g_k^{-1})\,h^{-1}$ for some $h \in G$. α^* is equivalent to the representation of $Z_{k^*} \cong Z(g_k^{-1}) = Z_k$ *dual* to α. Let $x \in Z_{k^*}$, then

$$\mathrm{tr}_{\alpha^*,k^*}(x) = \mathrm{tr}(\pi_{V^{k^*}_{\alpha^*}}(x)) = \mathrm{tr}(\pi_{(V^k_\alpha)^*}(h^{-1}xh)) = \mathrm{tr}(\pi_{V^k_\alpha}(h^{-1}x^{-1}h)) = \mathrm{tr}_{\alpha,k}(h^{-1}x^{-1}h)\ .$$

In particular, we have $\mathrm{tr}_{\alpha^*,k^*}(g_{k^*}) = \mathrm{tr}_{\alpha,k}(g_k)$.

Lemma 3.12. *Let G be a finite group. The S-matrix of* $\mathrm{Rep}(D(G))$ *is given by*

$$S_{(\alpha,k),(\beta,l)} = \frac{|G|}{|Z_k|\,|Z_l|} \sum_{\substack{h\in G \\ hg_lh^{-1}\in Z(g_{k^*})}} \mathrm{tr}_{\alpha^*,k^*}\left(h\,g_l\,h^{-1}\right)\,\mathrm{tr}_{\beta,l}\left(h^{-1}\,g_{k^*}\,h\right)$$

Proof. According to (9) we need to determine $\theta_{V^*_{\alpha,k}\otimes V_{\beta,l}}$ first. Since we are only interested in its trace, we will compute $\theta_{V_{\alpha^*,k^*}\otimes V_{\beta,l}}$. Let $v_s\otimes w_t \in V_{\alpha^*,k^*}\otimes V_{\beta,l}$

$$\begin{aligned}\theta_{V_{\alpha^*,k^*}\otimes V_{\beta,l}}(v_s\otimes w_t) &= \Delta(\theta^{-1})(v_s\otimes w_t) = \sum_{x,y\in G}(x\,\delta_y)\ v_s\otimes(x\,\delta_{xy^{-1}})\ w_t \\ &= (ts)\,v_s\otimes(ts)\,w_t = \left(\tau^{-1}_{tst^{-1}}\,ts\,\tau_s\,v\right)_{tst^{-1}}\otimes\left(\bar\tau^{-1}_{tsts^{-1}t^{-1}}\,ts\,\bar\tau_t\,w\right)_{tsts^{-1}t^{-1}}\ .\end{aligned}$$

In view of the direct sum decomposition (8), we only need to consider those vectors $v_s\otimes w_t$, whose image lies again in $V_{\alpha^*,k^*}(s)\otimes V_{\beta,l}(t)$. By the above computation this corresponds to the condition that $t\in G$ commutes with $s\in G$. Taking the trace of $\theta_{V_{\alpha^*,k^*}\otimes V_{\beta,l}}$ therefore amounts to summing over the traces of the restriction of this map to those subspaces:

$$\begin{aligned}\mathrm{tr}\left(\theta_{V_{\alpha^*,k^*}\otimes V_{\beta,l}}\right) &= \sum_{\substack{s\in C_{g_{k^*}},t\in C_{g_l}\\ st=ts}} \mathrm{tr}_{\alpha^*,k^*}\left(\tau^{-1}_{tst^{-1}}\,ts\,\tau_s\right)\,\mathrm{tr}_{\beta,l}\left(\bar\tau^{-1}_{tsts^{-1}t^{-1}}\,ts\,\bar\tau_t\right) \\ &= \sum_{\substack{s\in C_{g_{k^*}},t\in C_{g_l}\\ st=ts}} \mathrm{tr}_{\alpha^*,k^*}\left(\tau_s^{-1}\,ts\,\tau_s\right)\,\mathrm{tr}_{\beta,l}\left(\bar\tau_t^{-1}\,ts\,\bar\tau_t\right) \\ &= \sum_{\substack{s\in C_{g_{k^*}},t\in C_{g_l}\\ st=ts}} \mathrm{tr}_{\alpha^*,k^*}\left(\tau_s^{-1}\bar\tau_t\,g_l\,\bar\tau_t^{-1}\,\tau_s\,g_{k^*}\right)\,\mathrm{tr}_{\beta,l}\left(g_l\,\bar\tau_t^{-1}\,\tau_s\,g_{k^*}\,\tau_s^{-1}\,\bar\tau_t\right) \\ &= \frac{\mathrm{tr}_{\alpha^*,k^*}(g_{k^*})}{\dim(V^{k^*}_{\alpha^*})}\,\frac{\mathrm{tr}_{\beta,l}(g_l)}{\dim(V^l_\beta)} \sum_{\substack{s\in C_{g_{k^*}},t\in C_{g_l}\\ st=ts}} \mathrm{tr}_{\alpha^*,k^*}\left(h_{s,t}\,g_l\,h^{-1}_{s,t}\right)\,\mathrm{tr}_{\beta,l}\left(h^{-1}_{s,t}\,g_{k^*}\,h_{s,t}\right)\end{aligned}$$

where we defined $h_{s,t} = \tau_s^{-1}\bar\tau_t$. By the trace property, we could replace $h_{s,t}$ by $x^{-1}h_{s,t}y = x^{-1}\tau_s^{-1}\,\bar\tau_t\,y = (\tau_s\,x)^{-1}\,\bar\tau_t\,y$ with $x\in Z_{k^*}$ and $y\in Z_l$ without changing the factors under the sum. Additionally summing over x and y corresponds to a sum over all elements of G by lemma 3.7 a). Therefore the above corresponds to

$$= \theta_{\alpha,k}\,\theta_{\beta,l}\,\frac{1}{|Z_k|}\,\frac{1}{|Z_l|}\sum_{\substack{a,b\in G,\\ a^{-1}b\,g_l\,b^{-1}a\in Z_{k^*}}} \mathrm{tr}_{\alpha^*,k^*}\left(a^{-1}b\,g_l\,b^{-1}a\right)\,\mathrm{tr}_{\beta,l}\left(b^{-1}a\,g_{k^*}\,a^{-1}b\right)$$

$$= \theta_{\alpha,k}\,\theta_{\beta,l}\,\frac{|G|}{|Z_k|\,|Z_l|}\sum_{\substack{h\in G,\\ h\,g_l\,h^{-1}\in Z_{k^*}}} \mathrm{tr}_{\alpha^*,k^*}\left(h\,g_l\,h^{-1}\right)\,\mathrm{tr}_{\beta,l}\left(h^{-1}\,g_{k^*}\,h\right)$$

from which the assertion follows. □

To calculate the structure constants p^+ and p^- of $\mathrm{Rep}(D(G))$ and to finally see the invertibility of S we need some prerequisites from character theory – the orthogonality relations of characters. Let G be a finite group and denote by I the set that labels the isomorphism classes of irreducible representations. Choose a representative π_i for each $i \in I$ and let π and π' be arbitrary representations. Then

$$\frac{1}{|G|}\sum_{h\in G}\mathrm{tr}_{\pi^*}(h)\,\mathrm{tr}_{\pi'}(hg) = \frac{\mathrm{tr}_{\pi}(g)}{\mathrm{tr}_{\pi}(e)}\,\delta_{\pi,\pi'}\ , \tag{10}$$

$$\frac{1}{|Z(g)|}\sum_{i\in I}\mathrm{tr}_{\pi_i^*}(g)\,\mathrm{tr}_{\pi_i}(h) = \delta^G_{g,h} := \begin{cases} 1 & \text{if } g \text{ and } h \text{ are conjugate in } G\\ 0 & \text{else}\end{cases}\ . \tag{11}$$

Using (11) our calculation for $\theta_{\alpha,k}$ shows that

$$p^+ = \sum_{\alpha,k}\theta_{\alpha,k}\,\dim(V_{\alpha,k})^2 = \sum_{\alpha,k}\frac{\mathrm{tr}_{\alpha,k}(g_k)}{\dim(V_\alpha^k)}\,|C_k|^2\,\dim(V_\alpha^k)^2 = \sum_k |C_k|^2\sum_\alpha \mathrm{tr}_{\alpha,k}(g_k)\,\mathrm{tr}_{(\alpha,k)^*}(e)$$
$$= \sum_k |C_k|^2\,|Z_k|\,\delta_{k,1} = |G|\,|C_1| = |G|$$

and since $\theta_{\alpha,k}^{-1} = \frac{\mathrm{tr}_{\alpha,k}(g_k^{-1})}{\dim(V_\alpha^k)}$ we also have $p^- = |G|$.

Lemma 3.13. *Let G be a finite group and let S be the S-matrix of $D(G)$. Then we have*

$$S^2 = |G|^2\,C$$

where $C_{(\alpha,k),(\gamma,m)} = \delta_{\alpha^*,\gamma}\,\delta_{k^*,m}$. In particular, S has non-vanishing determinant and therefore is invertible.

Proof. We will deduce this from the following form of $S_{(\alpha,k),(\beta,l)}$ which we derived during the proof of lemma 3.12:

$$S_{(\alpha,k),(\beta,l)} = \sum_{\substack{s\in C_{g_{k^*}},t\in C_{g_l}\\ st=ts}} \mathrm{tr}_{\alpha^*,k^*}\left(\tau_s^{-1}\,t\,\tau_s\right)\,\mathrm{tr}_{\beta,l}\left(\bar\tau_t^{-1}\,s\,\bar\tau_t\right)\ .$$

Then we have

$$
\begin{aligned}
&\sum_{\beta,l} S_{(\alpha,k),(\beta,l)}\, S_{(\beta,l),(\gamma,m)} \\
=&\sum_{\beta,l} \sum_{\substack{s\in C_{k^*},t\in C_l\\ st=ts}} \sum_{\substack{u\in C_{l^*},v\in C_m\\ uv=vu}} \mathrm{tr}_{\alpha^*,k^*}\left(\tau_s^{-1}\, t\, \tau_s\right)\, \mathrm{tr}_{\beta,l}\left(\bar\tau_t^{-1}\, s\, \bar\tau_t\right)\, \mathrm{tr}_{\beta^*,l^*}\left(\sigma_u^{-1}\, v\, \sigma_u\right)\, \mathrm{tr}_{\gamma,m}\left(\bar\sigma_v^{-1}\, u\, \bar\sigma_v\right)
\end{aligned}
$$

Let $h\in G$ be such that $h\, g_{l^*}\, h^{-1}=g_l^{-1}$. A small calculation shows that $h\,\sigma_u\, h^{-1}=\bar\tau_{h\,u\,h^{-1}}\, a$ for some $a\in Z_l$. Instead of summing over $u\in C_{l^*}$ we will sum over $\widetilde{u}=huh^{-1}\in C_{g_l^{-1}}$. Observe that $\widetilde{u}^{-1}\in C_l$. Since we then form the sum over all $t\in C_l$ and all $\widetilde{u}\in C_{g_l^{-1}}$ for all l, we might as well sum over $t\in Z(s), \widetilde{u}\in Z(v)$ if we introduce a factor $\delta^G_{t,\widetilde{u}^{-1}}$. Denote the trace on the dual representation of V^l_β by $\mathrm{tr}_{(\beta,l)^*}$. Then the above simplifies to

$$
=\sum_{\beta} \sum_{\substack{s\in C_{k^*}\\ t\in Z(s)}} \sum_{\substack{v\in C_m\\ \widetilde{u}\in Z(v)}} \delta^G_{t,\widetilde{u}^{-1}}\, \mathrm{tr}_{\alpha^*,k^*}\left(\tau_s^{-1}\, t\, \tau_s\right)\, \mathrm{tr}_{\beta,l}\left(\bar\tau_t^{-1}\, s\, \bar\tau_t\right)\, \mathrm{tr}_{(\beta,l)^*}\left(\bar\tau_{\widetilde{u}}^{-1}\, v\, \bar\tau_{\widetilde{u}}\right)\, \mathrm{tr}_{\gamma,m}\left(\bar\sigma_v^{-1}\, \widetilde{u}\, \bar\sigma_v\right)
$$

Due to the factor $\delta^G_{t,\widetilde{u}^{-1}}$ we must have $\widetilde{u}=a\,t^{-1}\,a^{-1}$ for $a\in G/Z(t)$. Observe that $\bar\tau_{a\,t^{-1}\,a^{-1}}= a\,\bar\tau_{t^{-1}}\, b = a\,\bar\tau_t\, b$ for some $b\in Z_l$. Thus, if we set $\widetilde{v}=a^{-1}\,v\,a$ the above simplifies further to

$$
\begin{aligned}
&=\sum_{\beta} \sum_{\substack{s\in C_{k^*}\\ t\in Z(s)}} \sum_{\substack{\widetilde{v}\in C_m\\ a\in G/Z(t)}} \mathrm{tr}_{\alpha^*,k^*}\left(\tau_s^{-1}\, t\, \tau_s\right)\, \mathrm{tr}_{\beta,l}\left(\bar\tau_t^{-1}\, s\, \bar\tau_t\right)\, \mathrm{tr}_{(\beta,l)^*}\left(\bar\tau_t^{-1}\, \widetilde{v}\, \bar\tau_t\right)\, \mathrm{tr}_{\gamma,m}\left(\bar\sigma_{\widetilde{v}}^{-1}\, t^{-1}\, \bar\sigma_{\widetilde{v}}\right) \\
&=\sum_{\substack{s\in C_{k^*}\\ t\in Z(s)}} \sum_{\widetilde{v}\in C_m} |C_t|\,|Z^t_s|\,\delta^{Z(t)}_{s,\widetilde{v}}\, \mathrm{tr}_{\alpha^*,k^*}\left(\tau_s^{-1}\, t\, \tau_s\right)\, \mathrm{tr}_{\gamma,m}\left(\bar\sigma_{\widetilde{v}}^{-1}\, t^{-1}\, \bar\sigma_{\widetilde{v}}\right)\ ,
\end{aligned}
$$

where Z^t_s denotes the centralizer of $s\in Z(t)$. Note that this sum only yields non-vanishing terms if $\widetilde{v}\in C_{k^*}$ due to the factor $\delta^{Z(t)}_{s,\widetilde{v}}$. Therefore we have $\bar\sigma_v=\tau_v$. Moreover, like above we have $\widetilde{v}=c\,s\,c^{-1}$ for some $c\in Z(t)/Z^t_s$. Thus, after a transformation as above, we have

$$
\begin{aligned}
&=\sum_{\substack{s\in C_{k^*}\\ t\in Z(s)}} |C_t|\,|Z(t)|\,\delta_{k^*,m}\, \mathrm{tr}_{\alpha^*,k^*}\left(\tau_s^{-1}\, t\, \tau_s\right)\, \mathrm{tr}_{(\gamma,m)^*}\left(\tau_s^{-1}\, t\, \tau_s\right) \\
&=|G|\,\delta_{k^*,m} \sum_{\substack{s\in C_{k^*}\\ t\in Z(s)}} \mathrm{tr}_{\alpha^*,k^*}\left(\tau_s^{-1}\, t\, \tau_s\right)\, \mathrm{tr}_{(\gamma,m)^*}\left(\tau_s^{-1}\, t\, \tau_s\right) \\
&=|G|\,\delta_{k^*,m} \sum_{s\in C_{k^*}} |Z(s)|\,\delta_{\alpha^*,\gamma} = |G|^2\,\delta_{k^*,m}\,\delta_{\alpha^*,\gamma}\ . \qquad \square
\end{aligned}
$$

We have finally proven the following:

Corollary 3.14. *The representation category* $\mathrm{Rep}(D(G))$ *of the quantum double* $D(G)$ *of a finite group* G *is a modular tensor category.*

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