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**Proper Actions, Nonlinearity and Homotopy
Theory**

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Proper Actions, Nonlinearity and Homotopy Theory

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We describe some operator theoretical features of equivariant stable homotopy. We pursue mainly three groups of targets in this. First, we give *geometrical* meaning to our theory by describing its cocycles. Second, we pursue the development of an Index theory which compares these constructions, having the hope of applications as a secondary, though important goal. Finally, we allow by these means the study of more general group actions, which are relevant in other fields of Mathematics.

Our ideas can be traced back to several sources. On the one side, index theoretical results by Dold [Dol74], but also the almost unknown but foundational work of A. S. Schwartz [Sva64] in nonlinear functional analysis can be considered as special cases of the situation described here. More recently, Stefan Bauer and Mikio Furuta used related methods to define invariants of four manifolds, which initially motivated our interest in these questions. Some results, in particular proposition 11, are just parametrized versions of results in [BF04]. The third source of our interest is the work of N.C Phillips on equivariant K -theory for proper actions, mainly in [Phi89]. We use his construction in a crucial way.

We begin by describing cocycles for equivariant cohomotopy. They are defined in terms of certain nonlinear operators on real G -Hilbert bundles, so we briefly recall first some well known facts in linear functional analysis parametrized over a space. A comprehensive treatment of them is given in the book [Phi89]. For matters of reality and functional analysis we refer to the text [Sch93], in particular to connections with Kasparov theory.

We point out that these constructions require proper actions of locally compact, second countable groups as an input. In the first part of this work we restrict our attention to proper actions of lie groups, though we also explain the possibility of modifying Phillips' slightly more general arguments to this setting.

0.1 Cocycles for equivariant cohomotopy

We recall first some basic definitions and technical facts of equivariant topology.

Definition 1. Let G be a second countable, locally compact Hausdorff group. Recall that a G -space is proper if the map

$$G \times X \xrightarrow{\theta_X} X \times X \\ (g, x) \mapsto (x, gx)$$

is proper.

The notion of a G -CW complex [Lüc89], p. 8 will be of relevance in this work. Recall that a G -CW complex structure on the pair (X, A) consists of a filtration of the G -space $X = \cup_{-1 \leq n} X_n$ beginning mit A and for which every space is inductively obtained from the previous one by attaching cells in pushout diagrams of the type

$$\begin{array}{ccc} \coprod_i S^{n-1} \times G/H_i & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_i D^n \times G/H_i & \longrightarrow & X_n \end{array}$$

We say that a proper G -CW complex is finite if it consists of a finite number of cells $G/H \times D^n$.

The following result enumerates some facts which we will need in this work, which are proven in chapter one of [Lüc89]:

Proposition 1. Let (X, A) be a proper G -CW pair

1. The inclusion $A \rightarrow X$ is a closed cofibration.
2. A is a neighborhood G -deformation retract, in the sense that there exists a neighborhood $A \subset U$, which is a G -equivariant deformation retract. The neighborhood can be chosen to be closed or open.

Some equivariant Hilbert bundles are the main objects on which our constructions begin. Let us recall basic terminology of [Phi89], which works in a slightly more general context.

Definition 2. Let X be a locally compact, Hausdorff proper G -space. A Banach bundle over X is a locally trivial fiber bundle E with fiber on a Banach space H , whose structure group is the set of all isometric linear bijections of H with the strong topology.

If H is a (real) Hilbert space, we will speak of a Hilbert bundle. A G -Hilbert bundle is a map $p : E \rightarrow X$ such that there is a continuous action of the locally compact group G in the total space, the map p -called the projection- is assumed to be G -equivariant and the action on the total space is given by fiberwise linear isometries.

Let E and F be Hilbert bundles over X . A linear morphism from E to F is an equivariant, continuous function $t : E \rightarrow F$ covering the identity on X , and for which the fiberwise adjoint t^* defined by $\langle t^*x, y \rangle = \langle x, ty \rangle$ is continuous. The support of a morphism t is the set $\{x \in X \mid tx \neq 0\}$

A linear morphism $t : E \rightarrow F$ is a compact morphism if it is fiberwise compact in the usual sense (that is, for every x , $t_x \cdot E_x \rightarrow F_x$ maps bounded sets to conditionally compact sets) and in addition, for every point $x \in X$, there exist local trivializations $a : E|_U \rightarrow U \times E_x$, $b : F|_U \rightarrow U \times F_x$ such that $bt a^{-1} : U \times E_x \rightarrow U \times F_x$ is given by an expression $(y, x) \mapsto (y, \psi(x))$ for a norm continuous map $\psi : U \rightarrow L(E_x, F_x)$ into the bounded linear maps between the fibers over x .

A linear morphism $t : E \rightarrow F$ is said to be Fredholm if there exists a morphism $s : F \rightarrow E$ such that $st - 1$ and $ts - 1$ are compact morphisms with compact supports. A Fredholm morphism is said to be essentially unitary if one can take $t = s^*$ in the definition. We recall the existence of G -invariant Riemannian metrics on vector bundles over proper G -spaces, which is proved for instance in [Pal61] as consequence of the slice theorem.

Remark 1. 1. By a theorem of Kuiper, all Hilbert bundles with separable infinite dimensional fibers are trivial. However, we will not use this fact.

2. Note that we choose the structure group of our bundles to be the isometric bijections with the strong operator topology. Other choices, like the weak topology would not give *enough vector bundles*, as pointed out by Phillips in [Phi89], chapter 9.

3. Note that we forced the existence of adjoints for our morphisms. This is technically convenient.
4. Well known facts from functional analysis go through “fibrewise”. For instance, a Fredholm morphism has always fiberwise finite dimensional kernel and cokernel, and the index is a virtual G -vector bundle of finite dimension.

We now enumerate a collection of facts on linear morphisms and G -Hilbert Bundles which we will use later.

Proposition 2 (Proper stabilization theorem). Let E be a G -Hilbert bundle over a proper G -space. Denote by \mathcal{H} the numerable Hilbert space consisting of the numerable sum of the space of square integrable functions in G , in symbols $\mathcal{H} = \bigoplus_{n=1}^{\infty} L^2(G)$. Let $\mathcal{H} \times X$ be the associated trivial Hilbert G -bundle. There exists an equivariant linear isomorphism of vector bundles

$$E \oplus \mathcal{H} \times X \cong \mathcal{H} \times X$$

Proof. Theorem 2.9, p. 29 of [Phi89], Theorem 2.1.4, p. 58 of [Sch93]. We point out that Phillips realizes this isomorphism to be an adjointable morphism between the G - $C_0(X)$ -Hilbert modules $\Gamma(E) \otimes_{C_0(X)} \mathcal{H} \times X$ and $\mathcal{H} \otimes C_0(X)$. The identification of such a morphism with an isomorphism of G -Hilbert bundles is consequence of lemma 1.9 in [Phi89]. \square

Recall N.C. Phillip’s definition of equivariant K -theory for proper actions of locally compact groups, [Phi89].

Definition 3. $K_0^G(X)$ is represented by cocycles (E, F, l) , where E and F are G -Hilbert bundles and $l : E \rightarrow F$ is a fiberwise *linear* Fredholm morphism. A cocycle is said to be *trivial* if l is fibrewise unitary. Two cocycles $(E_i, F_i, l_i)_{i=0,1}$ are equivalent if there exists a trivial cocycle τ such that $l_0 \oplus \tau = l_1 \oplus \tau$ is homotopic to a trivial morphism.

There exists at least one equivalent approach to the definition of equivariant K -theory in this context. We just recall one of them as it is needed as a technical modification in one of our arguments.

- $K_0^G(X)$ can be realized as the set of essentially unitary fredholm linear morphisms. Two such cocycles are said to be equivalent if they become homotopic via an essentially unitary homotopy after adding a trivial cocycle(Theorem 4.7, page 56 in [Phi89].)

We now enumerate two consequences of the proper stabilization theorem, which are fundamental for verifying excision in Phillips’ construction of equivariant K -theory.

Proposition 3. Let $i : U \rightarrow X$ be the inclusion of a G -invariant, open subset of X . Let (E, F, t) be a linear cocycle over U such that t is fibrewise bounded and has a bounded fredholm inverse. Then, there exists a linear cocycle $(X \times \mathcal{H}, X \times \mathcal{H}, r)$ $i^*(r)$ and t agree after adding a unitary linear cocycle.

Proof. Proposition 5.9, p. 74 in [Phi89]. We recall that the constructed classes agree after application of the proper stabilization theorem. \square

Proposition 4. Let X be a proper G -CW complex and $\varphi : E \rightarrow E$ be a fibrewise fredholm operator defined on the space $X \times I$. Denote by $\varphi_0 : E_0 \rightarrow E_0$ the restriction to $X \times \{0\}$. Then:

1. There exists a unitary cocycle ρ between (E, E, φ) and (E_0, E_0, φ_0) . Moreover, the isomorphism can be taken to be unitary over a fixed, invariant subspace $A \subset X$.
2. Let $A \subset X$ be a G -subcomplex and $l : F|_A \rightarrow E|_A$ be a bounded morphism. Then, there exists a linear cocycle (l', E, F) , defined over X such that $i^*(l')$ and l are equivalent.
3. Let $A \subset X$ be a G -invariant closed subset and $U \subset X$ an open neighborhood of which A is a deformation retract. Suppose that $(U \times K, U \times K, l)$ is an essentially unitary cocycle over U , where K is a strong unitary G -representation in a hilbert space. Then, there exists an essentially unitary cocycle $(X \times H, X \times H, F)$ such that $i^*(F) = l$.

Proof. 1. As the involved bundles are locally trivial, the total space of E carries the weak topology with respect to the set $p^{-1}(X_i)$, where X_i is an element of the filtration in the basis G -CW complex structure cfr. lemma 1.26 in [Lüc89]. Hence, the statement reduces to the case where $(X, A) = (G/H \times D^n, G/H \times S^{n-1})$ and E has the form $G \times \underset{H}{\mathcal{H}}$, for a given strong, unitary and continuous H -representation \mathcal{H} in a separable hilbert space. Let $\mathcal{U}_c(\mathcal{H})$ be the subspace of the H -equivariant, unitary operators u in \mathcal{H} , for which the conjugation with an arbitrary element $h^{-1}uh$ is a continuous operator (recall that this is a contractible space after results of Segal, it is an ANR). Giving an isomorphism ρ as described above amounts to give a map $\rho \in \text{Map}(D^n \times I, \mathcal{U}_c(\mathcal{H}))$ which is the identity on $G/H \times S^{n-1} \times \{0\}$. There is no obstruction for doing this because the inclusion $G/H \times S^{n-1} \rightarrow G/H \times D^n$ is a cofibration and $\mathcal{U}_c(\mathcal{H})$ is contractible. \square

Remark 2. Note the technical difficulties arising in the extension of linear morphisms over arbitrary spaces. As explained in point 3, we are only able to extend morphisms which are defined on globally trivial hilbert bundles. The attempt to overcome this difficulty by extending a section of a bundle instead of a morphisms breaks down essentially because the morphisms as defined here are the sections of a bundle whose fiber is the space of operators $L(H_1, H_2)$ with the $*$ -strong topology, which is not even a Banach bundle.

The second ingredient for our construction of cocycles for stable cohomotopy are some basic notions of nonlinear functional analysis. This is a complete subject itself, whose developements are certainly less known by ortodoxe homotopy theorists. More or less encyclopedical references to this topic are the books [Ber77] and [Dei85]. The reader interested in its relationship to Hopf bifurcation problems should read [Ize05].

Relying on our interests, we shall change between two equivalent approaches to this. One of them, the “no-zeros on the boundary- picture” will be useful in applications and has its origin in the theory of partial differential equations. For our purposes in algebraic topology, we shall give the following, technically more convenient

Definition 4. Suppose that $k : H \rightarrow H$ is a (possibly nonlinear) compact map in a (real) Hilbert space, in the sense that k sends bounded sets into relatively compact sets. A map $f = t + k : H \rightarrow H$ is a compact perturbation of the selfadjoint Fredholm linear map f if the preimages of bounded sets under f are bounded. Let $t : E \rightarrow F$ be a Fredholm morphism between the G -Hilbert bundles over the proper G -space X . A compact perturbation of t is a continuous, equivariant map $f : E \rightarrow F$, which has fibrewise the form $t + k$ for a compact perturbation of f .

Definition 5. Let X be a locally compact, proper G -space and let G be a locally compact group. Let (E, F, l) be a linear, real cocycle in the sense of Phillips.

A cocycle for the equivariant Fix-theory of X , $\text{Fix}_G^l(X)$ is a four-tuple (E, F, l, c) where

- E, F are real G -Hilbert bundles over X , with a linear, selfadjoint Fredholm morphism $\tilde{l} : E \rightarrow F$, which is equivalent to l in the sense of Phillips.
- A compact perturbation of l, c , such that the map $l + c$ is proper and extends to a map between one-point compactification bundles.

Two cocycles E, F, l, c and E', F', l', c' are equivalent if there is a linear, unitary cocycle (H, H', k) such that $E \oplus H, F \oplus H, l \oplus k$ and $E' \oplus H', F' \oplus H', l' \oplus k$ are isomorphic as linear cocycles, by an isomorphism which preserves the compact perturbations.

Two cocycles E, F, l, c and E, F, l, c' are homotopic if there exists a homotopy $H : S^E \times I \rightarrow S^F$, pointed over every fiber and relative to l such that $H|_{0} = l + c, H|_{1} = l + c'$.

The set $\text{Fix}_G^l(X)$ is called the equivariant Fix-theory of X in degree l .

Remark 3 (The “no-zeros in the boundary” -picture). Let X be a locally compact, Hausdorff proper G -space. Let E and F be real G -Hilbert bundles over X . Let $t : E \rightarrow F$ be a Fredholm morphism between them. A compact perturbation of t is a continuous equivariant map which is fibrewise of the form $t+k$, where k is a *non necessary linear*, compact fibrewise map (it maps fibrewise bounded sets to relatively compact subsets) defined on the unitary disk $D(E) = \{x \in E \mid |x| \leq 1\}$. We introduce the notation $C_t(D, F)$ for the set of compact perturbations of the Fredholm operator t , with the topology which is given by the supremum norm, and obviate the corresponding fibrewise definition. Moreover, we shall restrict us to the subspace $C_t^{\neq 0}(D, F)$ consisting of those compact perturbations for which $cx = 0$ has no solutions on the boundary of the disk D . It is known that all (Fréchet) smooth maps $f \in C_t^{\neq 0}(D, F)$ are proper, as it is proven as a consequence of the implicit function theorem in [Sma65].

Let $f = t + k$ be a compact perturbation of a Fredholm map, as given in the definition. Then f extends to a map of one point compactifications of the Hilbert

space $f : S^H \rightarrow S^H$. Moreover, homotopy classes of pointed nonlinear maps are in correspondence with nonzero compact homotopy classes of perturbations defined on the unitary disk. The same holds parametrized and equivariant over a proper G -space.

This is essentially due to the fact that the maps defined in the previous construction amount to maps of pairs $(D(H), \partial D(H)) \rightarrow (H, H - \{0\})$, which are equivalent via excision and homotopy equivalence to $(S^H, \{\infty\})$, via the intermediary pairs $(S^H, H - \{0\})$ and $(S^H, S^H - \text{Int}D(H))$. Note that the point at infinity in the pointed picture is identified with the point $(0, 1)$ in the *no-zeros on the boundary* picture, if we identify the pair $D(H) \subset H \oplus \mathbb{R}$ in the usual way, and everything holds fibrewise and equivariant over a proper G -space.

We describe now an additive structure in equivariant Fix-theory.

Let $(E_0, F_0 l_0, c_0)$ and $(E_1, F_1 l_1, C_1)$ be cocycles in the equivariant Fix-theory of a given degree l .

Let us suppose without loss of generality that we have representatives of the form (E_0, F_0, l, c_0) and (E_1, F_1, l, c_1) . Let $X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ be the trivial bundle and \cdot Denote by $S^{\mathbb{R}}$ the one point compactification bundle. Define the pinching map $S^{\mathbb{R}} \rightarrow S^{\mathbb{R}} \vee S^{\mathbb{R}}$ as the obvious extension of the map which sends the positive ray under the map $\ln(x)$ and $-\ln(-x)$ in the negative ray.

The sum of two cocycles is represented by the cocycle $(E_0 \oplus \mathbb{R}, F_0 \oplus \mathbb{R}, l \oplus \text{id}, c)$, where $l \oplus \text{id} + c : S^{E_0 \oplus \mathbb{R}} \rightarrow S^{F_0 \oplus \mathbb{R}}$ is given as the following composition

$$S^{E_0} \wedge_X S^{\mathbb{R}} \xrightarrow{\text{id} \wedge_X \nabla} S^{E_0} \wedge_X S^{\mathbb{R}} \vee_X S^{\mathbb{R}} \xrightarrow{\cong} S^{E_0 \oplus \mathbb{R}} \vee_X S^{E_0 \oplus \mathbb{R}} \xrightarrow{(l \oplus \text{id} + c_0) \wedge_X (l \oplus \text{id} + c_1)} S^{F_0 \oplus \mathbb{R}}$$

The zero element is represented by a cocycle $(E \oplus \mathbb{R}, F \oplus \mathbb{R}, l, c)$ such that c extends to a map sending fibrewise S^E to the point at infinity. .

The inverse of an element (E, F, l, c) is represented by the element $(E, F, l, -c)$.

Proposition 5. The operations described above turn the equivariant Fix-theory in degree l into an abelian group.

There is a relative version for pairs (X, A) of proper G -CW complexes. An element in Fix_l is represented by a compact perturbation of a fibrewise perturbation of a fredholm operator $\varphi + c : E \rightarrow F$, which extends to the one-point compactification bundles being constant over the subspace A , with the value at infinity. Note that this is consistent with the usual identification of X with the pair (X, ϕ) .

Alternatively, in the “no-zeros in the boundary”-picture, a cocycle for the equivariant Fix-theory can be chosen to be a compact perturbation of a fredholm map $\varphi + c : DE \rightarrow F$ without zeros on the boundary, which is constant over A with value $(0, 1)$. We now construct a multiplicative structure on the equivariant Fix-theory.

$$\cup : \text{Fix}_G^{l_1} \times \text{Fix}_G^{l_2} \rightarrow \text{Fix}_G^{l_1+l_2}(X, A_1 \cup A_2)$$

Consider for this representing elements $u_i = \varphi_i + c_i \in \text{Fix}_G^{l_i}(X, A_i)$ for $i \in \{1, 2\}$, where c_i is a compact map accepting fibrewise an extension to the one-point compactification, constant over A_i with the value at infinity. $u_1 \cup u_2$ is the cocycle defined as $(E_1 \oplus E_2, \varphi_1 \oplus \varphi_2, C)$ where the map C is such that $C : (e_1, e_2) \mapsto (c_1(e_1), c_2(e_2))$. Note that this map allows an extension to the one-point compactification.

Proposition 6. Let (X, A) be a proper G -CW pair. There exists a natural sequence

$$\mathrm{Fix}_l(X, A) \xrightarrow{\rho^*} \mathrm{Fix}_l(X) \xrightarrow{i^*} \mathrm{Fix}_l(A)$$

which is exact in the middle, where ρ and i denote the inclusion of A into X and X into (X, A) , respectively.

Proof. That $i^* \circ \rho^* = 0$ is clear from the definitions. Let now $\varphi + c : E \rightarrow F$ be a cocycle for which $i^*([\varphi + c])$ is compactly homotopic to the trivial morphism over A .

In view of proposition 2, we can choose a representative (which we denote by the same symbols) for which both E and F are the trivial G -Hilbert bundle $\mathcal{H} \times X \rightarrow X$ and c is constant over A with value ∞ . Using proposition 3, we can assume up to equivalence that the Fredholm linear operator extends to a once $\tilde{\varphi}$ in all of X .

Suppose that there is a homotopy $h_t : i^*S^E \times I \rightarrow i^*S^F$ defined over A which begins with $i^*[\varphi + c]$ and ends with a map $\varphi + c$ which sends the space E to the base point at infinity. As $X - A$ is build up of a finite number of equivariant cells, one can argue inductively to extend h to a map $H : S^E \rightarrow S^F$, defined on all X such that $h|_{A \times I} = h$. H determines a homotopy between certain element $\rho^*(\tilde{\varphi} + \tilde{c})$ defined over X and $\varphi + c$. □

Proposition 7. Let (X, B) be a proper, finite G -CW pair obtained as the pushout with respect to the cellular map $(f, F) : (X_0, A) \rightarrow (X, B)$ as in the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{F} & X \end{array}$$

then the map $(f, F)^* : \mathrm{Fix}_G^l(X, B) \rightarrow \mathrm{Fix}_G^l(X_0, A)$ induces a natural isomorphism.

Proof. Let $(E, F\varphi + C) \in \mathrm{Fix}_G^l(X_0, A)$. Due to proposition 3, it is possible to assume that there exists a linear morphism $\tilde{\varphi} : E \oplus E' \rightarrow F \oplus E'$ defined over X such that $F^*(\tilde{\varphi})$ and φ differ by addition of an unitary cocycle. As $c|_A = \infty$, it is possible to extend c to a map \tilde{c} defined on X for which $\tilde{c}|_B = \infty$. Then $(F^*, f^*) : \mathrm{Fix}_G^*(X, B) \rightarrow \mathrm{Fix}_G^*(X_0, A)$ sends $\tilde{\varphi} + \tilde{c}$ to $\varphi + c$. This proves surjectivity. To prove injectivity, recall that if $\varphi + h_t : E \times I \rightarrow E$ is a nullhomotopy starting with $(F^*, f^*)(\varphi + c)$, ending with the constant ∞ . As before, as c is constant over B and $(F^*, f^*)(c)$ is trivial over A , it is possible to extend the map h_t to a homotopy \tilde{h}_t defined over X which is trivial over B and which begins with $\varphi + c$, ends with a constant map. this shows that the map is injective. □

The induction structure is defined as follows. To illustrate the proof and avoid technicalities, we restrict ourselves to the case where G is a Lie group.

Proposition 8. Let $\alpha : H \rightarrow G$ be a lie group homomorphism. Then there exists a group homomorphism

$$\mathrm{Fix}_G^*(\mathrm{ind}_\alpha(X, A)) \rightarrow \mathrm{Fix}_H^*(X, A)$$

satisfying

1. Bijectivity. If $\ker(\alpha)$ acts freely on (X, A) , then the map is an isomorphism.
2. Compatibility with the boundary homomorphisms. $\delta_H^n \circ \text{Ind}_\alpha = \text{ind}_\alpha \circ \delta_G^n$.
3. Functoriality. If $\beta : G \rightarrow K$ is a group homomorphism, then the diagram commutes:
4. Compatibility with conjugation. For any $g \in G$, the homomorphism

$$\text{ind}_{c(g):G \rightarrow G} \text{Fix}_G^n(X, A) \rightarrow \text{Fix}_G^n(\text{ind}_{c(g):G \rightarrow G}(X, A))$$

agrees with the map $\pi_G^n(f_2)$, where $f_2 : (X, A) \rightarrow \text{ind}_{c(g):G \rightarrow G}$ sends x to $(1, g^{-1}x)$ and $c(g)$ is the conjugation isomorphism in G

Proof. 1. Let $\varphi + c : E \rightarrow E$ be a compact perturbation over the space $(\text{ind}_\alpha X, A)$. The map $i : X \rightarrow \text{ind}_\alpha(X)$ ($x \mapsto (1_G, X)$) induces a group homomorphism $\text{Fix}_G^*(\text{ind}_\alpha X, A) \rightarrow \text{Fix}_H^*(X, A)$. An inverse is given by the map which associates to a linear cocycle (E, F, φ) the cocycle $(E/H, F/H, \varphi/H)$. It is easy to show that this is the case for the perturbation and that this still satisfies the boundedness condition. We point out that this only makes sense for lie groups, and cannot be generalized to the more general context of locally compact groups, see remark below.

2. Follows from the naturality of the induced bundle constructions.
3. Follows from the functoriality of the induced vector bundle construction.
4. Compatibility with conjugation. Follows from element chasing in the diagram

$$\begin{array}{ccc} \text{Fix}_G^l(\text{ind}_{c(g)}(X, A)) & \xrightarrow{\text{ind}_{c(g)}} & \text{Fix}_G^l(X, A) \\ \uparrow & \nearrow \text{Fix}_G^n(f_2) & \\ \text{Fix}_G^l(\text{ind}_{c(g)}(X, A)) & & \end{array}$$

where $f_2 : (X, A) \rightarrow \text{ind}_{c(g)}(X, A)$ is given by $x \mapsto (1, g^{-1}x)$ and $c(g)(g) = gg'g^{-1}$.

□

Remark 4. N.C. Phillips proves in [Phi89], Theorem 8.1, p. 123-132 a more general result, allowing proper actions of locally compact groups as input, instead of only lie groups. It is certainly plausible to try to generalize them, though we prefer to keep the arguments in this note as easy as possible. The main point in this is that we cannot guarantee the local triviality of bundles $E/H \rightarrow X/H$ unless H is lie, see [Pal61].

We now prove following properties which turn the equivariant Fix-theory into a G -cohomology theory, the first of which is

Proposition 9 (homotopy invariance). Let $f_0, f_1 : (X, A) \rightarrow (Y, B)$ be two G -maps of pairs of proper G -CW complexes. If they are homotopic, then $\text{Fix}_G^*(f_1) = \text{Fix}_G^*(f_2)$.

Proof. in view of the naturality of the construction, this amounts to prove that $\text{Fix}_G^*(h) = \text{id}$ for the map $h : (X, A) \times I \rightarrow (X, A) \times I$ given by $(x, t) \mapsto (x, 0)$. Let (E, F, l, c) be a nonlinear cocycle representing an element in $\text{Fix}_G^*(X, A \times I)$. In the notation of proposition 4, there exist unitary morphisms $u : E \rightarrow h^*(E)$, $v : F \rightarrow h^*(F)$ covering the identity $X \times I \rightarrow X \times I$ such that the restrictions to E_0, F_0 are the respective identities. Note that the composition $f = h^*(E) \xrightarrow{u^{-1}} E \xrightarrow{\varphi+c} F \xrightarrow{v^{-1}} h^*(F)$ is homotopic to $h^*(\varphi + c)$ relative to $h^*(\varphi)$. After checking out the conditions for the definition of Fix_G^* , one has that the equivalence classes $\text{Fix}_G^*(h)(\varphi + c) = [f] = [\varphi + c]$ agree. \square

We now construct a suspension isomorphism $\sigma_n^{X,A} : \pi_G^n(X, A) \rightarrow \pi_G^{n+1}((X, A) \times (I, \{0, 1\}))$. Given $\varphi + c : E \rightarrow F \in \pi_G^n((X, A))$, form the bundle $E' = E \oplus \mathbb{R}$, denote by p the fibrewise projection on $E' \rightarrow E$ and define the map $\sigma_n(\varphi + c) : DE' \times I \rightarrow E' \times I$ defined as $(e, t) \mapsto p \circ \varphi + (\log(t) - (\log(\frac{1}{t})))c(p(v))$. By this means, we obtain a compact map which extends to the fibrewise one-point compactifications, being trivial on the required subspace. Given an element $\varphi + c \in \pi_G^{n+1}((X, A) \times I, \{0, 1\})$, consider a unitary, fibrewise linear cocycle $u : E \rightarrow E_0 \times I$ $v : F \rightarrow F_0$ covering the identity $X \times I \rightarrow X \times I$, which restricted over the subspace $A \times I \cup X \times \{0, 1\}$ is the identity map. The map constructed as $E_0 \times I \xrightarrow{u^{-1}} E \xrightarrow{\varphi+c} F \xrightarrow{v} F_0 \times I$ determines an inverse for $\sigma_n(X, A)$.

A coboundary map is defined as the composition

$$\begin{aligned} \text{Fix}_G^n(A) &\xrightarrow{\sigma_G^n(A)} \text{Fix}_G^{n+1}(A \times I, A \times \{0, 1\}) \\ &\xrightarrow{\text{Fix}_G^{n+1}(i_1)^{-1}} \text{Fix}_G^{n+1}(X \cup_{A \times \{0\}} A \times I, X \amalg A \times \{1\}) \\ &\xrightarrow{\text{Fix}_G^{n+1}(i_2)} \text{Fix}_G^{n+1}(X \cup_{A \times \{0\}} A \times I, A \times \{1\}) \\ &\xrightarrow{\text{Fix}_G^{n+1}(\text{pr}_1)^{-1}} \text{Fix}_G^{n+1}(X, A) \end{aligned}$$

Where the map $\text{Fix}_G^{n+1}(i_1)$ is bijective by excision and $\text{Fix}_G^{n+1}(\text{pr}_1)$ is bijective because of homotopy invariance.

Several approaches have been proposed towards the definition of equivariant cohomotopy theory for proper actions. The recent work of Lück [Lüc05] using finite dimensional bundles deals with the difficulties appearing in the case where a discrete group acts on a finite G -CW complex. We briefly recall this approach

Fix an equivariant, proper G -CW complex. Form the category $\text{SPHB}^G(X)$ having as objects the G -sphere bundles over X . A morphism from $\xi : E \rightarrow X$ to $\mu : F \rightarrow X$ is a bundle map $S^\xi \rightarrow S^\mu$ covering the identity in X , which preserves fiberwise the basic points. A homotopy between the morphisms u_0, u_1 is a G -bundle map $h : S^\xi \times [0, 1] \rightarrow S^\mu$ from the bundle $S^\xi \times [0, 1] \rightarrow [0, 1] \times X$ to the bundle S^μ covering the projection $X \times [0, 1] \rightarrow X$ and preserving the base points on every fiber such that its restriction to $X \times \{i\}$ is u_i for $i = 0, 1$. Let $\underline{\mathbb{R}}^n$ be the trivial vector bundle over X , which is furnished with the trivial

action of G . Two morphisms of the form

$$S^{\xi_i \oplus \mathbb{R}^{k_i}} \rightarrow S^{\xi_i \oplus \mathbb{R}^{k_i+n}}$$

are said to be equivalent if there are objects μ_i in $\text{SPHB}^G(X)$ and an isomorphism of vector bundles $\nu : \mu_0 \oplus \xi_0 \cong \mu_1 \oplus \xi_1$ such that the following diagram of morphisms in $\text{SPHB}^G(X)$ commutes up to homotopy

$$\begin{array}{ccc} S^{\mu_0 \oplus \mathbb{R}^{k_1}} \wedge_X S^{\xi_0 \oplus \mathbb{R}^{k_0}} & \xrightarrow{\text{id} \wedge_X u_0} & S^{\mu_0 \oplus \mathbb{R}^{k_1}} \wedge_X S^{\xi_0 \oplus \mathbb{R}^{k_0+n}} \\ \sigma_1 \downarrow & & \downarrow \sigma_2 \\ S^{\mu_0 \oplus \xi_0 \oplus \mathbb{R}^{k_0+k_1}} & & S^{\mu_0 \oplus \xi_0 \oplus \mathbb{R}^{k_0+k_1+n}} \\ S^{\nu \oplus \text{id}} \downarrow & & \downarrow S^{\nu \oplus \text{id}} \\ S^{\mu_1 \oplus \xi_1 \oplus \mathbb{R}^{k_0+k_1}} & & S^{\mu_1 \oplus \xi_1 \oplus \mathbb{R}^{k_0+k_1+n}} \\ \sigma_3 \downarrow & & \downarrow \sigma_4 \\ S^{\mu_1 \oplus \mathbb{R}^{k_0}} \wedge_X S^{\xi_1 \oplus \mathbb{R}^{k_1}} & \xrightarrow{\text{id} \wedge_X u_1} & S^{\mu_1 \oplus \mathbb{R}^{k_0}} \wedge_X S^{\xi_1 \oplus \mathbb{R}^{k_1+n}} \end{array}$$

where the isomorphisms σ_i are determined by the fiberwise defined homeomorphism $S^{V \oplus W} \approx S^V \wedge S^W$ and the associativity of smash products, which holds for every pair of representations V, W .

Definition 6. Let X be a G -CW complex. We define its n -th G -equivariant stable cohomotopy group, $\pi_G^n(X)$ as the set of homotopy classes of equivalence classes of morphisms $u : S^{\xi \oplus \mathbb{R}^k} \rightarrow S^{\xi \oplus \mathbb{R}^{k+n}}$ under the above mentioned relation. For a G -CW pair, (X, A) we define $\pi_G^n(X, A)$ as the equivalence classes of morphisms which are trivial over A , i.e. those which are given by a representative $u : S^{\xi \oplus \mathbb{R}^k} \rightarrow S^{\xi \oplus \mathbb{R}^{k+n}}$ which satisfies that over every point $a \in A$, the map $u_a : S^{\xi_a \oplus \mathbb{R}^k} \rightarrow S^{\xi_a \oplus \mathbb{R}^{k+n}}$ is constant with value the base point. For a pair of bundle morphisms $u : S^{\xi \oplus \mathbb{R}^k} \rightarrow S^{\xi \oplus \mathbb{R}^{k+n}}$, $v : S^{\xi' \oplus \mathbb{R}^k} \rightarrow S^{\xi' \oplus \mathbb{R}^{k+n}}$, the sum is defined as the homotopy class of the morphism

$$\begin{aligned} u : S^{\xi \oplus \xi' \oplus \mathbb{R}^k} \wedge_X S^{\mathbb{R}} &\xrightarrow{\text{id} \wedge_X \nabla} S^{\xi \oplus \xi' \oplus \mathbb{R}^k} \wedge_X (S^{\mathbb{R}} \vee_X S^{\mathbb{R}}) \xrightarrow{\sigma_3} \\ &(S^{\xi \oplus \mathbb{R}^k} \wedge_X S^{\mathbb{R}}) \vee_X (S^{\xi' \oplus \mathbb{R}^k} \wedge_X S^{\mathbb{R}}) \xrightarrow{(u \wedge_X \text{id}) \vee_X (v \wedge_X \text{id})} \\ &S^{\xi \oplus \xi' \oplus \mathbb{R}^{k+n}} \wedge_X S^{\mathbb{R}} \end{aligned}$$

where σ_3 is the canonical isomorphism given by the fiberwise distributivity and associativity isomorphisms and ∇ denotes the pinching map $S^{\mathbb{R}} \rightarrow S^{\mathbb{R}} \vee S^{\mathbb{R}}$. The relative version for elements lying in the group of a pair, $\pi_G^n(X, A)$ translates word by word when one sets all sphere bundles and morphisms to be trivial over A .

Our approach extends this notion and solves the principal problem of this construction, namely: the the lack of finite dimensional G -vector bundles to represent excisive G -cohomology theories. The crucial result in this is the construction of an index theory in the context of parametrized nonlinear analysis. The immediate goal is

Theorem 1. Let G be a discrete group acting on a G -CW pair (X, A) . Denote by l a selfadjoint fredholm morphism whose fibrewise index is a trivial virtual vector bundle of dimension p . The *parametrized Schwartz index* defines an isomorphism

$$\text{Fix}_G^l(X, A) \rightarrow \pi_G^p(X, A)$$

We give two proofs of theorem 1, relying in the two approaches to nonlinear analysis: the *no-zeros on the boundary* and the proper maps one. Because of its simplicity, we give now the proof relying on the first one. We point out that the techniques in the first case can only be used for fredholm operators of positive fredholm index, whereas the second one is valid more generally, at the price of technicalities. In both cases, the initial point is an approximation lemma for compact perturbation of fredholm maps, which takes the following form:

Proposition 10. Suppose that $U \subset E$ is a locally trivial bundle with fiber a bounded subset of the hilbert bundle E . Suppose that $c : U \rightarrow F$ is a bounded, fibrewise compact map with target on a G -Banach bundle. Given $\epsilon > 0$, there exist a continuous, bounded fibrewise mapping $t_\epsilon : U \rightarrow Z_\epsilon$ such that

- Z_ϵ is a subset of a finite dimensional subspace of F .
- $|t_\epsilon x - tx| \leq \epsilon$ on every fiber and the image of f_ϵ is contained in the convex hull of $f(U)$.

Proof. It is essentially proved in page 89 of [Ber77] in the non- equivariant and non parametrized case, so we just sketch the modification we need. First of all, we assume the existence of an invariant riemannian metric on F , which can be done since the action on the basis space is proper and the group is locally compact.

Since the closure of the image of U under c is compact, the image over every point can be covered by a finite union of balls of radius ϵ and centers y_1, \dots, y_n . Moreover, in presence of the invariant riemannian metric, this can be done fibrewise over X . Let Z_ϵ be the G -subbundle of Y which is fibrewise spanned by the y_i . We now construct an invariant partition of unity on U as follows. Over a point in X , define for $0 \leq i \leq k$ and $x \in U$ $\mu_i(x) = \max\{0, \epsilon - |cx - y_i|\}$. We note that this is a G -invariant function since the norm is it. Put $\lambda_i = \frac{\mu_i}{\sum \mu_i}$. and notice that $\sum \lambda_i = 1$. Define $c_\epsilon(x) = \sum_i \lambda_i y_i$. This defines an equivariant map because of the linear action of the group on every fiber. Now, notice that

$$|cx - c_\epsilon x| = |\sum_i \lambda_i(x)(cx - y_i)| \leq \sum_i \lambda_i(x) |cx - y_i| \leq \epsilon$$

□

Another notion which will be of capital importance is that of a compact homotopy of deformations. This is the right notion to study deformations of nonlinear operators in Hilbert spaces (and bundles).

Definition 7. Let f be a fixed continuous, equivariant map between G -Banach spaces X and Y . Suppose that S is a closed subset of X . Then g_0 and g_1 are compactly homotopic to f in S (relative to f) if there exists a compact map $h : s \times I \rightarrow Y$ with $g_0 = f + h(x, 0)$ and $g_1 = f + h(x, 1)$. We shall denote the compact homotopy classes of perturbations relative to f in S by

$C_f(S, Y)/\simeq$, and we write $C_f^{\neq 0}(S, Y)$ for the subset consisting of the homotopy classes of nonzero perturbations in the strong sense, namely, there exist no point in S on which the compact perturbation vanishes. In the following, we will fix our attention in the set $S = D$ defined by the vectors of length at most 1. We obviate the use of an invariant Riemannian metric to make sense of these notions *fibrewise* in a G -Hilbert bundle.

We are now able to state our main result in this section, whose unparametrized and non-equivariant version can be found in [Šva64] or [Ber77], p. 257 for a modern reference in English.

Theorem 2. Let X be a finite proper G -CW complex with G a discrete group. Suppose that l is a linear Fredholm operator on the Hilbert bundle E such that the index $\text{Ker}(l) - \text{Coker}(l)$ is a finite dimensional, trivial vector bundle (of virtual dimension p). There exists a bijective correspondence between the set of compact homotopy classes of compact perturbations of l which are nonzero in the boundary of $D(E)$ and the p -th equivariant cohomotopy group of Lück:

$$C_l^{\neq 0}(D, E)/\simeq \xrightarrow{\cong} \pi_G^p(X)$$

Before we begin with the proof, we point out that the equivariant cohomotopy groups for discrete groups and finite G -CW complexes were defined in [Lüc05] as certain equivalence classes of morphisms between one-point compactifications of finite dimensional vector bundles. By technical reasons which will become clear in the proof of the theorem, we find it easier to handle these groups as equivalence classes of spherical bundles inside finite dimensional vector bundles with a G -invariant, Riemannian metric.

Proof. 1. Step 1. Let $f \in C_l^{\neq 0}(D, E)$. We can assume that $f = L + C$, where C has a finite dimensional range contained in a finite dimensional subbundle l_{n+1} of dimension $n + 1$ in E , and L is fiberwise surjective ($n \geq p + 1$).

2. Since L is fiberwise Fredholm with index p , we may write $E = \text{Ker}L \oplus X_1$, where the Kernel is a finite dimensional vector bundle ξ of dimension p and $L : X_1 \rightarrow E$ is a fiberwise linear bounded homeomorphism with inverse L^{-1} .

3. Restrict now f to $\partial D_{n+p} \cap \{\text{Ker}l \oplus L^{-1}(l_{n+1})\}$. Note that since f is nonzero in the boundary and compact, there exists $\alpha > 0$ such that $\inf_{x \in \partial D} |fx| \geq \alpha$ on every fiber. This follows because a compact perturbation of a Fredholm operator is locally proper, see [Sma65], theorem 1.6. So we can consider the spherical morphism $\tilde{f} : S^{\xi \oplus l_{n+1}} \rightarrow S^{l_{n+1}}$ which is fiberwise given as $\frac{f}{|f|}$. This does not depend on the choices of subspaces containing the image of ∂D , for if l_n and l_m are finite dimensional bundles containing $C(\partial D)$, and \tilde{f} and \tilde{g} are the spherical morphisms obtained by the previous procedure, they can be seen to be of the form $h \wedge_x \text{id}_{\eta_n} : S^{l_n \cap l_m \oplus \eta_n} \rightarrow S^{l_n}$, respectively $h \wedge_x \text{id}_{\eta_m} : S^{l_n \cap l_m \oplus \eta_m} \rightarrow S^{l_n}$, where the morphism $h : \partial D \cap \{\text{Ker}L \cap L^{-1}(l_n \cap l_m)\} \rightarrow S^{l_n \cap l_m}$ is obtained by the normalization procedure. (That is, f and g are suspensions of the same mapping by certain vector bundles). This is well defined at the level of compact homotopy classes, for a compact homotopy $h = l + C$ joining f_0 and f_1 can always be chosen to

have a fixed finite dimensional range for all $t \in I$. The restriction to the corresponding finite dimensional bundle gives a fibrewise homotopy between the spherical morphisms associated to f_0 and f_1 .

4. To finish the proof, we verify that two perturbations f_0, f_1 with the same associated spherical morphism are homotopic. We can assume, both the range of $f_0 - l$ and $f_1 - l$ are defined in the same finite dimensional bundle l . We use the homotopy h between the spherical morphisms to produce a fibrewise compact homotopy $H = L + h : S^{l \oplus \xi} \times I \rightarrow S^l$ between f_0 and f_1 . We have to extend the compact homotopy on the finite dimensional sphere to a compact homotopy in ∂D . For this, we use a fibrewise and equivariant extension theorem. We begin with the equivariant (not fiberwise!) extension theorem of Antonyan [Ant85]. Let $x \in X$ and consider the map defined on the fibre of $h_x : S_x^{l \oplus \xi} \rightarrow E_x$. Note that $S_x^{l \oplus \xi} \times I$ is a closed invariant subset of the metrizable G -space ∂D , where of course, the metric is determined by an invariant scalar product on the hilbert space E_x . As E_x is a locally convex space, the main theorem of [Ant85] applies and gives an extension $H : \partial D \times I \rightarrow E_x$ such that $\sup_{\partial D_x \times I} |H| = \sup_{S_x^{l \oplus \xi} \times I} |h|$. This gives a compact homotopy between f_{0_x} and f_{1_x} . We use now a G -invariant partition of unity over X to glue the homotopies together and we get the fibrewise compact homotopy. □

This finishes the first proof of theorem 1. In the second case, the process of *normalizing* is replaced by the following retraction lemma:

Lemma 1. Let $\tau \subset E$ be a finite dimensional G - vector bundle over the proper G -space X . Let $l : E \rightarrow E$ be a linear fredholm morphism. Denote by S^{τ^\perp} the fiber bundle which is determined by the unit sphere in the orthogonal complement of τ . Then, there exists a fibrewise retraction $\rho_\tau : l^{-1}(\tau) \rightarrow S^E - \tau^\perp$.

Proof. □

The approximation lemma is now as follows:

Proposition 11. Let $f = l + c : E \rightarrow E$ be a fibrewise compact perturbation of the linear fredholm morphism l defined in a G - hilbert bundle E over the proper G -space X . Then, there exist finite dimensional G -vector bundles $\xi \subset E$ such that

- $\xi \oplus l(E) \cong E$.
- Given a fibrewise inclusion as orthogonal summand in a finite dimensional vector bundle $\xi \rightarrow \tau$, $\xi \oplus \zeta = \tau$, the restricted map $f : S^{l^{-1}(\xi)} \rightarrow S^E$ sends the unit sphere S^{ξ^\perp} to the orthogonal complement of ξ . The map $\rho_\xi(f)$ has image in a
- The maps $\rho_\tau f|_{l^{-1}(\tau)}$ and $id_\zeta \wedge_X \rho_\xi$ are G -homotopic over X .

Proof. 1. Denote by D the unit ball bundle in E . Due to the boundedness condition, the map $f^{-1}(D_x)$ is bounded over any point $x \in X$. Hence, the closure C_x of its image under the compact map c is fibrewise compact. Cover C_x with a finite number of balls of radius $\epsilon_x \leq \frac{1}{4}$ centered at points

v_{i_x} . Construct the vector bundle η which is fibrewise spanned by these vectors. Form the finite dimensional G -vector bundle $\xi = \eta \oplus l(E)^\perp$. It is clear that $\xi \oplus l(E) \cong E$.

2. If $w_x \in S_x^{\tau^\perp}$ is in the image of $f|_{S^{l^{-1}(\tau)}}$, then $f^{-1}(w_x) \cap l^{-1}(\tau)$ will be mapped under $f|_{l^{-1}}$ to a subspace of $\tau_x + C_x$. So, w_x will be contained in $S_x^{\tau^\perp} \cap \tau_x + C_x$, which is not possible, because the distance between these subspaces is greater than $1 - \epsilon_x > \frac{3}{4}$.
3. In view of the slice theorem and the local triviality of the G -hilbert bundles involved, we can cover the space X with invariant neighborhoods for which there is a map $U_x \rightarrow G/H$, and the bundle over U_x is the pullback of the bundle $G \times_{\underset{H}{E_x}} \rightarrow G/H$, where E_x is some strong, norm continuous representation of H in a hilbert space. Hence, we can restrict ourselves to bundles over an orbit. In the notation of part 2, there is a retraction $\rho_{\tau_x} : S^{\tau_x} \rightarrow S^{E_x} - S^{\tau_x^\perp}$, and consider the isomorphism $l^{-1}(\tau_x) \cong \zeta_x \oplus l^{-1}(\xi_x)$ given by $w'_x \mapsto (l \circ (1 - \text{pr}_{l^{-1}(\xi_x)})w'_x, \text{pr}_{l^{-1}(\xi_x)}w'_x)$. We claim that after this isomorphism, the maps $\text{id}_{\zeta_x} \rho_{\xi_x}(f|_{l^{-1}(\xi_x)})$ and $f|_{S^{l^{-1}(\tau_x)}}$ are homotopic. Consider for this a ball $D \subset E$ which contains the inverse image $f^{-1}(D_1(0))$ of the unitary ball. We define the homotopy $h : D \times I \rightarrow S^E - S^{\tau^\perp}$ as follows

$$h(w_x, t) = \begin{cases} l + [(1 - 3t)\text{id}_{E_x} + (3t)\text{pr}_{\xi_x}] \circ c & t \in [0, \frac{1}{3}] \\ l + \text{pr}_{\xi_x} \circ c[(2 - 3t)\text{id}_{l^{-1}(\xi_x)} + (3t - 1)\text{pr}_{l^{-1}(\xi_x)}] & t \in [\frac{1}{3}, \frac{2}{3}] \\ \text{pr}_{\zeta_x} \circ l + [(3 - 3t)\text{pr}_{\xi_x} + (3t - 2)\rho_{\zeta_x} \circ (l + c) \circ \text{pr}_{l^{-1}(\zeta_x)}] & t \in [\frac{2}{3}, 1] \end{cases}$$

Since $S_x^E - D \cap \tau_x^\perp$ is contractible, the homotopy above can be extended to a homotopy $S^{l^{-1}(\tau)} \times I \rightarrow S^E - S^{\tau^\perp} \simeq S^\tau$, as needed. \square

Definition 8. Let G be a discrete group acting on the proper G -CW complex X . Denote by (E, F, l, k) a non-linear cocycle for the equivariant Fix-theory over the proper G -space X , where l is a linear morphism whose index bundle is trivial of dimension p . Let ξ be a finite dimensional G -vector bundle as constructed in proposition 11. The parametrized Schwartz index is the element.

$$[p_\xi(l + k)|_{l^{-1}(\xi)}] : S^{l^{-1}(\xi)} \rightarrow S^{l(E)^\perp} \in \pi_G^{\text{ind}}(l)(X)$$

in the equivariant cohomotopy group as, introduced by Lück in [Lüc05]. This construction is well defined as consequence of part 3 of proposition 11. and gives an isomorphism of both theories,

This finishes the second proof of theorem 1

Remark 5 (The spectral sequence). Let X be a proper G -CW complex. There is an equivariant Atiyah-Hirzebruch spectral sequence which converges to $\pi_G^n(X)$ and whose E^2 -term is given in terms of Bredon cohomology

$$E_2^{p,q} = H_{\mathbb{Z}\text{Or}G}^p(X, \pi_?^q)$$

Applied to the universal proper G -space $\underline{E}G$:

$$E_2^{p,q} = H_{\mathbb{Z}SUB_{COM}(G)}^p(\underline{E}G, \pi_?^q)$$

where $\pi_?^0$ is the contravariant coefficient system $H \mapsto \pi_H^0$.

There is a canonical identification $H_{\mathbb{Z}SUB_{COM}(G)}^0(\underline{E}G, \pi_?^0) \cong \varprojlim_{H \in COM} \pi_H^0$.

The edge homomorphism of the spectral sequence defines a map $\text{edge}^G : \pi_G^0(\underline{E}G) \rightarrow A_{inv}(G)$. Several known results of the spectral sequence go through. Among them, as in case of discrete groups, the edge homomorphism is a rational isomorphism.

Definition 9 (An operator theoretical Burnside Ring). Let G be a locally compact group. The operator theoretical burnside ring of G , $A^{op}(G)$ is the 0-dimensional equivariant Fix-theory of the classifying space of proper actions $\underline{E}G$. In symbols

$$A^{op}(G) = \text{Fix}_G^0(\underline{E}G)$$

Example 1 (The group $Sl_2(\mathbb{R})$). Recall that the group $Sl_2(\mathbb{R})$ is defined to be the group of real 2×2 -matrices with determinant 1. It is a Lie group of dimension 3 and has one connected component. The maximal compact subgroup is $S^1 = SO_2$.

As $Sl_2(\mathbb{R})$ is almost connected, a model for $E_{COM}Sl_2$ is $Sl_2(\mathbb{R})/SO_2 \approx \mathbb{R}^2$, which can be handled as the upper-half plane model for the 2-dimensional hyperbolic space. Note that this is a zero-dimensional proper CW -complex. From the equivariant Atiyah- Hirzebruch Spectral Sequence follows that the edge homomorphism

$$\text{edge}^{Sl_2(\mathbb{R})} : \pi_{Sl_2(\mathbb{R})}^0(\underline{E}GSl_2(\mathbb{R})) \rightarrow \lim_{inv H \in COM(Sl_2(\mathbb{R}))} \pi_H(pt)$$

is an isomorphism. On the other hand, since S^1 is a final object in the category of compact subgroups of $Sl_2(\mathbb{R})$, we have

$$A^{op}(Sl_2(\mathbb{R})) \cong A(S^1)$$

$$A(S^1) \cong \mathbb{Z}$$

is a well known fact.

0.2 Examples in the topology of four manifolds.

We present an application of our work. Instead of expecting striking results out of them, the author would like to modestly put the ideas involved in the constructions into the contexts they first appeared, in an attempt to be respectful and grateful toward other fields of mathematics.

The first one is an extension of results due to Markus Szymik (not yet published) and Stephan Bauer [BF04] in the context of gauge theory. The author is deeply grateful to the Gauge theoretical community in Germany, in particular Markus Szymik and Raphael Zentner, who were always *hilfsbereit* and shared their knowledge with him. An introduction to this topic are for instance the work of Moore [Moo01], the book [Nic00] and the notes [Tel].

We recall briefly that in the context of smooth, riemannian oriented manifolds, the existence of a $\text{Spin}^c(4)$ -structure is always guaranteed. This amounts to a map from a Spin^c principal bundle Q together with a bundle map $Q \rightarrow P$ to the frame bundle of the tangential bundle. The identification of the group $\text{Spin}^c(4)$ with the subgroup $\{u_+, u_- \mid u_-, u_+ \in \text{U}(2), \det(u_+) = \det(u_-)\}$ allows to define positive, respectively, negative spinor bundles $S^+, S^- = Q \times_{\rho^{+, -}} \mathbb{C}^2$,

where $\rho_{+,-} : \text{Spin}^c \rightarrow \text{U}(2)$ are the respective projections. Using quaternionic multiplication, it is possible to furnish $S := S^+ \oplus S^-$ with the structure of a module over the Clifford algebra of the cotangential bundle $T^*(X) \times S^{+,-} \rightarrow S^{-,+}$. Clifford identities give a linear map $\rho : \Lambda^2 \rightarrow \text{End}_C(S^+)$ whose kernel is the bundle of anti-selfdual 2-forms and whose image is the bundle of trace free skew hermitian endomorphisms. For any spin^c -connection A , define the associated Dirac operator D as the composition $\Gamma(S^+) \xrightarrow{\nabla_{A+a}} \Gamma(S^+) \otimes \Lambda^1(T^*M) \xrightarrow{\gamma} \Gamma(S^-)$, where γ denotes Clifford multiplication.

The *monopole map* $\tilde{\mu}$ is defined for four-tuples (A, ϕ, a, f) of a Spin^c connection A , a positive spinor ϕ , a 1-form a and a locally constant function f on M as

$$\mu : \text{Conn} \times \Gamma(S^+) \oplus \Omega^1(M) \oplus H^0(M) \rightarrow \text{Conn} \times \Gamma(S^-) \oplus \Omega^+(M) \oplus \Omega^0(M) \oplus H^1(M)$$

$$(A, \phi, a, f) \mapsto (A, D_{A+a}\phi, F_{A+a}^+ - \sigma(\phi), d^*(a) + f, a_{\text{harm}})$$

where σ is the tracefree endomorphism $(-i)(\phi \otimes \phi^*) - \frac{1}{2} |\phi|^2 \text{id}$. Given a point in M , the based gauge group \mathcal{G}_x is the kernel of the evaluation map at x . $\text{map}(X, \mathbb{S}^1) \rightarrow \mathbb{S}^1$. The subspace $A + \ker(d)$ is invariant under the free action of the based Gauge group. The quotient is isomorphic to the *Picard torus*, $\mathfrak{Pic}(X) = H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$. Let \mathcal{A} and \mathcal{B} be the quotients

$$a + \ker d \times \Gamma(S^+) \oplus \Omega^1(X) \oplus H^0(X, \mathbb{R})/G_x$$

respectively

$$a + \ker d \times \Gamma(S^-) \oplus \Omega^+(X) \oplus \Omega^0(X, \mathbb{R}) \oplus H^1(X, \mathbb{R})/G_x$$

the quotient map $\mu : \tilde{\mu}/G_x : \mathcal{A} \rightarrow \mathcal{B}$ has by definition a presentation as a fibrewise compact perturbation of a Fredholm operator. It is proper after a result of Bauer and Furuta, [BF04], which essentially uses estimates determined by the Weitzenböck formula. This gives rise to a cocycle. $\mathcal{A}, \mathcal{B}, D_A + d + d^*, c = F_A^+ + a \cdot \phi + \sigma(\phi)$, where σ is the selfdual tracefree endomorphism $\phi \mapsto (-i\phi \otimes \phi^* - \frac{1}{2} |\phi|^2)$.

Example 2 (A generalized Bauer-Furuta invariant for proper actions on 4-manifolds). Suppose G is a (possibly noncompact) Lie group acting properly and cocompactly on the smooth Spin^c -manifold M . Assume furthermore, that the group preserves the orientation and by means of isomorphisms of complex spin structures, and respecting the connection. As the action is proper, it can be assumed that G preserves the metric. Let \mathbb{G} be the group of pairs (φ, u) , where φ is a G -equivariant diffeomorphism which preserves both the metric and the orientation and $u : f^*(\sigma_M) \rightarrow \sigma_M$ is an isomorphism of the spin-c principal bundle. In particular, this gives a description of \mathbb{G} in the middle of the following exact sequence

$$1 \rightarrow S^1 \rightarrow \mathbb{G} \rightarrow G \rightarrow 1$$

In this situation, the class of is denoted by $m_{\mathbb{G}}(X, \sigma_X) \in \text{Fix}_G^{\text{ind}(\lambda)}(\mathfrak{Pic}(X))$ and is called the parametrized Bauer-Furuta invariant. The restriction map

$$\text{res}_{\mathbb{G}}^{S^1} \text{Fix}_{\mathbb{G}}^{\text{ind}(l)}(\mathfrak{Pic}(X)) \rightarrow \text{Fix}_{S^1}^{\text{ind}(l)}(\text{Pic}(X) \cong \pi_{S^1}^{\text{ind}(l)}(\mathfrak{Pic}(X)))$$

maps $m_{\mathbb{G}}$ to the S^1 -equivariant cohomotopical Bauer-Furuta invariant defined in [BF04]. If, moreover, vector bundles do suffice to represent the equivariant KO -theory of the Picard torus, the element λ can be identified after choice of an invariant riemannian metric on the manifold with the difference of the complex virtual index bundle of the dirac operator and the trivial bundle of the space of selfdual harmonic two-forms.

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