Noncommutative manifolds and elliptic functions

Alain Connes

Abstract

I'll describe my joint work with Michel Dubois-Violette whose primary goal is to give a complete description of noncommutative three-dimensional spherical manifolds, a noncommutative version of the sphere S^3 defined by basic K-theoretic equations. We find a 3-parameter family of deformations S^3_u of the standard 3-sphere S^3 and a corresponding 3-parameter deformation of the 4-dimensional Euclidean space $\mathbb{R}^4$. For generic values of the deformation parameters we show that the obtained algebras of polynomials on the deformed $\mathbb{R}^4_u$ only depend on two parameters and are isomorphic to the algebras introduced by Sklyanin in connection with the Yang-Baxter equation. It follows that different S^3_u can span the same $\mathbb{R}^4_u$. This equivalence generates a foliation of the parameter space Σ. This "scaling" foliation admits singular leaves reduced to a point. These critical points are either isolated or fall in two 1-parameter families $C_\pm \subset \Sigma$. Up to the simple operation of taking the fixed algebra by an involution, these two families are identical and we concentrate first on C_+. For $u \in C_+$ the above isomorphism with the Sklyanin algebra breaks down and the corresponding algebras are special cases of θ-deformations, a notion which we generalize in any dimension and various contexts, and study in some details. Here, and this point is crucial, the dimension is not an artifact, i.e. the dimension of the classical model, but is the Hochschild dimension of the corresponding algebra which remains constant during the deformation. Besides the standard noncommutative tori, examples of θ-deformations include the recently defined noncommutative 4-sphere S^4_θ as well as m-dimensional generalizations, noncommutative versions of spaces $\mathbb{R}^m$ and quantum groups which are deformations of various classical groups. We develop general tools such as the twisting of the Clifford algebras in order to exhibit the spherical property of the hermitian projections corresponding to the noncommutative $2n$-dimensional spherical manifolds S^{2n}_θ. A key result is the differential self-duality properties of these projections which generalize the self-duality of the round instanton. In the generic case there are two elliptic curves present, the leaf of the scaling foliation on the one hand and the characteristic variety of the quadratic algebra on the other.

We show that these two curves have the same j-invariant but complementary λ-invariant. The degenerate cases shows that the two families of elliptic curves are intrisically different. The evaluation of the pairing of the volume Hochschild cycle with the fundamental class in cyclic cohomology leads to extremely involved computations with elliptic functions and modular forms.