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**The Best Constant in the
Topchii-Vatutin Inequality
for Martingales**

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The Best Constant in the Topchii-Vatutin Inequality for Martingales

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Consider the class of even convex functions $\phi : \mathbb{R} \rightarrow [0, \infty)$ with $\phi(0) = 0$ and concave derivative on $(0, \infty)$. Given any ϕ -integrable martingale $(M_n)_{n \geq 0}$ with increments $D_n \stackrel{\text{def}}{=} M_n - M_{n-1}$, $n \geq 1$, the Topchii-Vatutin inequality [10] asserts that

$$E\phi(M_n) - E\phi(M_0) \leq C \sum_{k=1}^n E\phi(D_k)$$

with $C = 4$. It is proved here that the best constant in this inequality is $C = 2$ for general ϕ -integrable martingales $(M_n)_{n \geq 0}$, and $C = 1$ if $(M_n)_{n \geq 0}$ is further nonnegative or having symmetric conditional increment distributions.

1. INTRODUCTION AND RESULT

Let $(M_n)_{n \geq 0}$ be a martingale with increments $D_n \stackrel{\text{def}}{=} M_n - M_{n-1}$, $n \geq 1$, and associated absolute maxima $M_n^* \stackrel{\text{def}}{=} \max_{0 \leq k \leq n} |M_k|$, $n \geq 0$. Let further \mathcal{G}_0 be the class of even convex functions $\phi : \mathbb{R} \rightarrow [0, \infty)$ with $\phi(0) = 0$ and \mathcal{G}_1 its subclass of $\phi \in \mathcal{G}_0$ with a concave derivative on $(0, \infty)$. Note that the latter class comprises the functions $\phi(x) = |x|^p$ for $p \in [1, 2]$ as well as $\phi(x) = (|x| + a)^p \log^r(|x| + a) - a^p \log^r a$ for $p \in [1, 2)$, $r > 0$ and $a > 0$ sufficiently large. The following convex function inequality is due to Topchii and Vatutin [10]: There exists a finite positive constant C such that for all $\phi \in \mathcal{G}_1$, all martingales $(M_n)_{n \geq 0}$ and all $n \geq 1$

$$E\phi(M_n) - E\phi(M_0) \leq C \sum_{k=1}^n E\phi(D_k). \quad (1.1)$$

More precisely, they showed (1.1) be true with $C = 4$ and $M_0 = 0$. If $\phi(x) = |x|$ or $\phi(x) = x^2$, then it is well-known that (1.1) holds true with $C = 1$ and that this value cannot be improved. We shall prove in this note that the best constant for general $\phi \in \mathcal{G}_1$ and general ϕ -integrable martingales is $C = 2$, but that $C = 1$ is optimal when imposing certain additional restrictions on the class of considered martingales. The result is stated as the following theorem.

THEOREM 1. *If $0 \neq \phi \in \mathcal{G}_1$ and $M = (M_k)_{0 \leq k \leq n}$ is a ϕ -integrable martingale, then*

$$E\phi(M_n) - E\phi(M_0) < 2 \sum_{k=1}^n E\phi(D_k). \quad (1.2)$$

The constant 2 is sharp in the sense that, for each $\varepsilon \in (0, 1)$, there exists a bounded martingale M and some $\phi \in \mathcal{G}_1$ such that

$$E\phi(M_n) - E\phi(M_0) \geq (2 - \varepsilon) \sum_{k=1}^n E\phi(D_k). \quad (1.3)$$

If M is nonnegative or having symmetric conditional increment distributions, then inequality (1.1) holds true with $C = 1$.

An analogue of (1.1) for the maximum M_n^* can be quite easily inferred from the following Burkholder-Davis-Gundy inequality (see e.g. [2, Thm. 1 on p. 425]): Let $\nu > 0$ and $\mathcal{G}_0^{(\nu)}$ be the class of all $\phi \in \mathcal{G}_0$ satisfying $\phi(2x) \leq \nu\phi(x)$ for all x . Then there exists a constant $C_\nu^* \in (0, \infty)$ such that for all $\phi \in \mathcal{G}_0^{(\nu)}$ and all martingales $(M_n)_{n \geq 0}$ having $M_0 = 0$

$$E\phi(M_n^*) \leq C_\nu^* E\phi \left(\left(\sum_{k=1}^n D_k^2 \right)^{1/2} \right). \quad (1.4)$$

This inequality applies to class \mathcal{G}_1 because $\mathcal{G}_1 \subset \mathcal{G}_0^{(4)}$ as will be shown in Lemma 2 at the end of Section 2. Defining $\psi(t) \stackrel{\text{def}}{=} \phi(t^{1/2})$, the same lemma will further show that ψ is concave

and subadditive on $[0, \infty)$, that is $\psi(\sum_{k=1}^n x_k) \leq \sum_{k=1}^n \psi(x_k)$ for all $x_1, \dots, x_n \geq 0$ and $n \in \mathbb{N}$. Utilizing this last fact on the right hand side in (1.4), we obtain

$$E\phi(M_n^*) \leq C_4^* \sum_{k=1}^n E\phi(D_n). \quad (1.5)$$

Let us finally mention that sharp inequalities similar to those considered here were derived in a recent paper by de la Peña, et al. [3] for infinite degree order statistics.

2. PROOF OF THEOREM 1

The proof of Theorem 1 and in particular of the sharpness of the constant $C = 2$ in (1.1) are heavily based on several reductions, the main one being that it suffices to consider only certain extremal elements $\phi \in \mathcal{G}_1$. This was also used in [1] and [9] for the study of odd functional moments of positive random variables with a decreasing density. The general background is that the class of increasing convex (or concave) functions $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ as well as many important subclasses like \mathcal{G}_1 form a convex cone for which Choquet theory tells us that each element ϕ can be written as an integral of its extremal elements with respect to some measure on $[0, \infty]$ (depending on ϕ). For the given classes these integral representations are obtained by simple partial integration. The following lemma provides the result for the class \mathcal{G}_1 and exemplifies the general procedure.

LEMMA 1. *For each $\phi \in \mathcal{G}_1$, there exists a unique finite measure Q_ϕ on $[0, \infty]$ such that*

$$\phi(x) = \int_{[0, \infty]} \phi_t(x) Q_\phi(dt), \quad x \geq 0, \quad (2.1)$$

where $\phi_0(x) = |x|$, $\phi_\infty(x) = x^2$, and

$$\phi_t(x) \stackrel{\text{def}}{=} \begin{cases} x^2, & \text{if } |x| \leq t \\ 2xt - t^2, & \text{if } |x| > t \end{cases} \quad (2.2)$$

for $t \in (0, \infty)$.

Note that the functions ϕ_t also arise in problems of robust estimation and are known in statistics as Huber function or Huber's ρ -functions, see e.g. [5], [6].

PROOF. Each $\phi \in \mathcal{G}_1$ has a concave derivative ϕ' with $\phi'_+(0) \stackrel{\text{def}}{=} \lim_{x \rightarrow +0} \phi'(x) \geq 0$ and thus also a nonincreasing second right derivative ϕ''_+ with asymptotic value $\phi''_+(\infty) \stackrel{\text{def}}{=} \lim_{x \rightarrow \infty} \phi''_+(x) \geq 0$. Therefore $\Lambda_{\phi'}((x, \infty)) \stackrel{\text{def}}{=} \phi''_+(x) - \phi''_+(\infty)$ for $x \geq 0$ defines a measure on $(0, \infty)$. Put

$$\mathcal{G}_1^* \stackrel{\text{def}}{=} \{\phi \in \mathcal{G}_1 : \phi'_+(0) = 0, \phi''_+(\infty) = 0\}.$$

and $\phi^*(x) \stackrel{\text{def}}{=} \phi(x) - \phi'_+(0)|x| - \phi''_+(\infty)x^2/2$ which is an element of \mathcal{G}_1^* . Partial integration now gives for $x > 0$

$$\begin{aligned} \phi'(x) - \phi'_+(0) - \phi''_+(\infty)x &= \int_0^x (\phi''_+(y) - \phi''_+(\infty)) dy \\ &= \int_0^x \int_{(y, \infty)} \Lambda_{\phi'}(dt) dy \\ &= \int_{(0, \infty)} (x \wedge t) \Lambda_{\phi'}(dt) \end{aligned}$$

and then also

$$\begin{aligned} \phi^*(x) &= \int_0^x (\phi'(y) - \phi'_+(0) - \phi''_+(\infty)y) dy \\ &= \int_{(0, \infty)} \int_0^x (y \wedge t) dy \Lambda_{\phi'}(dt) \\ &= \int_{(0, \infty)} \phi_t(x) Q_{\phi^*}(dt), \end{aligned}$$

where $Q_{\phi^*} \stackrel{\text{def}}{=} \Lambda_{\phi'}/2$. We conclude (2.1) with $Q_\phi \stackrel{\text{def}}{=} \phi'_+(0)\delta_0 + \frac{1}{2}\phi''_+(\infty)\delta_\infty + Q_{\phi^*}$. \diamond

PROOF OF THEOREM 1. The following reduction arguments will show that it suffices to prove

$$E\phi_1(s + D) \leq \phi_1(s) + C E\phi_1(D) \quad (2.3)$$

for all $s \geq 0$ and all centered random variables D having a two point distribution, where $C = 2$ in the general case, while $C = 1$ if D is symmetric or $s + D \geq 0$. Of course, ϕ_1 is the function defined by (2.2). Note that in terms of the martingales under consideration the former means nothing but a reduction to martingales of the form $(M_0, M_1) = (s, s + D)$.

FIRST REDUCTION. As noted above, for each $\phi \in \mathcal{G}_1$ the even function $\phi^*(x) = \phi(x) - \phi'_+(0)|x| - \phi''_+(\infty)x^2/2$ is an element of \mathcal{G}_1^* . Since

$$\begin{aligned} E\phi(M_n) &= E\phi^*(M_n) + \phi'_+(0) E|M_n| + \frac{\phi''_+(\infty)}{2} EM_n^2 \\ &\leq E\phi^*(M_n) + \phi'_+(0) \left(E|M_0| + \sum_{k=1}^n E|D_k| \right) + \frac{\phi''_+(\infty)}{2} \left(EM_0^2 + \sum_{k=1}^n ED_k^2 \right) \end{aligned}$$

it suffices to prove Theorem 1 for functions $\phi \in \mathcal{G}_1^*$.

SECOND REDUCTION. Using (2.2), $\phi_t(x) = t^2\phi_1(x/t)$ for all $t \in (0, \infty)$ and $Q_\phi(\{0, \infty\}) = 0$ if $\phi'_+(0) = 0$ and $\phi''_+(\infty) = 0$ (see at the end of the proof of Lemma 1), we infer for each $\phi \in \mathcal{G}_1^*$

$$E\phi(M_n) = \int_{(0, \infty)} E\phi_t(M_n) Q_\phi(dt) = \int_{(0, \infty)} t^2 E\phi_1(M_n/t) Q_\phi(dt).$$

Since $(M_k/t)_{0 \leq k \leq n}$ is still a martingale, it suffices to prove Theorem 1 with $\phi = \phi_1$.

THIRD REDUCTION. By conditioning

$$\begin{aligned} & E\phi_1(M_n) - E\phi_1(M_{n-1}) - C E\phi_1(D_n) \\ &= \int \left(E(\phi_1(s + D_n) | M_{n-1} = s) - \phi_1(s) - C E(\phi_1(D_n) | M_{n-1} = s) \right) P(M_{n-1} \in ds) \end{aligned}$$

where, given $M_{n-1} = s$, D_n has conditional mean 0. This reduces the proof to that of (2.3) for any centered random variable D and any $s \in \mathbb{R}$. We may further restrict to $s \geq 0$ because $E\phi_1(s + D) = E\phi_1(-s - D)$ and $-D$ is also centered.

FOURTH REDUCTION. Finally, since every centered distribution is a mixture of centered two point distributions, we conclude that it is indeed enough to prove (2.3) for all $s \geq 0$ and all centered D taking only two values, see e.g. Hoeffding [4].

In the following we simply write f' and always mean f'_+ in those cases where left and right derivatives are different.

PROOF OF (2.3) WITH $C = 1$ FOR SYMMETRIC D . Suppose D has distribution $(\delta_{-a} + \delta_a)/2$ for some $a \geq 0$ and let

$$\Delta(s) \stackrel{\text{def}}{=} E\phi_1(s + D) - E\phi_1(D) - \phi_1(s), \quad s \geq 0.$$

Then

$$\begin{aligned} \Delta(s) &= \frac{\phi_1(s + a) + \phi_1(s - a)}{2} - \phi_1(a) - \phi_1(s), \\ \Delta'(s) &= \frac{\phi_1'(s + a) + \phi_1'(s - a)}{2} - \phi_1'(s), \\ \Delta''(s) &= \frac{\phi_1''(s + a) + \phi_1''(s - a)}{2} - \phi_1''(s) \end{aligned}$$

for $s \geq 0$. In particular $\Delta(0) = \Delta'(0) = 0$ and $\Delta''(0) \leq 0$. Note that

$$\phi_1'(x) \stackrel{\text{def}}{=} \begin{cases} 2x, & \text{if } |x| \leq 1 \\ 2 \operatorname{sign}(x), & \text{if } |x| \geq 1 \end{cases} \quad \text{and} \quad \phi_1''(x) = 2 \mathbf{1}_{[-1,1]}(x) \quad \mathbb{A}\text{-a.e.}$$

where \mathbb{A} denotes Lebesgue measure on \mathbb{R} and $\mathbf{1}_B$ the indicator function of a set B . Hence, if $a \in [0, 1]$, then \mathbb{A} -a.e.

$$\Delta''(s) = \begin{cases} 0, & \text{if } 0 \leq s \leq 1 - a \text{ or } s > a + 1 \\ -1, & \text{if } 1 - a < s \leq 1 \\ 1, & \text{if } 1 < s \leq a + 1 \end{cases},$$

while in case $a \in (1, 2]$

$$\Delta''(s) = \begin{cases} -2, & \text{if } 0 \leq s \leq a - 1 \\ -1, & \text{if } a - 1 < s \leq 1 \\ 1, & \text{if } 1 < s \leq a + 1 \\ 0, & \text{if } s > a + 1 \end{cases},$$

and in case $a > 2$

$$\Delta''(s) = \begin{cases} 0, & \text{if } 1 < s \leq a - 1 \text{ or } s > a + 1 \\ -2, & \text{if } 0 \leq s \leq 1 \\ 1, & \text{if } a - 1 < s \leq a + 1 \end{cases}.$$

We also have that $\Delta(s)$ and $\Delta'(s)$ vanish at $s = 0$ and (by linearity of ϕ_1 on $(1, \infty)$) for sufficiently large s . From this we see that Δ' is everywhere nonpositive and unimodal which in turn yields $\Delta(s) \leq 0$ for all $s \geq 0$ and thus (2.3) with $C = 1$.

PROOF OF (2.3) WITH $C = 1$ FOR NONNEGATIVE $s + D$. Let D be a centered random variable with distribution $p\delta_{-a} + q\delta_b$ for $a, b \geq 0$, hence $p + q = 1$ and $qb - pa = 0$. The function Δ now takes the form

$$\Delta(s) = p\phi_1(s - a) + q\phi_1(s + b) - p\phi_1(-a) - q\phi_1(b) - \phi_1(s)$$

and has derivative $\Delta'(s) = p\phi_1'(s - a) + q\phi_1'(s + b) - \phi_1'(s)$. By concavity of ϕ_1' on $[0, \infty)$,

$$\Delta'(s) \leq \phi_1'(s - pa + qb) - \phi_1'(s) = 0$$

for all $s \geq a$. Consequently, $E\phi_1(s + D) \leq E\phi_1(D) + \phi_1(s)$ follows for all $s > a$ if this is true for $s = a$.

If $s = a \leq 1$, then $\phi_1(s) = s^2$ whence $\phi_1(s + x) - \phi_1(x) \leq (s + x)^2 - x^2 = s(2x + s)$ for all $x \geq -s$ implies the asserted inequality, namely

$$E\phi_1(s + D) - E\phi_1(D) \leq sE(2D + s) = s^2.$$

Now fix $s = a \geq 1$, note that $ED = 0$ implies $p = \frac{b}{s+b}$, and look at $\Delta(s)$ as a function $G(b)$, say, of b . We obtain

$$\begin{aligned} G(b) &= q\phi_1(s + b) - p\phi_1(-s) - q\phi_1(b) - \phi_1(s) \\ &= q\phi_1(s + b) - q\phi_1(b) - (1 + p)\phi_1(s) \\ &= \frac{s\phi_1(s + b) - s\phi_1(b) - (s + 2b)\phi_1(s)}{s + b}. \end{aligned}$$

This implies in case $b \geq 1$

$$G(b) = \frac{s(2(s + b) - 1) - s(2b - 1) - (s + 2b)(2s - 1)}{s + b} = \frac{s + 2b - 4sb}{s + b} \leq 0,$$

and in case $0 < b < 1$

$$G(b) = \frac{s(2(s + b) - 1) - sb^2 - (s + 2b)(2s - 1)}{s + b} = \frac{-2b(s - 1) - sb^2}{s + b} \leq 0.$$

So we have again shown that (2.3) holds with $C = 1$.

PROOF OF (2.3) WITH $C = 2$ FOR GENERAL D . The assertion to prove may be rephrased in terms of $\Delta(s)$ as

$$\Delta(s) \leq E\phi_1(D) = p\phi_1(s-a) + q\phi_1(s+b)$$

for all $s \geq 0$. Since $s = 0$ is trivial, fix an arbitrary $s > 0$, let D have distribution $p\delta_{-a} + q\delta_b$ and suppose $\theta \stackrel{\text{def}}{=} a - s \geq 0$ (only this case needs to be considered after the previous part of the proof). Note that $ED = 0$ implies $b = (p/q)a$ and thus $D \stackrel{d}{=} p\delta_{-s-\theta} + q\delta_{(p/q)(s+\theta)}$. In order to prove (2.3) with $C = 2$, fix any $p \in (0, 1)$ and consider

$$\begin{aligned} H(\theta) &\stackrel{\text{def}}{=} E\phi_1(s+D) - 2E\phi_1(D) - \phi_1(s) \\ &= p\left(\phi_1(\theta) - 2\phi_1(s+\theta)\right) + q\left(\phi_1\left(s + \frac{p}{q}(s+\theta)\right) - 2\phi_1\left(\frac{p}{q}(s+\theta)\right)\right) - \phi_1(s) \end{aligned}$$

for $\theta \geq 0$. Since $s + D \geq 0$ if $\theta = 0$, we infer $H(0) \leq -E\phi_1(D) < 0$ from the previous part of the proof. Differentiation with respect to θ gives

$$\begin{aligned} H'(\theta) &= p\left(\phi_1'(\theta) - 2\phi_1'(s+\theta)\right) + p\left(\phi_1'\left(s + \frac{p}{q}(s+\theta)\right) - 2\phi_1'\left(\frac{p}{q}(s+\theta)\right)\right) \\ &= p\left(\left(\phi_1'\left(s + \frac{p}{q}(s+\theta)\right) - \phi_1'\left(\frac{p}{q}(s+\theta)\right)\right) - \left(\phi_1'(s+\theta) - \phi_1'(\theta)\right) \right. \\ &\quad \left. - \left(\phi_1'(s+\theta) + \phi_1'\left(\frac{p}{q}(s+\theta)\right)\right)\right). \end{aligned} \tag{2.4}$$

The function ϕ_1' is monotone and is subadditive as a nonnegative concave function on $[0, \infty)$. It follows that

$$\phi_1'(s+\theta) + \phi_1'\left(\frac{p}{q}(s+\theta)\right) \geq \phi_1'(s+\theta + \frac{p}{q}(s+\theta)) \geq \phi_1'(s + \frac{p}{q}(s+\theta))$$

and thereby in (2.4)

$$H'(\theta) \leq -p\left(\phi_1'\left(\frac{p}{q}(s+\theta)\right) + (\phi_1'(s+\theta) - \phi_1'(\theta))\right) \leq 0.$$

Consequently, H is nonincreasing on $[0, \infty)$ with $H(0) < 0$ and therefore everywhere negative. This proves (2.3) with $C = 2$ and strict inequality.

ATTAINING THE BOUND IN (2.3) WITH $C = 2$. We finally have to provide examples showing that the bound $C = 2$ is sharp. Let $s \geq 1$ and D be distributed as $\frac{b}{a+b}\delta_{-a} + \frac{a}{a+b}\delta_b$ for some $a \geq 1 + s$ and $b \in [0, 1]$. Then

$$\begin{aligned} &E\phi_1(s+D) - \phi_1(s) - (2-\varepsilon)E\phi_1(D) \\ &= \frac{1}{a+b}\left(b(2a-2s-1) + a(2s+2b-1) - (a+b)(2s-1) - (2-\varepsilon)(b(2a-1) + ab^2)\right) \\ &= \frac{b}{a+b}\left(2 - 2ab - 4s + \varepsilon(2a-1+ab)\right). \end{aligned}$$

Now it is easily seen that, for any $\varepsilon > 0$, a positive value is obtained when choosing $b = 1/a$ and a sufficiently large. The proof of Theorem 1 is herewith complete. \diamond

Recall that $\mathcal{G}_0^{(\nu)}$ denotes the class of all $\phi \in \mathcal{G}_0$ satisfying $\phi(2x) \leq \nu\phi(x)$ for all x . We claimed in the Introduction that $\mathcal{G}_1 \subset \mathcal{G}_0^{(4)}$ as well as $\mathcal{G}_1 \subset \mathcal{G}_2$, where \mathcal{G}_2 denotes the subclass of \mathcal{G}_0 containing those ϕ for which $\psi(x) \stackrel{\text{def}}{=} \phi(x^{1/2})$ is concave on $[0, \infty)$. These claims are finally confirmed in the subsequent lemma.

LEMMA 2. $\mathcal{G}_1 \subset \mathcal{G}_2$ and $\mathcal{G}_1 \subset \mathcal{G}_0^{(4)}$.

PROOF. Note that each nonnegative concave function f on $[0, \infty)$ is subadditive and that $f(x)/x$ is nonincreasing (because $-f$ is evidently star-shaped, see [8, p. 453]). Given any $\phi \in \mathcal{G}_1$, use this for $f = \phi'$ to see that the pertinent ψ is indeed concave because $\psi'(x) = \frac{\phi'(x^{1/2})}{2x^{1/2}}$. Moreover, the subadditivity of ϕ' on $[0, \infty)$ implies $\phi'(2x) \leq 2\phi'(x)$ and thus

$$\phi(2x) = \int_0^{2x} \phi'(t) dt = \int_0^x 2\phi'(2t) dt \leq \int_0^x 4\phi'(t) dt = 4\phi(x)$$

for all $x \geq 0$. ◇

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NOTE ADDED IN PROOF

In a recent paper, Li [7, Thm. 2.1] proved the following large deviation inequality for a martingale $(M_n)_{n \geq 0}$ with $M_0 = 0$: If $1 < p \leq 2$ and $K \stackrel{\text{def}}{=} \sup_{n \geq 1} E|M_n|^p < \infty$, then

$$P\left(\max_{1 \leq i \leq n} |M_n| > nx\right) \leq CKn^{1-p}x^{-p}$$

for all $x > 0$, $n \geq 1$ and $C = (18pq^{1/2})^p$, where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. A combination of Doob's maximal inequality (see [2, p. 255]) with our Theorem 1 for $\phi(x) = |x|^p$ immediately shows that Li's inequality actually holds true with the considerably smaller constant $C = 2$.

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