Chapter 1
Introduction

What goes around, comes around.

In Applied Probability, one is frequently facing the task to determine the asymptotic behavior of a real-valued stochastic processes $(R_t)_{t\in\mathbb{T}}$ in discrete $(\mathbb{T}=\mathbb{N}_0)$ or continuous $(\mathbb{T}=\mathbb{R}_\geq)$ time which bears a *regeneration scheme* in the following sense: for an increasing sequence $0<\nu_1\le\nu_2,\ldots$ either

- $\{R_{\nu_k+t}:0\le t<\nu_{k+1}-\nu_k\},\ k\ge 1$ [type A]

or

- $\{R_{\nu_k+t}-R_{\nu_k}:0\le t<\nu_{k+1}-\nu_k\},\ k\ge 1$ [type B]

are independent and identically distributed (iid) random elements, often called *cycles* hereafter. The ν_k, which may be random or deterministic, are called *regeneration epochs* and constitute a so-called *renewal process*, that is a nondecreasing sequence of nonnegative random variables with iid increments. The last assertion is a necessary consequence of each of the two above regeneration properties. Intuitively speaking, regeneration means that $(R_t)_{t\in\mathbb{T}}$, possibly after being reset to 0 (type B), restarts or *regenerates* at $\nu_1,\nu_2,\ldots$ in a distributional sense. For example, if R_t denotes the number of waiting customers at time t in a queuing system, then (under suitable model assumptions) a regeneration scheme of type A is obtained with the help of the epochs ν_k where an arriving customer finds the system idle. For another example, let R_t be the stock level at time t in an (s,S)-inventory model with maximal stock level S and critical level $0<s<S$. Whenever incoming demands cause the stock level to fall below s, denote these epochs as $\nu_1,\nu_2,\ldots$, the stock is immediately refilled to go back to S. If the times between demands and the demand sizes are iid, then it is not difficult to verify that the ν_k are regeneration epochs for (R_t). Simple symmetric random walk $(R_n)_{n\ge0}$ on the integer lattice provides a more theoretical example. Here R_n denotes the position at time n of a particle which in each step moves to one of the two neighboring sites with probability $\frac{1}{2}$ each. It is known and will in fact be shown in ??????? that with probability one this particle visits any site $k\in\mathbb{Z}$ infinitely often. Hence, the epochs $\nu_1,\nu_2,\ldots$ where it returns to 0 provide a regeneration scheme of type A for $(R_n)_{n\ge0}$. A regeneration scheme of type B can also be given for this sequence by letting ν_k be the first time where the particle hits k for any $k\in\mathbb{N}$. In a similar vein, Brownian motion $(R_t)_{t\ge0}$ with positive drift μ, i.e.

a Gaussian process with stationary independent increments, continuous trajectories, $R_0 = 0$, $\mathbb{E}R_t = \mu t$ and $\mathbb{V}\mathrm{ar} R_t = \sigma^2 t$ for some $\sigma^2 > 0$, regenerates in the sense of type B at the consecutive hitting epochs of the line $x \mapsto \mu x$.

Drawing conclusions from the existence of a regeneration scheme for a given stochastic process may be viewed as the ultimate goal of renewal theory, but in a narrower and more classical sense it deals with the analysis of those sequences that are at the bottom of such schemes, namely sums of iid real-valued random variables, called *random walks*, including the afore-mentioned renewal sequences as a special case by having nonnegative increments. However, unlike classical limit theorems which provide information on the asymptotic behavior of a random walk after a suitable normalization, renewal theory strives for the fine structure of random walks by exploring its ubiquitous regenerative pattern. The present text puts a strong emphasis on this latter....

1.1 The renewal problem: a simple example to begin with

Despite its simplistic nature, the following example provides a good framework to motivate some of the most basic questions in connection with a renewal process. Suppose we are given an infinite supply of light bulbs which are used one at a time until they fail. Their lifetimes are denoted as $X_1, X_2, \ldots$ and assumed to be iid random variables with positive mean μ. If the first light bulb is installed at time $S_0 := 0$, then

$$S_n \;:=\; \sum_{k=1}^{n} X_k \quad \text{for } n \geq 1$$

denotes the time at which the n^{th} bulb fails and is replaced with a new one. In other words, each S_n marks a renewal epoch. Some of the natural problems that come to mind for this model are the following:

(Q1) Is the number of renewals up to time t, denoted as $N(t)$, almost surely finite for all $t > 0$? And what about its expectation $\mathbb{E}N(t)$?

(Q2) What is the asymptotic behavior of $t^{-1}N(t)$ and its expectation as $t \to \infty$, that is the long run average (expected) number of renewals per unit of time?

(Q3) What can be said about the long run behavior of $\mathbb{E}(N(t+h) - N(t))$ for any fixed $h > 0$?

The stochastic process $(N(t))_{t \geq 0}$ is called the *renewal counting process* associated with $(S_n)_{n \geq 1}$ and may be formally defined as

$$N(t) \;:=\; \sup\{n \geq 0 : S_n \leq t\} \quad \text{for } t \geq 0. \tag{1.1}$$

An equivalent definition is

$$N(t) \;:=\; \sum_{n \geq 1} \mathbf{1}_{[0,t]}(S_n)$$

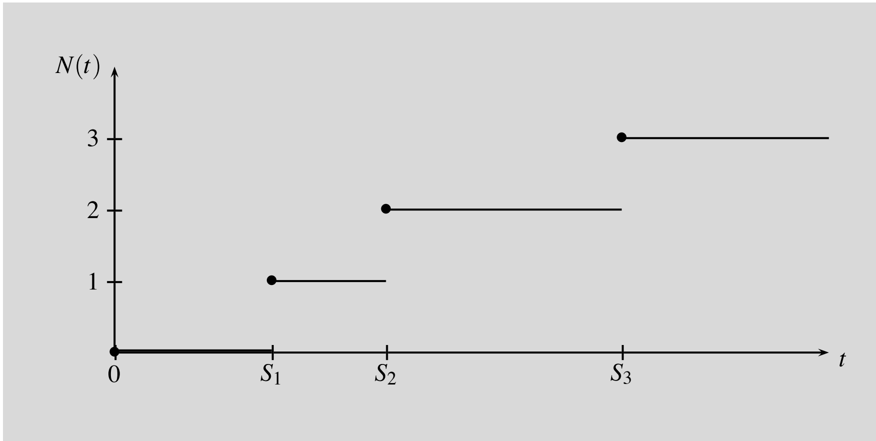

Fig. 1.1 The renewal counting process $(N(t))_{t\geq 0}$ with renewal epochs $S_1, S_2, \ldots$

and has the advantage that it immediately extends to general measurable subsets A of $\mathbb{R}_\geq$ by putting

$$N(A) \;:=\; \sum_{n\geq 1} \mathbf{1}_A(S_n) \;=\; \sum_{n\geq 1} \delta_{S_n}(A). \tag{1.2}$$

Here $\mathbf{1}_A$ denotes the indicator function of A and δ_{S_n} the Dirac measure at S_n. Ignoring measurability aspects here [☞ Subsection 2.1.4], it is clear that $N = \sum_{n\geq 1} \delta_{S_n}$ does in fact constitute a *random counting measure*, also called *point process*, on $(\mathbb{R}_\geq, \mathscr{B}(\mathbb{R}_\geq))$. By further defining its intensity measure

$$\mathbb{U}(A) \;:=\; \mathbb{E}N(A) \;=\; \sum_{n\geq 1} \mathbb{P}(S_n \in A) \quad (A \in \mathscr{B}(\mathbb{R}_\geq)) \tag{1.3}$$

we arrive at the so-called *renewal measure* of $(S_n)_{n\geq 1}$ which measures the expected number of renewals in a set and is one of the central objects in renewal theory. Its "distribution function"

$$[0,\infty) \;\ni\; t \;\mapsto\; \mathbb{U}(t) \;:=\; \mathbb{U}([0,t]) \;=\; \sum_{n\geq 1} \mathbb{P}(S_n \leq t) \tag{1.4}$$

is called *renewal function* of $(S_n)_{n\geq 1}$ and naturally of particular interest.

Turning to question (Q1), we directly infer

$$N(t) < \infty \quad \text{a.s. for all } t \geq 0 \tag{1.5}$$

because $S_n \to \infty$ a.s. as a consequence of the strong law of large numbers (SLLN). The question whether $N(t)$ has finite expectation as well requires only little more work and follows with the help of a stochastic comparison argument.

Proposition 1.1.1. *If $(S_n)_{n\geq 1}$ is a renewal process, then its renewal function is everywhere finite, i.e.*

$$\mathbb{U}(t) \ < \ \infty \quad \textit{for all } t \geq 0.$$

Proof. Since $\mathbb{P}(X_1 = 0) < 1$ there exists a constant $c > 0$ such that $p := \mathbb{P}(X_1 \leq c) < 1$. Consider the renewal process (S'_n) with increments defined as $X'_n := c\mathbf{1}_{\{X_n > c\}}$ for $n \geq 1$, thus $S'_n = \sum_{k=1}^n X'_k$ for $n \geq 1$. This process moves as follows: Each jump of size c is preceded by a random number of zeroes having a geometric distribution with parameter $1 - p = \mathbb{P}(X'_1 = c)$. Consequently, if $N', \mathbb{U}'$ have the obvious meaning, then $N'(nc)$ equals n plus the sum of $n+1$ (actually independent) geometric random variables with parameter p giving

$$\mathbb{U}'(nc) \ = \ \mathbb{E}N'(nc) \ = \ n + (n+1)\frac{p}{1-p} \ = \ \frac{n+p}{1-p} \quad \text{for all } n \in \mathbb{N}.$$

The assertion now follows because $X'_n \leq X_n$ for each $n \in \mathbb{N}$ clearly implies $N(t) \leq N'(t)$ and hence $\mathbb{U}(t) \leq \mathbb{U}'(t)$ for all $t \geq 0$. □

With the previous result at hand we can turn to question (Q2) on the long run average number of renewals per time unit, also called *renewal rate*. Ignoring the random oscillations in the replacement scheme it is natural to expect that this rate should be μ^{-1}, as every installed light bulb is expected to burn for a time interval of length μ. The following result provides a positive answer for $t^{-1}N(t)$ and is based on a neat probabilistic argument that goes back to DOOB [12]. The corresponding assertion for $t^{-1}\mathbb{U}(t)$ is also valid, but its proof requires more work and will be given later [☞ Thm. 2.4.3].

Proposition 1.1.2. *If $(S_n)_{n\geq 1}$ is a renewal process with mean interrenewal time $0 < \mu \leq \infty$, then*

$$\lim_{t\to\infty} \frac{N(t)}{t} \ = \ \frac{1}{\mu} \quad \textit{a.s.}$$

with the usual convention that $\infty^{-1} := 0$.

Proof. Using $N(t) \to \infty$ a.s. in combination with the SLLN for $(S_n)_{n\geq 1}$ we infer that both, $N(t)^{-1}S_{N(t)}$ and $N(t)^{-1}S_{N(t)+1}$ converge a.s. to μ as $t \to \infty$. Moreover, (1.1) implies $S_{N(t)} \leq t < S_{N(t)+1}$ for all $t \geq 0$. Consequently,

$$\frac{S_{N(t)}}{N(t)} \ \leq \ \frac{t}{N(t)} \ \leq \ \frac{S_{N(t)+1}}{N(t)} \quad \text{for all } t \geq 0$$

provides the assertion upon letting t tend to infinity. □

Asking for the expected number of renewals in a bounded interval of length h (question (Q3)) the heuristic argument given before the previous result suggests that it should be approximately equal to $\mu^{-1}h$, that is

$$\mathbb{U}((t,t+h]) \;=\; \mathbb{U}(t+h)-\mathbb{U}(t) \;\approx\; \frac{h}{\mu} \tag{1.6}$$

at least for large values of t. On the other hand, it should not take by surprise that for this to show the random fluctuations of $N(t+h)-N(t)$ must be considered more carefully. In fact, an answer to (Q3) cannot be provided at this point and is one of the highly nontrivial blockbuster results to be derived in ???????

1.2 The Poisson process: a nice example to learn from

So far we have not addressed the question whether the distribution of $N(t)$ or its expectation $\mathbb{U}(t)$ may be computed explicitly in closed form. Let F and F_n denote the cdf of X_1 and S_n, respectively, hence $F_1 = F$ and $F_n = F^{*n}$ for each $n \in \mathbb{N}$, where F^{*n} denotes the n-fold covolution of F defined recursively as

$$F^{*n}(t) \;=\; \int_{[0,t]} F^{*(n-1)}(t-x)\,F(dx) \quad \text{for all } t \geq 0.$$

Now observe that, by (1.1),

$$\{N(t)=n\} \;=\; \{S_n \leq t < S_{n+1}\} \;=\; \{S_n \leq t\}\backslash\{S_{n+1} \leq t\}$$

and thus

$$\mathbb{P}(N(t)=n) \;=\; \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t) \;=\; F_n(t) - F_{n+1}(t). \tag{1.7}$$

for all $n \in \mathbb{N}_0$ and $t \geq 0$. Furthermore, by (1.4)

$$\mathbb{U}(t) \;=\; \sum_{n\geq 1} \mathbb{P}(S_n \leq t) \;=\; \sum_{n\geq 1} F_n(t) \quad \text{for all } t \geq 0. \tag{1.8}$$

This shows that closed form expressions require an explicit knowledge of *all* $F_n(t)$ as well as of their infinite sum which is true only in very few cases. The most important one of these will be discussed next.

Suppose that F is an exponential distribution with parameter $\theta > 0$, that is

$$F(t) \;=\; 1 - e^{-\theta t} \quad (t \geq 0).$$

It is well-known that S_n then has a Gamma distribution with parameters n and θ, the density of which (with respect to Lebesgue measure $\lambda\!\!\!\lambda_0$) is

$$f_n(x) \;=\; \frac{\theta^n x^{n-1}}{(n-1)!}\, e^{-\theta x} \quad (x \ge 0)$$

for each $n \in \mathbb{N}$. In order to find $\mathbb{P}(N(t) = n)$, we consider $g_n(t) := e^{\theta t}\, \mathbb{P}(N(t) = n)$ for $t \ge 0$ and any fixed $n \in \mathbb{N}$. A conditioning argument yields

$$\begin{aligned} g_n(t) \;&=\; e^{\theta t}\, \mathbb{P}(S_n \le t < S_n + X_{n+1}) \\ &=\; e^{\theta t} \int_0^t \mathbb{P}(X_1 > t - x)\, f_n(x)\, dx \\ &=\; \int_0^t e^{\theta x} f_n(x)\, dx \\ &=\; \int_0^t \frac{\theta^n x^{n-1}}{(n-1)!}\, dx \quad \text{for all } t \ge 0, \end{aligned}$$

whence

$$\mathbb{P}(N(t) = n) \;=\; \frac{(\theta t)^n}{n!} e^{-\theta t} \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}.$$

For $n = 0$ we obtain more easily

$$\mathbb{P}(N(t) = 0) \;=\; \mathbb{P}(S_1 > t) \;=\; 1 - F(t) \;=\; e^{-\theta} \quad \text{for all } t \ge 0.$$

We have thus shown

Proposition 1.2.1. *If $(S_n)_{n \ge 1}$ is a renewal process having exponential increments with parameter θ, then $N(t)$ has a Poisson distribution with parameter θt for each $t > 0$, in particular*

$$\mathbb{U}(t) \;=\; \mathbb{E}N(t) \;=\; \theta t \quad \textit{for all } t \ge 0, \tag{1.9}$$

that is, $\mathbb{U}$ equals Lebesgue measure on $\mathbb{R}_>$.

We thus see that for the exponential case question (Q3) has an explicit answer in that (1.6) becomes an exact identity:

$$\mathbb{U}(t + h) - \mathbb{U}(t) \;=\; \frac{h}{\mu} \quad \text{for all } t \ge 0,\ h > 0.$$

The previous result allows us also to find the cdf of S_n $(n \in \mathbb{N})$, namely

$$F_n(t) \;=\; \mathbb{P}(N(t) \ge n) \;=\; e^{-\theta t} \sum_{k \ge n} \frac{(\theta t)^k}{k!} \quad \text{for all } t \ge 0. \tag{1.10}$$

As for the renewal counting process $\{N(t) : t \ge 0\}$, the fact that $N(t) \stackrel{d}{=} \mathit{Poisson}(\theta t)$ is actually a piece only of the following more complete result.

Theorem 1.2.2. *If $(S_n)_{n\geq 1}$ is a renewal process having exponential increments with parameter θ, then the associated renewal counting process $(N(t))_{t\geq 0}$ forms a* ***homogeneous Poisson process with intensity (rate)*** *θ, that is:*

(PP1) $N(0) = 0$.

(PP2) $(N(t))_{t\geq 0}$ *has independent increments, i.e.,*

$$N(t_1), N(t_2) - N(t_1), ..., N(t_n) - N(t_{n-1})$$

are independent random variables for each choice of $n \in \mathbb{N}$ *and* $0 < t_1 < t_2 < ... < t_n < \infty$.

(PP3) $(N(t))_{t\geq 0}$ *has stationary increments, i.e.,* $N(s+t) - N(s) \stackrel{d}{=} N(t)$ *for all* $s, t \geq 0$.

(PP4) $N(t) \stackrel{d}{=} Poisson(\theta t)$ *for each* $t \geq 0$.

We refrain from providing a proof of the result at this point [☞ ??????] and just mention that the crucial fact behind it is the lack of memory property of the exponential distribution.

1.3 Markov chains: a good example to motivate

A good motivation for the theoretical relevance of renewal theory in connection with stochastic processeses with inherent regeneration scheme is provided by a look at an important subclass, namely finite irreducible Markov chains.

A stochastic sequence $(M_n)_{n\geq 0}$ such that all M_n take values in a *finite* set $\mathscr{S}$ is called *finite Markov chain* if it satisfies the *Markov property*, viz.

$$\mathbb{P}(M_{n+1} = j | M_n = i, M_{n-1} = i_{n-1}, ..., M_0 = i_0) = \mathbb{P}(M_{n+1} = j | M_n = i)$$

for all $n \in \mathbb{N}_0$ and $i_0, ..., i_{n-1}, i, j \in \mathscr{S}$, and is *temporally homogeneous*, viz.

$$\mathbb{P}(M_{n+1} = j | M_n = i) \;=\; \mathbb{P}(M_1 = j | M_0 = i) \;=:\; p_{ij}$$

for all $n \in \mathbb{N}_0$ and $i, j \in \mathscr{S}$. The set $\mathscr{S}$ is called the *state space* of $(M_n)_{n\geq 0}$ and $\mathbf{P} = (p_{ij})_{i,j\in\mathscr{S}}$ its *(one-step) transition matrix*. We continue with a summary of some basic properties of such chains. A more detailed exposition will be provided in ????????

First of all, if τ is a stopping time for $(M_n)_{n\geq 0}$, i.e. τ takes values in $\mathbb{N}_0 \cup \{\infty\}$ and $\{\tau = n\} \in \sigma(M_0, ..., M_n)$ for all $n \in \mathbb{N}_0$, then the Markov property persists in the sense that

$$\mathbb{P}(M_{\tau+1}=j|M_\tau=i,M_{\tau-1},...,M_0,\tau<\infty) \;=\; \mathbb{P}(M_{\tau+1}=j|M_\tau=i) \;=\; p_{ij}$$

for all $i,j\in\mathscr{S}$. This is called the *strong Markov property*. For any path $i_0\to i_1\to ...\to i_n$ in $\mathscr{S}$ its probability is easily obtained by multiplying one-step transition probabilities, viz.

$$\mathbb{P}(M_1=i_1,...,M_n=i_n|M_0=i_0) \;=\; \lambda_{i_0}p_{i_0i_1}\cdot...\cdot p_{i_{n-1}i_n},$$

where $\lambda=\{\lambda_i : i\in\mathscr{S}\}$ denotes the distribution of M_0, called *initial distribution* of the chain. In the following, we make use of the common notation $\mathbb{P}_i:=\mathbb{P}(\cdot|M_0=i)$ and $\mathbb{P}_\lambda:=\sum_{i\in\mathscr{S}}\mathbb{P}_i$ for any distribution $\lambda=\{\lambda_i\}$ on $\mathscr{S}$. Hence $(M_n)_{n\ge 0}$ starts at i under $\mathbb{P}_i$ and has initial distribution λ under $\mathbb{P}_\lambda$.

Not surprisingly, temporal homogeneity extends to all n-step transition probabilities $(n\in\mathbb{N})$, i.e. $p_{ij}^{(n)}:=\mathbb{P}(M_n=j|M_0=i)=\mathbb{P}(M_{k+n}=j|M_k=i)$ for all $k\in\mathbb{N}_0$. Moreover, they satisfy the *Chapman-Kolmogorov equations*

$$p_{ij}^{(n)} \;=\; \sum_{k\in\mathscr{S}} p_{ik}^{(m)}p_{kj}^{(n-m)} \quad \text{for all } i,j\in\mathscr{S} \text{ and } m,n\in\mathbb{N}_0, \tag{1.11}$$

where $p_{ij}^{(0)}:=\delta_{ij}$. If $\mathbf{P}^{(n)}:=(p_{ij}^{(n)})_{i,j\in\mathscr{S}}$ denotes the n-step transition matrix, then these equations may be restated in matrix form as

$$\mathbf{P}^{(n)} \;=\; \mathbf{P}^{(m)}\mathbf{P}^{(n-m)} \quad \text{for all } m,n\in\mathbb{N}_0. \tag{1.12}$$

Consequently $\mathbf{P}^{(n)}=\mathbf{P}^n$ for each $n\in\mathbb{N}_0$, i.e. the transition matrices form a semigroup generated by $\mathbf{P}$. Let us note that, under $\mathbb{P}_\lambda$, the distribution of M_n is given by $\lambda\mathbf{P}^n$ for every $n\in\mathbb{N}_0$.

The chain $(M_n)_{n\ge 0}$ and its transition matrix $\mathbf{P}$ are called *irreducible* if all states *communicate* with respect to $\mathbf{P}$, where $i,j\in\mathscr{S}$ are said to be communicating if there exist $m,n\in\mathbb{N}_0$ such that $p_{ij}^{(m)}>0$ and $p_{ji}^{(n)}>0$. In other words, the chain can reach any state from any other state in a finite number of steps with positive probability. As a further prerequisite, some state properties must be defined. Let $T(i):=\inf\{n\ge 1 : M_n=i\}$ denote the first return time to $i\in\mathscr{S}$, where $\inf\emptyset:=\infty$. Then i is called

- recurrent if $\mathbb{P}_i(T(i)<\infty)=1$.
- transient if $\mathbb{P}_i(T(i)=\infty)>0$.
- positive recurrent if i is recurrent and $\mathbb{E}_iT(i)<\infty$.
- null recurrent if i is recurrent and $\mathbb{E}_iT(i)=\infty$.
- aperiodic if $\mathbb{P}_i(T(i)\in d\mathbb{N})<1$ for any integer $d\ge 2$.

It will be shown in ?????? that each of these properties is a *solidarity property* which means that it is shared by communicating states and thus by all states if the chain is irreducible. In the latter case we can therefore attribute any property to the chain as well. Putting $f_{ij}^{(n)}:=\mathbb{P}_i(T(j)=n)$, it is an easy exercise to verify that

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \quad \text{for all } i,j \in \mathscr{S} \text{ and } n \in \mathbb{N}. \tag{1.13}$$

Now consider, for any state $i \in \mathscr{S}$, the sequence $\{T_n(i)\}$ of successive return times, i.e. $T_1(i) := T(i)$ and

$$T_n(i) := \begin{cases} \inf\{k > T_{n-1}(i) : M_k = i\}, & \text{if } T_{n-1}(i) < \infty, \\ \infty, & \text{otherwise} \end{cases} \quad \text{for } n \geq 2.$$

The next lemma provides a first indication of how renewal theory enters in Markov chain analysis. A stochastic sequence with iid increments taking values in $\mathbb{R}_{\geq} \cup \{\infty\}$ and having positive mean is called *proper renewal process* if the increments are a.s. finite, and *terminating renewal process* otherwise.

Lemma 1.3.1. *For any $i \in \mathscr{S}$, the sequence $(T_n(i))_{n\geq 1}$ forms a renewal process under $\mathbb{P}_i$. It is proper if i is recurrent and terminating otherwise.*

Proof. Fix any $i \in \mathscr{S}$, write T_n for $T_n(i)$ and put $\beta := \mathbb{P}_i(T_1 < \infty)$. Use the Markov property and temporal homogeneity to find that

$$\begin{aligned} &\mathbb{P}_i(T_1 = m, T_2 - T_1 = n) \\ &\quad = \mathbb{P}_i(T_1 = m, M_m = i, T_2 - T_1 = n) \\ &\quad = \mathbb{P}(M_{m+n} = i, M_{m+k} \neq i \text{ for } 1 \leq k < n | M_m = i) \mathbb{P}_i(T_1 = m) \\ &\quad = \mathbb{P}_i(M_n = i, M_k \neq i \text{ for } 1 \leq k < n) \mathbb{P}_i(T_1 = m) \\ &\quad = \mathbb{P}_i(T_1 = m) \mathbb{P}_i(T_1 = n) \end{aligned}$$

for all $n, m \in \mathbb{N}$ and then also, after summation over $m, n \in \mathbb{N}$, that $\mathbb{P}_i(T_2 < \infty) = \beta^2$. This shows conditional independence and identical distribution (under $\mathbb{P}_i$) of T_1 and $T_2 - T_1$ given $T_2 < \infty$. For arbitrary $n \in \mathbb{N}$, it follows by an inductive argument that $T_1, T_2 - T_1, ..., T_n - T_{n-1}$ are conditionally iid given $T_n < \infty$ as well as $\mathbb{P}_i(T_n < \infty) = \beta^n$. The assertions of the lemma are now easily concluded. □

As an immediate consequence, we now obtain the following zero-one law.

Lemma 1.3.2. *If $i \in \mathscr{S}$ is recurrent, then*

$$\mathbb{P}_i(M_n = i \text{ infinitely often}) = 1,$$

while

$$\mathbb{P}_j(M_n = i \text{ infinitely often}) = 0$$

for all $j \in \mathscr{S}$ if i is transient.

Proof. Observe that $\{M_n = i \text{ infinitely often}\} = \{T_n(i) < \infty \text{ for all } n \in \mathbb{N}\}$. Now use the previous lemma to infer that

$$\mathbb{P}_i(T_n(i) < \infty \text{ for all } n \in \mathbb{N}) \;=\; \lim_{n\to\infty} \mathbb{P}_i(T(i) < \infty)^n$$

which clearly equals 1 if i is recurrent and 0 otherwise. If i is transient, the strong Markov property further implies

$$\mathbb{P}_j(M_n = i \text{ infinitely often}) \;=\; \mathbb{P}_j(T(i) < \infty)\,\mathbb{P}_i(M_n = i \text{ infinitely often}) \;=\; 0$$

for all $j \in \mathscr{S}$. □

We see from the previous result that any finite Markov chain has at least one recurrent state because otherwise all $i \in \mathscr{S}$ would be visited only finitely often which is clearly impossible if $|\mathscr{S}| < \infty$. Adding irreducibility as a further assumption, solidarity now leads to the following important conclusion:

Theorem 1.3.3. *Every irreducible finite Markov chain is recurrent.*

We can now turn to the most interesting question about the long run behavior of an irreducible finite Markov chain $(M_n)_{n\ge 0}$. Since M_n moves around in $\mathscr{S}$ visiting every state infinitely often and since, by the Markov property, the chain has no memory, one can expect M_n to converge in distribution to some limit law $\pi = \{\pi_i : i \in \mathscr{S}\}$ which does not depend on the initial distribution $\mathbb{P}(M_0 \in \cdot)$. An important invariance property of any such limit law π is stated in the following lemma. Put $\mathbb{P}_\lambda := \sum_{i\in\mathscr{S}} \lambda_i \mathbb{P}_i$ for any distribution $\lambda = (\lambda_i)_{i\in\mathscr{S}}$ on $\mathscr{S}$ and notice that $(M_n)_{n\ge 0}$ has initial distribution λ under $\mathbb{P}_\lambda$.

Lemma 1.3.4. *Suppose that, for some initial distribution $\lambda = (\lambda_i)_{i\in\mathscr{S}}$ on $\mathscr{S}$,*

$$\pi_i \;:=\; \lim_{n\to\infty} \mathbb{P}_\lambda(M_n = i) \tag{1.14}$$

exists for all $i \in \mathscr{S}$, i.e., $M_n \xrightarrow{d} \pi = (\pi_i)_{i\in\mathscr{S}}$. Then π is a left eigenvector of $\mathbf{P}$ for the eigenvalue 1, i.e. $\pi = \pi\mathbf{P}$, and

$$\mathbb{P}_\pi(M_n \in \cdot) \;=\; \pi \quad \textit{for all } n \in \mathbb{N}_0.$$

Any such π is called ***invariant*** *or* ***stationary distribution of*** *$(M_n)_{n\ge 0}$. If the chain is irreducible then all π_i are positive.*

Proof. Since $\pi\mathbf{P}^n$ equals the distribution of M_n under $\mathbb{P}_\pi$ as stated earlier, it suffices to prove the first assertion and the positivity of π in the irreducible case. But with $\mathscr{S}$ being finite condition (1.14) implies

$$\pi_j \;=\; \lim_{n\to\infty} \mathbb{P}_\lambda(M_{n+1}=j) \;=\; \sum_{i\in\mathscr{S}} \lim_{n\to\infty} \mathbb{P}_\lambda(M_n=i)p_{ij} \;=\; \sum_{i\in\mathscr{S}} \pi_i p_{ij}$$

for all $j\in\mathscr{S}$ which is the same as $\pi=\pi\mathbf{P}$. Now suppose that $(M_n)_{n\ge 0}$ is irreducible. As $\pi=\pi\mathbf{P}^n$ for all $n\in\mathbb{N}_0$, we see that, if $\pi_j=0$ for some $j\in\mathscr{S}$, then

$$0 \;=\; \pi_j \;=\; \sum_{i\in\mathscr{S}} \pi_i p_{ij}^{(n)} \quad \text{for all } n\in\mathbb{N}$$

and thus $p_{ij}^{(n)}=0$ for all π-positive i and all $n\in\mathbb{N}$. But this means that j cannot be reached from any π-positive i which is impossible by irredubility. □

In view of the previous lemma we are now facing two questions for a given irreducible finite Markov chain $(M_n)_{n\ge 0}$:

(Q1) Does $(M_n)_{n\ge 0}$ always have a stationary distribution π?
(Q2) Does (1.14) hold true for any choice of λ and with the same limit π?

Of course, a positive answer to (Q1) follows from a positive answer to (Q2) which, however, is not generally true and brings in fact aperiodicity into play. Namely, if the chain is not aperiodic, then it can be shown that (by solidarity) it has a unique period $d\ge 2$ in the sense that d is the maximal integer such that $\mathbb{P}_i(T(i)\in d\mathbb{N})=1$ for all $i\in\mathscr{S}$. This further entails $p_{ii}^{(n)}=0$ for all $i\in\mathscr{S}$ and all $n\in\mathbb{N}\backslash d\mathbb{N}$ [☞ ??????? for further details]. On the other hand, if (1.14) held true for any λ, we inferred upon choosing $\lambda=\delta_i$ that

$$\pi_i \;=\; \lim_{n\to\infty} p_{ii}^{(n)} \;=\; \liminf_{n\to\infty} p_{ii}^{(n)} \;=\; 0$$

which contradicts that all π_i must be positive.

Based on the previous observations we further confine ourselves now to aperiodic finite Markov chains $(M_n)_{n\ge 0}$. Then, for (Q2) to be answered affirmatively, it suffices to show that

$$\pi_j \;=\; \lim_{n\to\infty} p_{ij}^{(n)} \quad \text{for all } i,j\in\mathscr{S}.$$

But with the help of (1.13) and the dominated convergence theorem, this reduces to

$$\pi_j \;=\; \lim_{n\to\infty} \sum_{k\ge 1} \mathbf{1}_{\{1,\ldots,n\}}(k) f_{ij}^{(k)} p_{jj}^{(n-k)} \;=\; \lim_{n\to\infty} p_{jj}^{(n)} \quad \text{for all } j\in\mathscr{S}$$

and finally makes us return to renewal theory via the following observation: Since $\{M_n=j\}=\sum_{k\ge 1}\{T_k(j)=n\}$, where the summation indicates as usual the union of pairwise disjoint events, we infer that

$$p_{jj}^{(n)} \;=\; \sum_{k\ge 1} \mathbb{P}_j(T_k(j)=n) \;=\; \mathbb{U}_j(\{n\}), \quad \text{for all } j\in\mathscr{S} \text{ and } n\in\mathbb{N} \tag{1.15}$$

where $\mathbb{U}_j$ denotes the renewal measure of the discrete renewal process $(T_k(j))_{k\geq 1}$. Consequently, in order to find the limiting behavior of $p_{jj}^{(n)}$ we must find the limiting behavior of the renewal measure $\mathbb{U}_j$ along singleton sets tending towards infinity. This is in perfect accordance with (Q3) of Section 1.1 once observing that, due to the fact that the $T_k(j)$ are integer-valued, $\mathbb{U}_j(\{n\}) = \mathbb{U}_j(n) - \mathbb{U}_j(n-1)$ for all $n \in \mathbb{N}$.

1.4 Branching processes: a surprising connection

In this section, we will take a look at a very simple branching model of cell division. It may be surprising at first glance that branching as a typically exponential-type phenomenon can be studied with the help of renewal theory which rather deals with stochastic phenomena of linear type. Let it be said to all weisenheimers that this is not accomplished by just using a logarithmic transformation.

We consider a population of cells having independent lifetimes with a standard exponential distribution. At the end of its lifetime, each cell either splits into two new cells with probability p or dies with probability $1-p$ independent of all other cells alive. Suppose that at time $t=0$ the evolution starts with one cell having lifetime T and number of offspring Y, thus $\mathbb{P}(Y=2) = p = 1-\mathbb{P}(Y=0)$, and Y is independent of T. Let $Z(t)$ be the number of cells alive at time $t \in \mathbb{R}_{\geq}$. Then

$$Z(t) \;=\; \mathbf{1}_{\{T>t\}} \;+\; \mathbf{1}_{\{T\leq t, X=2\}}(Z_1(t-T)+Z_2(t-T)) \tag{1.16}$$

where $Z_i(t-T)$ denotes the size at time t of the subpopulation of cells stemming from the i^{th} daughter cell born at $T \leq t$ $(i=1,2)$. As following from the model assumptions, the $(Z_i(t))_{t\geq 0}$ are mutually independent copies of $(Z(t))_{t\geq 0}$ and further independent of (T,Y).

Our goal is to find the expected population size $M(t) := \mathbb{E}Z(t)$. Since T is independent of Y and $T \stackrel{d}{=} Exp(1)$, we infer from (1.16)

$$\begin{aligned} M(t) \;&=\; \mathbb{P}(T>t) + \int_{[0,t]} M(t-s)\, 2p\, \mathbb{P}(T \in ds) \\ &=\; e^{-t} + \int_0^t M(t-s)\, 2pe^{-s}\, ds \quad \text{for all } t \geq 0 \end{aligned}$$

which is an integral equation of convolutional type that may be rewritten as

$$M \;=\; \overline{F} + M * Q, \tag{1.17}$$

here with $\overline{F}(t) := e^{-t}$ for $t \geq 0$ and $Q(ds) := 2pe^{-s}\mathbf{1}_{(0,\infty)}(s)\, ds$. An equation of this type can be solved with the help of renewal theory as we will see in a moment and is therefore called *renewal equation*. Notice first that Q has total mass

$$\|Q\| \;=\; 2p \int_0^\infty e^{-s}\, ds \;=\; 2p \;=\; \mathbb{E}Y$$

and is thus a probability distribution only if $\mathbb{E}Y = 1$. On the other hand, a look at the mgf of Q, viz.

$$\phi_Q(t) \;=\; \int e^{st}\,Q(ds) \;=\; 2p\int_0^\infty e^{(t-1)s}\,ds \;=\; \frac{2p}{1-t} \quad (-\infty < t < 1),$$

shows the existence of a unique θ such that $\phi_Q(\theta) = 1$, namely $\theta = 1-2p$. Now observe that

$$e^{\theta t}M(t) \;=\; e^{(\theta-1)t} + \int_0^t e^{\theta(t-s)}M_\theta(t-s)\,2pe^{(\theta-1)s}\,ds \quad \text{for all } t \ge 0$$

which may be rewritten as

$$M_\theta \;=\; \overline{F}_\theta + M_\theta * Q_\theta, \tag{1.18}$$

where $M_\theta(t) := e^{\theta t}M(t)$, $\overline{F}_\theta(t) := e^{-2pt}$ and $Q_\theta(ds) := 2pe^{-2ps}\mathbf{1}_{(0,\infty)}(s)\,ds$. Since $\|Q_\theta\| = \phi_Q(\theta) = 1$ (by choice of θ), we see that a change of measure turns the original renewal equation (1.17) into an equivalent so-called *proper renewal equation* with a probability distribution as convolution measure, in fact $Q_\theta = Exp(2p)$. Let $(S_n)_{n\ge 1}$ be a renewal process with this increment distribution. Then (1.18) becomes

$$M_\theta(t) \;=\; \overline{F}_\theta(t) \;+\; \mathbb{E}M_\theta(t-S_1) \quad \text{for all } t \ge 0$$

and upon n-fold iteration

$$M_\theta(t) \;=\; \sum_{k=0}^{n-1}\mathbb{E}\overline{F}_\theta(t-S_k) \;+\; \mathbb{E}M_\theta(t-S_n) \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}.$$

It is intuitively clear and taken for granted here that M is continuous on $\mathbb{R}_\ge$ and thus bounded on compact subintervals. Consequently,

$$\lim_{n\to\infty}\mathbb{E}M_\theta(t-S_n) \;\le\; \lim_{n\to\infty}\mathbb{P}(S_n \le t)\max_{s\in[0,t]}M_\theta(s) \;=\; 0$$

because $S_n \to \infty$ a.s., and we therefore conclude

$$M_\theta(t) \;=\; \sum_{k\ge 0}\mathbb{E}\overline{F}_\theta(t-S_k) \;=\; \overline{F}_\theta(t) + \overline{F}_\theta * \mathbb{U}(t),$$

where $\mathbb{U} := \sum_{n\ge 1}\mathbb{P}(S_n \in \cdot)$ denotes the renewal measure of $(S_n)_{n\ge 1}$. But the latter has exponentially distributed increments with parameter $2p$ and so $\mathbb{U} = 2p\,\lambda\!\!\lambda_0$ on $\mathbb{R}_>$ by Proposition 1.2.1. This finally allows us to compute $\overline{F}_\theta * \mathbb{U}(t)$ explicitly leading to

$$M_\theta(t) \;=\; e^{-2pt} \;+\; \int_0^t 2pe^{-2p(t-s)}\,ds \;=\; 1 \quad \text{for all } t \ge 0$$

and therefore

$$M(t) \;=\; e^{(2p-1)t} \quad \text{for all } t \ge 0 \tag{1.19}$$

The parameter $2p-1$ thus giving the exponential rate of mean growth or decay of the population is called its *Malthusian parameter*.

Astute readers will have noticed that for this particularly nice example of a cell splitting model (1.19) could have been obtained far more easily by showing with the help of the memoryless property of the exponential distribution that $(e^{\theta t}Z(t))_{t\geq 0}$ is a martingale with $Z(0)=1$ and thus having constant expectation equal to one. However, if we replace the exponential lifetime distribution with an arbitrary distribution F with finite mean μ, then this latter argument breaks down while the renewal argument still works. In fact, we may even additionally assume an arbitrary offspring distribution $\{p_k\}$ to arrive at the following general renewal equation for $M(t)=\mathbb{E}Z(t)$:

$$M(t) \;=\; \overline{F}(t) \;+\; \int_{[0,t]} M(t-s)\, Q(ds) \quad \text{for all } t\geq 0 \tag{1.20}$$

where $Q(ds)=\mu F(ds)$. If $\mu\neq 1$ and thus Q is not a probability distribution, then a transformation of (1.20) into a proper one requires the existence of a (necessarily unique) θ such that $\phi_Q(\theta)=\mu\phi_F(\theta)=1$ which may fail if $\mu<1$. In the case where θ exists the result is as before that

$$M_\theta(t) \;=\; \overline{F}_\theta(t)+\overline{F}_\theta * \mathbb{U}(t) \tag{1.21}$$

where $\mathbb{U}$ denotes the renewal measure of a renewal process with increment distribution $Q_\theta(ds)=\mu e^{\theta s}F(ds)$. Unlike the exponential case, however, this does not generally lead to an explicit formula for $M(t)$ because $\mathbb{U}$ is not known explicitly. Instead, one must resort once again to asymptotic considerations as $t\to\infty$. For a further discussion of these aspects in this more general situation, we refer to ???????.

1.5 Collective risk theory: a classical application

This application is a relative of the previous one in that it eventually leads to a renewal equation that must be solved in order to gain information on the quantity of interest.

In collective risk theory, a part of nonlife insurance mathematics, the following problem is of fundamental interest: An insurance company earns premiums at a constant rate $c\in\mathbb{R}_>$ from a portfolio of insurance policies and faces negative claims from these of absolute sizes $X_1,X_2,\ldots$ at successive random epochs $0<T_1<T_2<\ldots$ Given an initial risk reserve $R(0)$, the risk reserve $R(t)$ at time t, i.e., the available capital at t to cover incurred future claims, is given by

$$R(t) \;=\; R(0)+ct-\sum_{k=1}^{N(t)} X_k \quad \text{for all } t\geq 0,$$

where $N(t) := \sum_{n\geq 1} \mathbf{1}_{\{T_n\leq t\}}$ denotes the number of claims up to time t. If $R(t)$ becomes negative, so-called *technical ruin* occurs. It is therefore a main concern of the insurance company to choose $R(0)$ and c in such a way that the probability for this event, called *ruin probability*, is small. Plainly, this requires a computation of this probability after the specification of a stochastic model for the bivariate sequence $(T_n,X_n)_{n\geq 1}$. Again, we do not strive for greatest generality in this introductory section but will instead discuss the problem in the framework of what is known today as the *Cramér-Lundberg model* which has its origin in a dissertation by F. LUNDBERG [22]:

(CL1) $(N(t))_{t\geq 0}$ is a homogeneous Poisson process with intensity λ or, equivalently, $T_1, T_2-T_1, \ldots$ are iid with $T_1 \stackrel{d}{=} Exp(\lambda)$.
(CL2) $X_1, X_2, \ldots$ are iid with common distribution F and finite positive mean μ.
(CL3) $(T_n)_{n\geq 1}$ and $(X_n)_{n\geq 1}$ are independent.

Put $Y_n := T_n - T_{n-1}$ for $n \in \mathbb{N}$ (with $T_0 = 0$) and let (X,Y) denote a generic copy of (X_n,Y_n) hereafter. Defining the epoch of technical ruin, viz.

$$\Lambda \;:=\; \inf\{t\geq 0 : R(t) < 0\} \quad (\inf\emptyset := \infty),$$

the task is to compute for a fixed premium rate c

$$\Psi(r) \;:=\; \mathbb{P}(\Lambda < \infty | R(0) = r) \quad \text{for } r > 0.$$

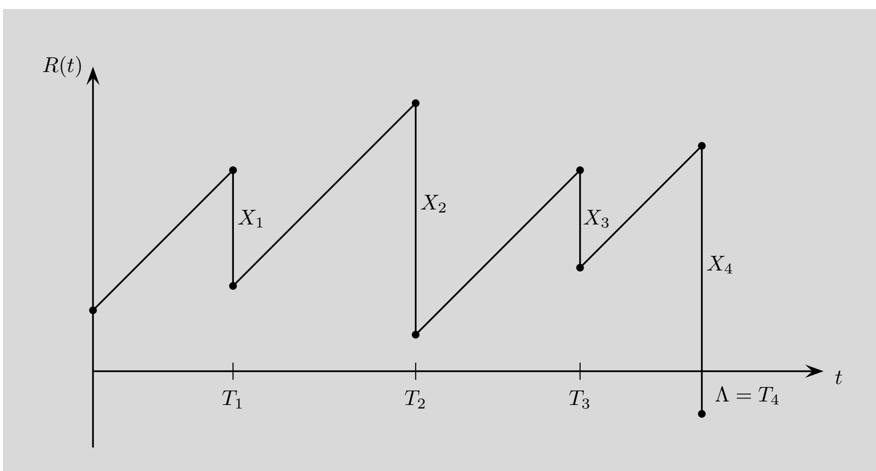

Fig. 1.2 The risk reserve process $(R(t))_{t\geq 0}$ with ruin epoch Λ

Let us begin with the observation that technical ruin can only occur at the epochs T_n, that is (given $R(0) = r$)

$$\Lambda \;=\; T_\tau, \quad \text{where } \tau \;:=\; \inf\{n \geq 1 : r + cT_n - S_n < 0\}$$

and $(S_n)_{n\geq 1}$ denotes the renewal process with increments $X_1, X_2, \ldots$ Hence

$$\Psi(r) \;=\; \mathbb{P}(\tau < \infty | R(0) = r) \quad \text{for } r \geq 0.$$

In the following considerations we keep $R(0) = r$ fixed and simply write $\mathbb{P}$ instead of $\mathbb{P}(\cdot|R(0) = r)$. Rewriting τ as

$$\tau \;=\; \inf\{n \geq 1 : S_n - cT_n > r\}$$

we see that ν is a so-called *first passage time* for the random walk $(S_n - cT_n)_{n\geq 1}$ with drift $\nu := \mathbb{E}(X - cY) = \mu - c\lambda^{-1}$. Hence

$$\Psi(r) = 1 \quad \text{for all } r > 0$$

if $\nu \geq 0$, because $S_n - cT_n \to \infty$ a.s. by the SLLN if $\nu > 0$, and $\limsup_{n\to\infty} S_n - cT_n = \infty$ a.s. by the Chung-Fuchs theorem [☞ Thm. 2.2.11] in the case $\nu = 0$. The interesting case to be discussed hereafter is therefore when

$$\text{(CL4)} \qquad \nu = \mathbb{E}(X - cY) = \mu - \frac{c}{\lambda} < 0$$

which means that the mean premium earned between two claim epochs is larger than the expected claim size. We first prove a renewal equation for $\overline{\Psi} := 1 - \Psi$.

Lemma 1.5.1. *Assuming (CL1–4), $\overline{\Psi}$ satisfies the renwal equation*

$$\overline{\Psi}(r) \;=\; \overline{\Psi}(0) \;+\; \int_0^r \overline{\Psi}(r-x)\, Q(dx) \quad \text{for all } r \geq 0. \tag{1.22}$$

where $Q(dx) := \frac{\lambda}{c}\,\mathbb{P}(X > x)\,dx$ on $\mathbb{R}_\geq$.

Proof. Since X and Y are independent with $X \stackrel{d}{=} F$ and $Y \stackrel{d}{=} Exp(\lambda)$, a conditioning argument leads to

$$\begin{aligned}
\overline{\Psi}(r) \;&=\; \mathbb{P}(S_n \leq r + cT_n \text{ for all } n \geq 1)\\
&=\; \int_{\{(x,y):x\leq r+cy\}} \mathbb{P}(x + S_n \leq r + c(y + T_n) \text{ for all } n \geq 1)\; \mathbb{P}(X \in dx, Y \in dy)\\
&=\; \int_0^\infty \int_{[0,r+cy]} \overline{\Psi}(r - x + cy)\,\lambda e^{-\lambda y}\, F(dx)\, dy\\
&=\; \int_r^\infty \frac{\lambda}{c} e^{-(\lambda/c)(y-r)} \int_{[0,y]} \overline{\Psi}(y-x)\, F(dx)\, dy
\end{aligned}$$

and thus shows the differentiability of $\overline{\Psi}$ with

$$\overline{\Psi}'(r) = -\frac{\lambda}{c}\int_{[0,r]}\overline{\Psi}(r-x)\,F(dx) + \int_r^\infty \frac{\lambda^2}{c^2}e^{-(\lambda/c)(y-r)}\int_{[0,y]}\overline{\Psi}(y-x)\,F(dx)\,dy$$
$$= -\frac{\lambda}{c}\int_{[0,r]}\overline{\Psi}(r-x)\,F(dx) + \frac{\lambda}{c}\overline{\Psi}(r)$$

for all $r \ge 0$. Consequently, we obtain upon integration

$$\overline{\Psi}(r)-\overline{\Psi}(0) = \frac{\lambda}{c}\left(\int_0^r\left(\overline{\Psi}(y)-\int_{[0,y]}\overline{\Psi}(y-x)\,F(dx)\right)\,dy\right)$$

which leaves us with the verification of

$$\int_0^r\left(\overline{\Psi}(y)-\int_{[0,y]}\overline{\Psi}(y-x)\,F(dx)\right)\,dy = \int_0^r\overline{\Psi}(r-x)\,\mathbb{P}(X>x)\,dx$$

for $r \ge 0$. But this follows from

$$\int_0^r\int_{[0,y]}\overline{\Psi}(y-x)\,F(dx)\,dy = \int_{[0,r]}\int_0^{r-x}\overline{\Psi}(y)\,dy\,F(dx)$$
$$= \int_0^r\overline{\Psi}(y)\,\mathbb{P}(X\le r-y)\,dy = \int_0^r\overline{\Psi}(r-y)\,\mathbb{P}(X\le y)\,dy$$

and $\int_0^r\overline{\Psi}(y)\,dy = \int_0^r\overline{\Psi}(r-y)\,dy$. □

Notice that Q as defined in the lemma has total mass $\|Q\| = \frac{\lambda}{c}\,\mathbb{E}X = \frac{\lambda\mu}{c} < 1$ because $\nu < 0$. This means that (1.22) is a so-called *defective renewal equation*. Since $\|Q^{*n}\| = \|Q\|^n \to 0$, we infer that the renewal measure $\mathbb{U}_Q := \sum_{n\ge1}Q^{*n}$ associated with Q is a finite measure with total mass $(1-\|Q\|)^{-1}\|Q\|$ and

$$\overline{\Psi}*Q^{*n}(r) = \int_{[0,r]}\overline{\Psi}(r-x)\,Q^{*n}(dx) \le \|Q\|^n \to 0 \quad \text{as } n\to 1$$

Hence, by a similar iteration argument as in the previous section we find that

$$\overline{\Psi}(r) = \overline{\Psi}(0)+\overline{\Psi}(0)*\mathbb{U}_Q(r) = \overline{\Psi}(0)+\int_0^r\overline{\Psi}(0)\,\mathbb{U}_Q(dx) = \overline{\Psi}(0)(1+\mathbb{U}_Q(r))$$

and thereupon that

$$\lim_{r\to\infty}\overline{\Psi}(r) = \overline{\Psi}(0)(1+\|U_Q\|) = \frac{\overline{\Psi}(0)}{1-\|Q\|} = \frac{c\overline{\Psi}(0)}{c-\lambda\mu}. \qquad (1.23)$$

On the other hand, we have $S_n - cT_n \to -\infty$ a.s. if $\nu < 0$ and therefore $\overline{\Psi}(r) \to 1$ as $r\to\infty$. By combining this with (1.23) and solving for $\overline{\Psi}(0)$ yields

$$\overline{\Psi}(0) = \frac{c-\lambda\mu}{c} = 1-\frac{\lambda\mu}{c} = 1-\|Q\|. \qquad (1.24)$$

Naturally, this is not the end of the story when striving for the asymptotic behavior of $\Psi(r)$ beyond the quite trivial statement that $\lim_{r\to\infty}\Psi(r)=0$ if $\nu<0$. As in the branching example of the previous section, we will now make use of a change of measure argument which, however, requires an additional condition on the distribution F of X. As before, let ϕ_F be the mgf of F. Note that $Q_\theta(dx)=e^{\theta x}Q(dx)$ has total mass [☞ formula ????? in Appendix ?]

$$\|Q_\theta\| \;=\; \frac{\lambda}{c}\int_0^\infty \frac{\lambda}{c}\,e^{\theta x}\,\mathbb{P}(X>x)\,dx \;=\; \frac{\lambda}{c}\phi_F(\theta) \;=\; \frac{\lambda}{c\theta}(\phi_F(\theta)-1)$$

which is less than $\|Q\|<1$ for all $\theta<0$ because ϕ_F is increasing on $\mathbb{R}_{\le}$. In order for finding a θ such that $\|Q_\theta\|=1$, this θ must therefore be positive if it exists at all. Therefore we introduce as a further condition:

(CL5) There exists $\theta>0$ such that $\phi_F(\theta)\;=\;1+\dfrac{\lambda}{c\theta}$.

With the extra condition (CL5) the ruin probability $\Psi(r)$ after multiplication with $e^{\theta r}$ satisfies a proper renewal equation as stated in the next proposition. In the case where X is also exponentially distributed this can be converted into an explicit formula for $\Psi(r)$, while in the general case a statement on its asymptotic behavior as for $r\to\infty$ is possible but must wait until ??????

Proposition 1.5.2. *Assuming (CL1–5), the ruin probability* $\Psi_\theta(r):=e^{\theta r}\Psi(r)$ *satisfies the proper renewal equation*

$$\Psi_\theta(r) \;=\; e^{\theta r}Q((r,\infty)) \;+\; \int_{[0,r]}\Psi_\theta(r-x)\,Q_\theta(dx) \quad \textit{for all } r\ge 0 \tag{1.25}$$

and thus

$$\Psi_\theta(r) \;=\; e^{\theta r}Q((r,\infty)) \;+\; \int_{[0,r]} e^{\theta(r-x)}Q((r-x,\infty))\;\mathbb{U}_\theta(dx), \tag{1.26}$$

where Q is as in Lemma 1.5.1 and $\mathbb{U}_\theta:=\sum_{n\ge 1}Q_\theta^{*n}$ *denotes the renewal measure associated with* Q_θ.

Proof. Rewriting (1.22) with the help of (1.24), we obtain

$$\begin{aligned}
\Psi(r) \;&=\; 1-\left(\overline{\Psi}(0) \;+\; \int_0^r \overline{\Psi}(r-x)\,Q(dx)\right)\\
&=\; \Psi(0)-Q(r) \;+\; \int_{[0,r]}\Psi(r-x)\,Q(dx)\\
&=\; Q((r,\infty)) \;+\;+\; \int_{[0,r]}\Psi(r-x)\,Q(dx) \quad \text{for all } r\ge 0
\end{aligned}$$

and then (1.25) after multiplication with $e^{\theta r}$. Validity of (1.26) follows now in the same manner as in the previous section when using that Ψ_θ is bounded on compact subintervals of $\mathbb{R}_\geq$ in combination with $Q_\theta^{*n}(r) = \mathbb{P}(S_n \leq r) \to 0$ as $n \to \infty$, where $(S_n)_{n\geq 1}$ denotes a renewal process with increment distribution Q_θ. □

Finally looking at the special case where $F = Exp(1/\mu)$, we first note that then

$$Q_\theta(dx) \ = \ \frac{\lambda}{c} e^{-((1/\mu)-\theta)x} dx$$

and thus equals an exponential distribution (with parameter λ/c) if

$$\theta \ = \ \frac{1}{\mu} - \frac{\lambda}{c} \ > \ 0$$

which is easily seen to be an equivalent version of (CL4). Hence (CL5) does automatically hold here once (CL4) is valid. Now use $\mathbb{U}_\theta = \frac{\lambda}{c} \lambda\!\!\lambda_0$ and $Q = \frac{\lambda\mu}{c} Exp(1/\mu)$ to infer with the help of Proposition 1.5.2 that

$$\Psi_\theta(r) \ = \ \frac{\lambda\mu}{c} e^{-(\lambda/c)r} \ + \ \frac{\lambda}{c} \int_0^r \frac{\lambda\mu}{c} e^{-(\lambda/c)x} \, dx \ = \ \frac{\lambda\mu}{c} \quad \text{for all } r \geq 0$$

and therefore

$$\Psi(r) \ = \ \frac{\lambda\mu}{c} e^{-((1/\mu)-(\lambda/c))r} \quad \text{for all } r \geq 0 \tag{1.27}$$

1.6 Queuing theory: a typical application

Queuing theory as an important branch of Applied Probability deals with the performance analysis of service facilities which are subject to random input. Here we consider a single server station who is facing (beginning at time $T_0 = 0$) arrivals of customers at random epochs $0 < T_1 < T_2 < \ldots$ with service requests of (temporal) size $B_1, B_2, \ldots$ Customers who find the server busy join a queue and are served in the order they have arrived (first in, first out). Typical performance measures are quantities like workload, queue length or sojourn times of customers in the system. They may be studied over time (transient analysis) or in the long run (steady state analysis). Typically, the complexity of queuing systems does not allow a transient analysis whence one usually resorts to a steady state analysis. Like for finite Markov chains, the idea is that after a relaxation period the system is approximately in stochastic equilibrium so that relevant quantities may be approximated by their value under the stationary distribution. The computation of such approximations often requires the use of renewal theory as we will briefly demonstrate in this section.

We consider the so-called *M/G/1-queue* specified by the following assumptions:

(M/G/1-1) The arrival process $(N(t))_{t\geq 0}$, where $N(t) := \sum_{n\geq 1} \mathbf{1}_{\{T_n \leq t\}}$ for $t \geq 0$, is a homogeneous Poisson process with intensity λ (Poisson input).

(M/G/1-2) The service times $B_1, B_2, \ldots$ are iid with common distribution G and finite positive mean μ.
(M/G/1-3) The sequences $(T_n)_{n\ge 1}$ and $(B_n)_{n\ge 1}$ are independent.
(M/G/1-4) There is one server and a waiting room of infinite capacity.
(M/G/1-5) The queue discipline is FIFO ("first in, first out").

The *Kendall notation* "M/G/1", which may be expanded by further symbols when referring to more complex systems, has the following meaning:

"M": The first letter refers to the arrival pattern, and "M" stands for "Markovian". This means that the interarrival times $A_n := T_n - T_{n-1}$ are iid with an exponential distribution which renders $(N(t))_{t\ge 0}$ a homogeneous Poisson process and, in particular, a continuous time Markov process.
"G": The second letter refers to the sequence of service times, and "G" stands for "general". This means that the service times B_n are iid with an arbitrary distribution on $\mathbb{R}_\ge$ with positive mean.
"1": The number in the third position refers to the number of servers (or counters).

In the following, we will focus on the analysis of the queue length in steady state. Suppose for simplicity that at time $T_0 = 0$ the system is empty. For $t \ge 0$, let $Q(t)$ be the queue length at t, i.e. the number of waiting customers in the system at this time including the one currently in service. This is a stochastic process with $Q(0) = 0$ (due to our previous assumption) and trajectories that are right continuous with left hand limits. It is also a pure jump process with jumps being of size $+1$ at an arrival epoch and -1 at a departure epoch. On the other hand, it is *not* a Markov process unless the B_n are also exponentially distributed because the future evolution of the process at any time t does not only depend on the past through $Q(t)$ but also the time the customer currently in service already spent at the counter. The classical way out of this dilemma is a resort to an embedded discrete Markov chain in the sense defined in Section 1.3 but with countable state space $\mathbb{N}_0$. It is obtained by looking at

$$Q_n \;:=\; Q(D_n) \quad (n \in \mathbb{N}_0)$$

where $D_0 := 0$ and the D_n for $n \ge 1$ denote the successive departure epoch of customers in the system (thus $D_1 = T_1 + B_1$). As one can easily see, the Q_n and D_n satisfy the recursive relations

$$Q_n \;=\; (Q_{n-1} - 1)^+ + K_n \quad \text{and} \tag{1.28}$$

$$D_n \;=\; (D_{n-1} \vee T_n) + B_n \quad \text{for } n \in \mathbb{N}, \tag{1.29}$$

where K_n is the number of customers that enter the system during the service of the n^{th} customer (with service time B_n), thus

$$K_n \;=\; N(D_n) - N(D_{n-1} \vee T_n) \quad \text{for } n \in \mathbb{N}.$$

By using the model assumptions, notably (M/G/1-3) and the properties of a Poisson process, it is not difficult to verify that the K_n are iid with distribution $\{\kappa_j : j \in \mathbb{N}_0\}$

given by

$$\kappa_j := \int_{\mathbb{R}_>} \mathbb{P}(N(y) = j)\, G(dy) = \int_{\mathbb{R}_>} e^{-\lambda y} \frac{(\lambda y)^j}{j!}\, G(dy).$$

In particular,

$$\mathbb{E}K_1 = \sum_{j \geq 1} j\kappa_j = \int_{\mathbb{R}_>} \mathbb{E}N(y)\, G(dy) = \int_{\mathbb{R}_>} \lambda y\, G(dy) = \lambda\mu =: \rho. \quad (1.30)$$

Moreover, K_n is independent of $Q_0, ..., Q_{n-1}$ which in combination with (1.28) easily proves:

Lemma 1.6.1. *The queue length process $(Q_n)_{n \geq 0}$ at departure epochs constitutes an irreducible and aperiodic discrete Markov chain with state space $\mathbb{N}_0$ and transition matrix*

$$\mathbf{P} = (p_{ij})_{i,j \in \mathbb{N}_0} = \begin{pmatrix} \kappa_0 & \kappa_1 & \kappa_2 & \kappa_3 & \dots \\ \kappa_0 & \kappa_1 & \kappa_2 & \kappa_3 & \dots \\ 0 & \kappa_0 & \kappa_1 & \kappa_2 & \dots \\ 0 & 0 & \kappa_0 & \kappa_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let us continue by finding the necessary and sufficient condition under which the queuing system is stable or, equivalently, the chain $(Q_n)_{n \geq 0}$ is positive recurrent. Here we should mention that all notions introduced in Section 1.3 for finite Markov chains like irreducibility, recurrence and aperiodicity carry over without changes to the case of countable state space. Observe that ρ as defined in (1.30) satisfies

$$\rho = \frac{\mathbb{E}B_1}{\mathbb{E}T_1} = \frac{\text{mean service time}}{\text{mean interarrival time}}. \quad (1.31)$$

It is called the *traffic intensity* of the system because it provides a measure of its throughput rate. Intuitively, one can expect stability of the system if ρ is less than 1 because then, on the average, the server works at a faster rate than customers enter the system. The following renewal theoretic analysis will confirm this assertion.

For the random walk $S_n := \sum_{j=1}^n (K_j - 1)$ $(n \in \mathbb{N}_0)$, we define the associated sequence $\{\sigma_n\}$ of so-called *descending ladder epochs* by $\sigma_0 := 0$ and, recursively,

$$\sigma_n := \begin{cases} \inf\{k > \sigma_{n-1} : S_k < S_{\sigma_{n-1}}\}, & \text{if } \sigma_{n-1} < \infty, \\ \infty, & \text{otherwise} \end{cases} \quad \text{for } n \geq 1.$$

Since $(S_n)_{n \geq 1}$ can obviously make downward jumps of size -1 only, we infer $S_{\sigma_n} = -n$ if $\sigma_n < \infty$. Furthermore, as $\mathbb{E}S_1 = \mathbb{E}K_1 - 1 = \rho - 1$, we see that $\sigma_n < \infty$ a.s.

for all $n \in \mathbb{N}$ and thus $\liminf_{n\to\infty} S_n = -\infty$ a.s. requires $\rho \le 1$ [by the SLLN or the Chung-Fuchs theorem 2.2.11]. More precisely, we will show in ?????? that $\rho < 1$ implies $\mathbb{E}\sigma_1 < \infty$, while $\rho = 1$ implies $\sigma_1 < \infty$ a.s. and $\mathbb{E}\sigma_1 = \infty$. Furthermore, in any of these two cases, the σ_n have iid increments, and so $\{\sigma_n\}$ forms a discrete renewal process. Let $\mathbb{U} = \sum_{n\ge 1} \mathbb{P}(\sigma_n \in \cdot)$ denote its renewal measure with *renewal counting density*

$$u_n \;:=\; \mathbb{U}(\{n\}) \quad \text{for } n \in \mathbb{N}_0.$$

With the help of the previous facts we are now able to prove the following result about $\{Q_n\}$.

Proposition 1.6.2. *The queue length process at departure epochs $(Q_n)_{n\ge 0}$ is a recurrent Markov chain iff $\rho \le 1$, and it is positive recurrent iff $\rho < 1$. Furthermore $\rho \le 1$ implies*

$$\mathbb{P}(Q_n = j) \;=\; \mathbb{P}(Q_n = j, \sigma_1 > n) \;+\; \sum_{k=0}^{n} \mathbb{P}(Q_k = j, \sigma_1 > k)\, u_{n-k} \qquad (1.32)$$

for all $j,n \in \mathbb{N}_0$, in particular $\mathbb{P}(Q_n = 0) = u_n$.

Proof. Recall that $Q_0 = 0$ is assumed, i.e. $\mathbb{P} = \mathbb{P}(\cdot|Q_0 = 0)$. It suffices to study the recurrence of state 0 as $(Q_n)_{n\ge 0}$ is irreducible. The crucial observation is that $\sigma_1 = \inf\{n \ge 1 : Q_n = 0\}$ and that $Q_n = S_n$ for $0 \le n < \sigma_1$. Indeed, if $\rho \le 1$ and thus $\mathbb{P}(\sigma_1 < \infty) = 1$, the recurrence of 0 follows just by definition. On the other hand, if $\rho > 1$ and thus $\mathbb{P}(\sigma_1 = \infty) > 1$, then there is a positive chance of never hitting 0 again after time 0 so that 0 must be transient. As mentioned above, $\rho < 1$ further ensures $\mathbb{E}\sigma_1 < \infty$ and thus the positive recurrence of the chain. Finally,

$$\begin{aligned}
\mathbb{P}(Q_n = j) \;&=\; \mathbb{P}(Q_n = j, \sigma_1 > n) \;+\; \sum_{m\ge 1} \mathbb{P}(Q_n = j, \sigma_m \le n < \sigma_{m+1}) \\
&=\; \mathbb{P}(Q_n = j, \sigma_1 > n) \;+\; \sum_{k=0}^{n} \sum_{m\ge 1} \mathbb{P}(Q_n = j, \sigma_m = n-k, \sigma_{m+1} - \sigma_m > k) \\
&=\; \mathbb{P}(Q_n = j, \sigma_1 > n) \;+\; \sum_{k=0}^{n} \mathbb{P}(Q_k = j, \sigma_1 > k) \sum_{m\ge 1} \mathbb{P}(\sigma_m = n-k)
\end{aligned}$$

for all $j,n \in \mathbb{N}_0$ shows (1.32). Here we have used for the last line that, with $\mathscr{F}_n := \sigma((Q_k,S_k) : 0 \le k \le n)$,

$$\begin{aligned}\mathbb{P}(Q_n = j, \sigma_m = n-k, \sigma_{m+1} - \sigma_m > k) \\ &= \int_{\{\sigma_m = n-k\}} \mathbb{P}(Q_n = j, \sigma_{m+1} - \sigma_m > k | \mathscr{F}_{n-k})\, d\mathbb{P} \\ &= \int_{\{\sigma_m = n-k\}} \mathbb{P}(Q_k = j, \sigma_1 > k | Q_0)\, d\mathbb{P} \\ &= \mathbb{P}(Q_k = j, \sigma_1 > k)\,\mathbb{P}(\sigma_m = n-k)\end{aligned}$$

for all $j, m, n \in \mathbb{N}_0$ and $0 \le k \le n$. □

It should be observed that (1.32) may be restated as $\mathbb{P}(Q_n = j) = g_j * \mathbb{U}(n)$, where

$$g_j(k) := \mathbb{P}(Q_k = j, \sigma_1 > k) \quad \text{for } j,k \in \mathbb{N}_0,$$

and that this could have also been deduced as in the previous two examples from the fact that $\mathbb{P}(Q_n = j)$ (as a function of n for fixed j) satisfies the *discrete renewal equation*

$$\mathbb{P}(Q_n = j) \;=\; g_j(n) \;+\; \sum_{k=0}^{n} \mathbb{P}(Q_{n-k} = j)\,\mathbb{P}(\sigma_1 = k) \quad \text{for all } n \in \mathbb{N}_0.$$

So we have a similar result for a quantity of interest as in the previous two sections [☞ (1.21) and (1.26)], here in a discrete setup because the renewal process pertaining to $\mathbb{U}$ is integer-valued. Owing to this fact an application of the dominated convergence theorem to (1.32) immediately leads to the following result on the asymptotic behavior of Q_n once the convergence of u_n as $n \to \infty$ has been proved, the limit actually being $(\mathbb{E}\sigma_1)^{-1}$.

Theorem 1.6.3. *If* $\rho < 1$, *then*

$$\pi_j \;:=\; \lim_{n\to\infty} \mathbb{P}(Q_n = j) \;=\; \frac{1}{\mathbb{E}\sigma_1} \sum_{k\ge 0} \mathbb{P}(Q_k = j, \sigma_1 > k) \tag{1.33}$$

for all $j \in \mathbb{N}_0$.

It should not take by surprise that $\pi = (\pi_j)_{j\ge 0}$, which obviously forms a probability distribution, is the unique stationary distribution of $(Q_n)_{n\ge 0}$. Using

$$\mathbb{P}(Q_k = j, \sigma_1 > k) = \sum_{n>k} \mathbb{P}(Q_n = j, \sigma_1 = n)$$

in (1.33), we further obtain after interchanging the order of summation that

$$\pi_j \;=\; \frac{1}{\mathbb{E}\sigma_1} \mathbb{E}\left(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}_{\{Q_n = j\}} \right) \quad \text{for all } j \in \mathbb{N}_0, \tag{1.34}$$

which is the *occupation measure* representation of π having the very intuitive interpretation that the stationary probability for a queue length of j (at departures of customers) is just the expected number of epochs this value is attained during a cycle, defined as a (discrete) time interval between two epochs where the system becomes idle (busy period).

1.7 Record values: yet another surprise

Consider a sequence $(X_n)_{n\geq 1}$ of iid nonnegative random variables with a continuous distribution F. We say that a *record* occurs at time n if X_n exceeds all preceding values of the sequence, i.e., if $X_n > \max_{1\leq k<n} X_k$. In this case n is a *record epoch* and X_n a *record value*. More formally, define $\sigma_1 := 1, R_1 := X_1$ and then, recursively,

$$\sigma_n := \inf\{k > \sigma_{n-1} : X_k > R_{n-1}\} \quad \text{and} \quad R_n := X_{\sigma_n} \quad \text{for } n \geq 2.$$

Clearly, the σ_n are the record epochs and the R_n the record values of the sequence $(X_n)_{n\geq 1}$. Our main concern here is to get information on how the R_n spread on the nonnegative halfline. At first glance this seems to be quite unrelated to renewal theory, for record values do not generally show the pattern of a renewal process. For instance, if F is the uniform distribution on $(0,1)$ so that all X_n take values in this interval, then it is quite clear that the R_n accumulate at 1 from below. On the other hand, and this is the crucial observation, information on the R_n may be gained also after the application of a function G to the X_n that leaves their order unchanged.

Lemma 1.7.1. *Under the stated assumptions, let $G : \mathbb{R}_\geq \to \mathbb{R}_\geq$ be a nondecreasing function such that*

$$\mathbb{P}(G(X_1) < G(X_2) | X_1 < X_2) = 1.$$

Then the record epochs for $(X_n)_{n\geq 1}$ and $(G(X_n))_{n\geq 1}$ are a.s. the same, and $(G(R_n))_{n\geq 1}$ is the sequence of record values associated with $(G(X_n))_{n\geq 1}$.

Proof. Easy. □

As on can easily see, the lemma particularly applies to F itself (viewed as a cdf), and since F is continuous, the iid $F(X_n)$ are uniformly distributed on $(0,1)$. By then applying the transformation $x \mapsto -\log(1-x)$, we arrive at the sequence

$$Y_n := -\log(1-F(X_n)) \quad (n \in \mathbb{N})$$

of iid standard exponentials, for

$$\mathbb{P}(Y_1 > t) \;=\; \mathbb{P}(F(X_1) > 1-e^{-t}) \;=\; e^{-1} \quad \text{for all } t > 0.$$

Consequently, the problem of studying record epochs and values of iid continuous random variables may be reduced to the study of the corresponding variables for iid standard exponentials, for which we have the following result based on the lack of memory property of the exponential distribution.

Proposition 1.7.2. *Let $(X_n)_{n\ge 1}$ be a sequence of iid standard exponentials with canonical filtration $(\mathscr{F}_n)_{n\ge 1}$, associated record epochs σ_n and record values R_n for $n\in\mathbb{N}$. Then the following assertions hold true:*

(a) $(R_n)_{n\ge 1}$ *is a renewal process with standard exponential increments.*

(b) *For each $n\ge 2$, the conditional distribution of the n^{th} interrecord time $\tau_n := \sigma_n - \sigma_{n-1}$ given $\mathscr{F}_{\sigma_{n-1}}$ is geometric on $\mathbb{N}$ with parameter e^{-r} if $R_{n-1}=r$. Moreover,*

$$\mathbb{P}(\tau_n = k) \;=\; \int_0^\infty (1-e^{-r})^{k-1} e^{-2r}\,\frac{r^{n-2}}{(n-2)!}\,dr$$

for all $k\in\mathbb{N}$.

Proof. Put $Y_n := R_n - R_{n-1}$ for $n\in\mathbb{N}$, where $R_0 := 0$.

(a) Clearly, $T_1 = Y_1 \stackrel{d}{=} Exp(1)$. Suppose now that we have already shown that $Y_1,\dots,Y_n$ are iid with a standard exponential distribution. As one can easily verify, the sequence $(X_{\sigma_n+k})_{k\ge 1}$ is independent of $\mathscr{F}_{\sigma_n}$ and a copy of $(X_k)_{k\ge 1}$ [☞ ???????]. Putting $E_k := \{\tau_{n+1} = k\}$ for $k\in\mathbb{N}$, we infer for each $t\ge 0$

$$\begin{aligned}\mathbb{P}(Y_{n+1} > t|\mathscr{F}_{\sigma_n}) \;&=\; \sum_{k\ge 1}\mathbb{P}(X_{\sigma_n+k} > R_n + t, X_{\sigma_n+j}\le R_n \text{ for } 1\le j<k|R_n)\\ &=\; \sum_{k\ge 1}\mathbb{E}(\mathbb{P}(X_{\sigma_n+k} > R_n+t|R_n,E_k)\,\mathbf{1}_{E_k}|R_n)\quad \text{a.s.}\end{aligned}$$

and since X_{σ_n+k} is independent of R_n and E_k for each k, we further obtain by invoking Lemma A.1.1 and using $X_{\sigma_n+k}\stackrel{d}{=} Exp(1)$ that

$$\mathbb{P}(X_{\sigma_n+k} > R_n+t|R_n,E_k) \;=\; e^{-t}\,\mathbf{1}_{E_k}\quad \text{a.s.}$$

Consequently,

$$\mathbb{P}(Y_{n+1} > t|\mathscr{F}_{\sigma_n}) \;=\; e^{-t}\sum_{k\ge 1}\mathbb{P}(E_k|R_n) \;=\; e^{-t}\quad \text{a.s.}$$

for all $t\ge 0$ which proves that Y_{n+1} is independent of $\mathscr{F}_{\sigma_n}$, particularly of $R_1,\dots,R_n$, and has a standard exponential distribution. Hence the assertion follows by induction over n.

(b) As $(X_{\sigma_{n-1}+k})_{k\geq 1}$ is independent of $\mathscr{F}_{\sigma_{n-1}}$, it is clear that the conditional distribution of τ_n given $\mathscr{F}_{\sigma_{n-1}}$ only depends on the current record at σ_{n-1}, i.e. R_{n-1}, and has a geometric distribution with parameter e^{-r} if $R_{n-1}=r$. The formula for the unconditional probabilities $\mathbb{P}(\tau_n=k)$ then follows by integrating agains the distribution of R_{n-1} which, by (a), is a Gamma law with parameters $n-1$ and 1. □

With the help of the previous result we infer from Theorem 1.2.2 that the *record counting process*

$$N(t) \;:=\; \sum_{n\geq 1}\mathbf{1}_{\{R_n\leq t\}}\quad (t\geq 0)$$

forms a homogeneous Poisson process with intensity 1, also called *standard Poisson process*, if the underlying X_n are standard exponentials. It is now straightforward to conclude a similar result in the general case by drawing on Lemma 1.7.1 and the subsequent discussion.

Theorem 1.7.3. *Let $(X_n)_{n\geq 1}$ be a sequence of iid nonnegtive random variables with continuous cdf F and associated record values $(R_n)_{n\geq 1}$. Let further $b:=\inf\{x\in\mathbb{R}: F(x)=1\}$ and ν be the measure on $\mathbb{R}_{\geq}$, defined by*

$$\Lambda((s,t]) := \log\left(\frac{1-F(s)}{1-F(t)}\right)\quad \textit{for } 0\leq s<t<b,$$

and vanishing outside $[0,b]$. Then the record counting process $(N(t))_{t\geq 0}$ forms a ***nonhomogeneous Poisson process with intensity measure*** *Λ, that is*

(NPP1) $N(0)=0$.
(NPP2) $(N(t))_{t\geq 0}$ *has independent increments.*
(NPP3) $N(t)-N(s)\overset{d}{=}$ *Poisson*$(\Lambda((s,t]))$ *for all* $0\leq s<t<\infty$, *where Poisson*$(0):=\delta_0$.

If F has $\lambda\!\!\lambda_0$-density f, then so does Λ, viz. $\lambda(t)=\frac{f(t)}{1-F(t)}\mathbf{1}_{(0,b)}(t)$, which is then called the ***intensity function*** *of $(N(t))_{t\geq 0}$.*

Proof. As before, put $G(t):=-\log(1-F(t))$ and let $\{\widehat{N}(t)\}$ be the record counting process of the iid standard exponentials $(G(X_n))_{n\geq 1}$, hence a standard Poisson process by Proposition 1.7.2. Now all assertions are immediate consequences of this fact together with the observation that

$$N(t)=\widehat{N}(G(t))\quad \text{for all } t\geq 0.$$

All further details can therefore be left to the reader. □

So we have arrived at the very explicit result that the record counting process of a sequence of iid nonnegtive random variables with continuous cdf F is always a nonhomogeneous Poisson process, and its “renewal function” $\mathbb{U}(t)=\mathbb{E}N(t)$ just equals $\Lambda([0,t]) = -\log(1-F(t))$ for $t \geq 0$. Notice that in the case where F has a $\lambda\!\!\lambda_0$-density f the intensity function $\lambda(t)$ of $(N(t))_{t\geq 0}$ is nothing but the *failure* or *hazard rate* of the sequence $(X_n)_{n\geq 1}$, that is

$$\lambda(t) \;=\; \lim_{h\downarrow 0}\mathbb{P}(X_1 \in (t,t+h]|X_1 > t) \quad \text{for } \lambda\!\!\lambda_0\text{-almost all } t \geq 0.$$