Chapter 2
Random walks and stopping times: classifications and preliminary results

Our motivating examples have shown that renewal theory is typically concerned with sums of iid *nonnegative* random variables, i.e. renewal processes, the goal being to describe the implications of their regenerative properties. On the other hand, it then appears to be quite natural and useful to choose a more general framework by considering also sums of iid *real-valued* random variables, called *random walks*. Doing so, the regenerative structure remains unchanged while monotonicity is lost. However, this will be overcome by providing the concept of *ladder variables*, therefore introduced and studied in Subsection 2.2.3. The more general idea behind this concept is to sample a random walk along a sequence of stopping times that renders a renewal process as a subsequence. This in turn is based on an even more general result which, roughly speaking, states that any finite stopping time for a random walk may be formally copied indefinitely over time and thus always leads to another imbedded random walk. The exact result is stated as Theorem 2.2.3 in Subsection 2.2.2. Further preliminary results besides some basic terminology and classification include the (topological) recurrence of random walks with zero-mean increments, some basic properties of renewal measures of random walks with positive drift, the connection of the renewal measure with certain first passage times including ladder epochs, and the definition of the *stationary delay distribution*.

2.1 Preliminaries and classification of random walks

2.1.1 Lattice-type of random walks

As already mentioned, any sequence $(S_n)_{n\ge 0}$ of real-valued random variables with iid increments $X_1,X_2,\ldots$ and initial value S_0 independent of these is called *random walk (RW)* hereafter, and S_0 its *delay*. If $\mu := \mathbb{E}X_1$ exists, then μ is called the *drift* of $(S_n)_{n\ge 0}$. In the case where all X_n as well as S_0 are nonnegative and μ is positive (possibly infinite), $(S_n)_{n\geq 0}$ is also called a *renewal process (RP)*. Finally, a RW or RP $(S_n)_{n\geq 0}$ is called *zero-delayed* if $S_0 = 0$ a.s., and *delayed* otherwise.

An important characteristic of a RW $(S_n)_{n\geq 0}$ in the context of renewal theory is its lattice-type. Let F denote the distribution of the X_n and F_0 the distribution of S_0. Since $(S_n)_{n\geq 0}$ forms an additive sequence with state space $\mathbb{G}_0 := \mathbb{R}$, and since $\mathbb{G}_0$ has closed subgroups $\mathbb{G}_d := d\mathbb{Z} := \{dn : n \in \mathbb{Z}\}$ for $d \in \mathbb{R}_>$ and $\mathbb{G}_\infty := \{0\}$, it is natural to ask for the smallest closed subgroup on which the RW is concentrated. Namely, if all the X_n as well as S_0 take only values in a proper closed subgroup $\mathbb{G}$ of $\mathbb{R}$, that is, $F(\mathbb{G}) = F_0(\mathbb{G}) = 1$, then the same holds true for all S_n and its accumulation points. The following classifications of F and $(S_n)_{n\geq 0}$ reflect this observation.

Definition 2.1.1. For a distribution F on $\mathbb{R}$, its *lattice-span* $d(F)$ is defined as

$$d(F) \ := \ \sup\{d \in [0,\infty] : F(\mathbb{G}_d) = 1\}.$$

Let $\{F_x : x \in \mathbb{R}\}$ denote the translation family associated with F, i.e., $F_x(B) := F(x+B)$ for all Borel subsets B of $\mathbb{R}$. Then F is called

– *nonarithmetic*, if $d(F) = 0$ and thus $F(\mathbb{G}_d) < 1$ for all $d > 0$.
– *completely nonarithmetic*, if $d(F_x) = 0$ for all $x \in \mathbb{R}$.
– *d-arithmetic*, if $d \in \mathbb{R}_>$ and $d(F) = d$.
– *completely d-arithmetic*, if $d \in \mathbb{R}_>$ and $d(F_x) = d$ for all $x \in \mathbb{G}_d$.

If X denotes any random variable with distribution F, thus $X - x \stackrel{d}{=} F_x$ for each $x \in \mathbb{R}$, then the previous attributes are also used for X, and we also write $d(X)$ instead of $d(F)$ and call it the lattice-span of X.

For our convenience, a nonarithmetic distribution is sometimes also called *0-arithmetic* hereafter. A random variable X is nonarithmetic iff it is not a.s. taking values only in a lattice $\mathbb{G}_d$, and it is completely nonarithmetic if this is not the case for any shifted lattice $x + \mathbb{G}_d$, i.e. any affine closed subgroup of $\mathbb{R}$, either. As an example of a nonarithmetic, but not completely nonarithmetic random variable we mention $X = \pi + Y$ with a standard Poisson variable Y. Then $d(X - \pi) = d(Y) = 1$. If $X = \frac{1}{2} + Y$, then $d(X) = \frac{1}{2}$ and $d(X - \frac{1}{2}) = 1$. In this case, X is $\frac{1}{2}$-arithmetic, but not completely $\frac{1}{2}$-arithmetic. The following simple lemma provides the essential property of a completely d-arithmetic random variable $(d \geq 0)$.

Lemma 2.1.2. *Let X,Y be two iid random variables with lattice-span d. Then $d \leq d(X-Y)$ with equality holding iff X is completely non- $(d = 0)$ or d-arithmetic $(d > 0)$.*

Proof. Let F denote the distribution of X,Y. The inequality $d \leq d(X-Y)$ is trivial, and since $(X+z)-(Y+z) = X-Y$, we also have $d(X+z) \leq d(X-Y)$ for all

$z \in \mathbb{R}$. Suppose X is *not* completely non- or d-arithmetic. Then $d(X+z) > d$ for some $z \in \mathbb{G}_d$ and hence also $c := d(X-Y) > d$. Conversely, if the last inequality holds true, then

$$1 \;=\; \mathbb{P}(X-Y \in \mathbb{G}_c) \;=\; \int_{\mathbb{G}_d} \mathbb{P}(X-y \in \mathbb{G}_c)\, F(dy)$$

implies

$$\mathbb{P}(X-y \in \mathbb{G}_c) \;=\; 1 \quad \text{for all } F\text{-almost all } y \in \mathbb{G}_d$$

and thus $d(X-y) \ge c > d$ for F-almost all $y \in \mathbb{G}_d$. Therefore, X cannot be completely non- or d arithmetic. ☐

We continue with a classification of RW's based on Definition 2.1.1.

Definition 2.1.3. A RW $(S_n)_{n\ge 0}$ with increments $X_1, X_2, \ldots$ is called

– *(completely) nonarithmetic* if X_1 is (completely) nonarithmetic.
– *(completely) d-arithmetic* if $d > 0$, $\mathbb{P}(S_0 \in \mathbb{G}_d) = 1$, and X_1 is (completely) d-arithmetic.

Furthermore, the lattice-span of X_1 is also called the lattice-span of $(S_n)_{n\ge 0}$ in any of these cases.

The additional condition on the delay in the d-arithmetic case, which may be restated as $d(S_0) = kd$ for some $k \in \mathbb{N} \cup \{\infty\}$, is needed to ensure that $(S_n)_{n\ge 0}$ is really concentrated on the lattice $\mathbb{G}_d$. The unconsidered case where $(S_n)_{n\ge 0}$ has d-arithmetic increments but non- or c-arithmetic delay for some $c \notin \mathbb{G}_d \cup \{\infty\}$ will not play any role in our subsequent analysis.

2.1.2 Making life easier: the standard model of a random walk

Quite common in the theory of Markov chains and also possible and useful here in various situations (as any RW is also a Markov chain with state space $\mathbb{R}$) is the use of a *standard model* for a RW $(S_n)_{n\ge 0}$ with a given increment distribution. In such a model we may vary the delay (initial) distribution by only changing the underlying probability measure and not the delay or the whole sequence itself.

Definition 2.1.4. Let $\mathscr{P}(\mathbb{R})$ be the set of all probability distributions on $\mathbb{R}$. We say that a RW $(S_n)_{n\ge 0}$ is given in a *standard model*

$$(\Omega, \mathfrak{A}, (\mathbb{P}_\lambda)_{\lambda \in \mathscr{P}(\mathbb{R})}, (S_n))$$

if each S_n is defined on $(\Omega,\mathfrak{A})$ and, for each $\mathbb{P}_\lambda$, has delay (initial) distribution λ and increment distribution F, that is $\mathbb{P}_\lambda(S_n \in \cdot) = \lambda * F^{*n}$ for each $n \in \mathbb{N}_0$. If $\lambda = \delta_x$, we also write $\mathbb{P}_x$ for $\mathbb{P}_\lambda$.

A standard model always exists: Let $\mathscr{B}(\mathbb{R})$ denote the Borel σ-field over $\mathbb{R}$ and fix any distribution F on $\mathbb{R}$. By choosing the coordinate space $\Omega := \mathbb{R}^{\mathbb{N}_0}$ with coordinate mappings $S_0, X_1, X_2, \ldots$, infinite product Borel σ-field $\mathfrak{A} = \mathscr{B}(\mathbb{R})^{\mathbb{N}_0}$ and product measure $\mathbb{P}_\lambda = \lambda \otimes F^{\mathbb{N}_0}$ for $\lambda \in \mathscr{P}(\mathbb{R})$, we see that $(S_n)_{n\geq 0}$ forms a RW with delay distribution λ and increment distribution F under each $\mathbb{P}_\lambda$ and is thus indeed given in a standard model, called *canonical* or *coordinate model*.

2.1.3 Classification of renewal processes: persistence vs. termination

Some applications of renewal theory lead to RP's having an increment distribution that puts mass on ∞. This has been encountered in Section 1.3 when defining the sequence of successive hitting times of a transient state of a finite Markov chain. From an abstract point of view, this means nothing but to consider sums of iid random variables taking values in the extended semigroup $\mathbb{R}_\geq \cup \{\infty\}$. Following FELLER[☞ [14, footnote on p. 115]], any distribution F on this set with $F(\{\infty\}) > 0$ and thus $\lim_{x\to\infty} F(x) < 1$ is called *defective*. A classification of RP's that accounts for this possibility is next.

Definition 2.1.5. A RP $(S_n)_{n\geq 0}$ with a.s. finite delay S_0 and increments $X_1, X_2, \ldots$ having distribution F and mean μ is called

– *proper* (or *persistent*, or *recurrent*) if F is nondefective, i.e. $F(\{\infty\}) = 0$.
– *terminating* (or *transient*) if F is defective.

In the proper case, $(S_n)_{n\geq 0}$ is further called

– *strongly persistent* (or *positive recurrent*) if $\mu < \infty$.
– *weakly persistent* (or *null recurrent*) if $\mu = \infty$.

Clearly, the terms "recurrent" and "transience" reflect the idea that any renewal process may be interpreted as sequence of occurrence epochs for a certain event which is then recurrent in the first, and transient in the second case. In the transient case, all S_n have a defective distribution and

$$\mathbb{P}(S_n < \infty) \;=\; \mathbb{P}(X_1 < \infty, \ldots, X_n < \infty) \;=\; \mathbb{P}(X_1 < \infty)^n \quad \text{for all } n \in \mathbb{N}.$$

Moreover, there is a *last renewal epoch*, defined as S_T with

$$T := \sup\{n \geq 0 : S_n < \infty\} \tag{2.1}$$

giving the number of finite renewal epochs. For the latter variable the distribution is easily determined.

Lemma 2.1.6. *Let $(S_n)_{n\geq 0}$ be a terminating renewal process with increment distribution F. Then the number of finite renewal epochs T has a geometric distribution with parameter $p := F(\{\infty\})$, that is, $\mathbb{P}(T = n) = p(1-p)^n$ for $n \in \mathbb{N}_0$.*

Proof. This follows immediately from $\mathbb{P}(T = 0) = \mathbb{P}(X_1 = \infty) = p$ and

$$\mathbb{P}(T = n) \;=\; \mathbb{P}(X_1 < \infty, \ldots, X_n < \infty, X_{n+1} = \infty) \;=\; (1-p)^n p$$

for all $n \in \mathbb{N}$. □

2.1.4 Random walk and renewal measure: the point process view

Although this is not a text on point processes, we will briefly adopt this viewpoint because it appears to be quite natural, in particular for a general definition of the renewal measure of a RW to be presented below. So let $(S_n)_{n\geq 0}$ be a RW given in a standard model and consider the associated *random counting measure* on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$

$$N \;:=\; \sum_{n\geq 0} \delta_{S_n}$$

which more explicitly means that

$$N(\omega, B) \;:=\; \sum_{n\geq 0} \delta_{S_n(\omega)}(B) \quad \text{for all } \omega \in \Omega \text{ and } B \in \mathscr{B}(\mathbb{R}).$$

By endowing the set $\mathscr{M}$ of counting measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with the smallest σ-field $\mathfrak{M}$ that renders measurability of all projection mappings

$$\pi_B : \mathscr{M} \to \mathbb{N}_0 \cup \{\infty\}, \quad \mu \mapsto \mu(B),$$

i.e. $\mathfrak{M} := \sigma(\pi_B : B \in \mathscr{B}(\mathbb{R}))$, we have that $N : (\Omega, \mathfrak{A}, \mathbb{P}_\lambda) \to (\mathscr{M}, \mathfrak{M})$ defines a measurable map and thus a random element in $\mathscr{M}$, called *point process*. For each $B \in \mathscr{B}(\mathbb{R})$, $N(B)$ is an ordinary random variable which counts the number of points S_n in the set B. Taking expectations we arrive at the so-called *intensity measure* of the point process N under $\mathbb{P}_\lambda$, namely

$$\mathbb{U}_\lambda(B) := \mathbb{E}_\lambda N(B) = \sum_{n\ge 0} \mathbb{P}_\lambda(S_n \in B) \quad \text{for } B \in \mathscr{B}(\mathbb{R}).$$

The following definition extends those already given in the Introduction to general RW's on the real line.

Definition 2.1.7. Given a RW $(S_n)_{n\ge 0}$ in a standard model, the intensity measure $\mathbb{U}_\lambda$ of the associated point process N under $\mathbb{P}_\lambda$ is called *renewal measure of* $(S_n)_{n\ge 0}$ *under* $\mathbb{P}_\lambda$ and

$$\mathbb{U}_\lambda(t) := \mathbb{U}_\lambda((-\infty,t]) \quad \text{for } t \in \mathbb{R}$$

the pertinent *renewal function*. If $\lambda = \delta_x$ for some $x \in \mathbb{R}$, we write $\mathbb{U}_x$ for $\mathbb{U}_{\delta_x}$. Finally, the stochastic process $(N(t))_{t\in\mathbb{R}}$ defined by $N(t) := N((-\infty,t])$ is called *renewal counting process of* $(S_n)_{n\ge 0}$.

A detail we should comment on at this point is the following: In all examples of the Introduction we have defined the renewal measure on the basis of the epochs S_n for $n \ge 1$, whereas here we also account for S_0 even if $S_0 = 0$. The reason is that both definitions have their advantages. On the other hand, the general definition above forces us from now on to clearly state that $\mathbb{U}$ is the renewal measure of $(S_n)_{n\ge 1}$ and not $(S_n)_{n\ge 0}$ in those instances where S_0 is not to be accounted for. Fortunately, this will only be necessary occasionally, one example being the Poisson process which by definition has $N(0) = 0$.

Owing to the fact that $\mathbb{P}_\lambda = \lambda * F^{*n}$ for each $n \ge 0$ and $\lambda \in \mathscr{P}(\mathbb{R})$, where F is as usual the distribution of the X_n and $F^{*0} := \delta_0$, we have that

$$\mathbb{U}_\lambda = \sum_{n\ge 0} \lambda * F^{*n} = \lambda * \sum_{n\ge 0} F^{*n} = \lambda * \mathbb{U}_0 \quad \text{for all } \lambda \in \mathscr{P}(\mathbb{R}). \tag{2.2}$$

Natural questions on $\mathbb{U}_\lambda$ to be investigated are:

(Q1) Is $\mathbb{U}_\lambda$ *locally finite*, i.e. $\mathbb{U}_\lambda(B) < \infty$ for all bounded $B \in \mathscr{B}(\mathbb{R})$?
(Q2) Is $\mathbb{U}_\lambda(t) < \infty$ for all $t \in \mathbb{R}$ if $S_n \to \infty$ a.s.?

For a RP $(S_n)_{n\ge 0}$, the two questions are equivalent as $\mathbb{U}_\lambda(t) = 0$ for $t < 0$, and we have already given a positive answer in Section 1.1 [☞ Prop. 1.1.1]. In the situation of a RW, it may therefore take by surprise that the answer to both questions is not positive in general.

2.2 Random walks and stopping times: the basic stuff

This section is devoted to some fundamental results on RW's and stopping times including the important concept of ladder variables that will allow us to study RW's with the help of embedded RP's. Since stopping often involves more than just a given RW $(S_n)_{n\ge 0}$, due to additionally observed processes or randomizations, it is reasonable to consider stopping times with respect to filtrations that are larger than the one generated by $(S_n)_{n\ge 0}$ itself.

2.2.1 Filtrations, stopping times and some fundamental results

In the following, let $(S_n)_{n\ge 0}$ be a RW in a standard model with increments $X_1, X_2, \ldots$ and increment distribution F. For convenience, it may take values in any $\mathbb{R}^d$, $d \ge 1$. We will use $\mathbb{P}$ for probabilities that do not depend on the distribution of S_0. Let further $(\mathscr{F}_n)_{n\ge 0}$ be a filtration such that

(F1) $(S_n)_{n\ge 0}$ is adapted to $(\mathscr{F}_n)_{n\ge 0}$, i.e., $\sigma(S_0, \ldots, S_n) \subset \mathscr{F}_n$ for all $n \in \mathbb{N}_0$.
(F2) $\mathscr{F}_n$ is independent of $(X_{n+k})_{k\ge 1}$ for each $n \in \mathbb{N}_0$.

Let also $\mathscr{F}_\infty$ be the smallest σ-field containing all $\mathscr{F}_n$. Condition (F2) ensures that $(S_n)_{n\ge 0}$ is a temporally homogeneous Markov chain with respect to $(\mathscr{F}_n)_{n\ge 0}$, viz.

$$\mathbb{P}(S_{n+1} \in B|\mathscr{F}_n) \;=\; \mathbb{P}(S_{n+1} \in B|S_n) \;=\; F(B - S_n) \quad \mathbb{P}_\lambda\text{-a.s.}$$

for all $n \in \mathbb{N}_0$, $\lambda \in \mathscr{P}(\mathbb{R}^d)$ and $B \in \mathscr{B}(\mathbb{R}^d)$. A more general, but in fact equivalent statement is that

$$\mathbb{P}((S_{n+k})_{k\ge 0} \in C|\mathscr{F}_n) \;=\; \mathbb{P}((S_{n+k})_{k\ge 0} \in C|S_n) \;=\; \mathbf{P}(S_n, C) \quad \mathbb{P}_\lambda\text{-a.s.}$$

for all $n \in \mathbb{N}_0$, $\lambda \in \mathscr{P}(\mathbb{R})$ and $C \in \mathscr{B}(\mathbb{R}^d)^{\mathbb{N}_0}$, where

$$\mathbf{P}(x, C) \;:=\; \mathbb{P}_x((S_k)_{k\ge 0} \in C) \;=\; \mathbb{P}_0((S_k)_{k\ge 0} \in C - x) \quad \text{for } x \in \mathbb{R}^d.$$

Let us recall that, if τ is any stopping time with respect to $(\mathscr{F}_n)_{n\ge 0}$, also called $(\mathscr{F}_n)$-*time* hereafter, then

$$\mathscr{F}_\tau \;=\; \{A \in \mathscr{F}_\infty : A \cap \{\tau \le n\} \in \mathscr{F}_n \text{ for all } n \in \mathbb{N}_0\},$$

and the random vector $(\tau, S_0, \ldots, S_\tau)\mathbf{1}_{\{\tau<\infty\}}$ is $\mathscr{F}_\tau$-measurable. The following basic result combines the strong Markov property and temporal homogeneity of $(S_n)_{n\ge 0}$ as a Markov chain with its additional *spatial homogeneity* owing to its iid increments.

Proposition 2.2.1. *Under the stated assumptions, let τ be a $(\mathscr{F}_n)$-time. Then, for all $\lambda \in \mathscr{P}(\mathbb{R}^d)$, the following equalities hold $\mathbb{P}_\lambda$-a.s. on $\{\tau < \infty\}$:*

$$\mathbb{P}((S_{\tau+n}-S_\tau)_{n\ge 0}\in\cdot\,|\mathscr{F}_\tau)=\mathbb{P}((S_n-S_0)_{n\ge 0}\in\cdot)=\mathbb{P}_0((S_n)_{n\ge 0}\in\cdot). \tag{2.3}$$
$$\mathbb{P}((X_{\tau+n})_{n\ge 1}\in\cdot\,|\mathscr{F}_\tau)=\mathbb{P}((X_n)_{n\ge 1}\in\cdot). \tag{2.4}$$

If $\mathbb{P}_\lambda(\tau<\infty)=1$*, then furthermore (under* $\mathbb{P}_\lambda$*)*

(a) $(S_{\tau+n}-S_\tau)_{n\ge 0}$ *and* $\mathscr{F}_\tau$ *are independent.*
(b) $(S_{\tau+n}-S_\tau)_{n\ge 0}\stackrel{d}{=}(S_n-S_0)_{n\ge 0}$.
(c) $X_{\tau+1},X_{\tau+2},\ldots$ *are iid with the same distribution as* X_1.

Proof. It suffices to prove (2.4) for which we pick any $k\in\mathbb{N}_0$, $n\in\mathbb{N}$, $B_1,\ldots,B_n\in\mathscr{B}(\mathbb{R}^d)$ and $A\in\mathscr{F}_\tau$. Using $A\cap\{\tau=k\}\in\mathscr{F}_k$ and (F2), it follows for each $\lambda\in\mathscr{P}(\mathbb{R}^d)$ that

$$\begin{aligned}&\mathbb{P}_\lambda(A\cap\{\tau=k,X_{k+1}\in B_1,\ldots,X_{k+n}\in B_n\})\\&\quad=\ \mathbb{P}_\lambda(A\cap\{\tau=k\})\,\mathbb{P}(X_{k+1}\in B_1,\ldots,X_{k+n}\in B_n)\\&\quad=\ \mathbb{P}_\lambda(A\cap\{\tau=k\})\,\mathbb{P}(X_1\in B_1,\ldots,X_n\in B_n),\end{aligned}$$

and this clearly yields the desired conclusion. □

Assuming $S_0=0$ hereafter, let us now turn to the concept of formally copying a stopping time τ for $(S_n)_{n\ge 1}$. The latter means that there exist $B_n\in\mathscr{B}(\mathbb{R}^{nd})$ for $n\ge 1$ such that

$$\tau\ =\ \inf\{n\ge 1:(S_1,\ldots,S_n)\in B_n\}, \tag{2.5}$$

where as usual $\inf\emptyset:=\infty$. With the help of the B_n we can copy this stopping rule to the *post-*τ *process* $(S_{\tau+n}-S_\tau)_{n\ge 1}$ if $\tau<\infty$. For this purpose put $S_{n,k}:=S_{n+k}-S_n$,

$$\begin{aligned}\mathbf{S}_{n,k}\ &:=\ (S_{n+1}-S_n,\ldots,S_{n+k}-S_n)\ =\ (S_{n,1},\ldots,S_{n,k})\quad\text{and}\\ \mathbf{X}_{n,k}\ &:=\ (X_{n+1},\ldots,X_{n+k})\end{aligned}$$

for $k\in\mathbb{N}$ and $n\in\mathbb{N}_0$.

Definition 2.2.2. Let τ be a stopping time for $(S_n)_{n\ge 1}$ as in (2.5). Then the sequences $(\tau_n)_{n\ge 1}$ and $(\sigma_n)_{n\ge 0}$, defined by $\sigma_0:=0$ and

$$\tau_n\ :=\ \begin{cases}\inf\{k\ge 1:\mathbf{S}_{\sigma_{n-1},k}\in B_k\}, & \text{if }\sigma_{n-1}<\infty\\ \infty, & \text{if }\sigma_{n-1}=\infty\end{cases}\qquad\text{and}\qquad \sigma_n\ :=\ \sum_{k=1}^n\tau_k$$

for $n\ge 1$ (thus $\tau_1=\tau$) are called the *sequence of formal copies of* τ and its associated *sequence of copy sums*, respectively.

The following theorem summarizes the most important properties of the τ_n, σ_n and $S_{\sigma_n}\mathbf{1}_{\{\sigma_n<\infty\}}$.

Theorem 2.2.3. *Given the previous notation, put further $\beta := \mathbb{P}(\tau < \infty)$ and $\mathbf{Z}_n := (\tau_n, \mathbf{X}_{\sigma_{n-1},\tau_n})$ for $n \in \mathbb{N}$. Then the following assertions hold true:*

(a) $\sigma_0, \sigma_1, \ldots$ *are stopping times for* $(S_n)_{n\ge 0}$.
(b) τ_n *is a stopping time with respect to* $(\mathscr{F}_{\sigma_{n-1}+k})_{k\ge 0}$ *and* $\mathscr{F}_{\sigma_{n-1}}$*-measurable for each* $n \in \mathbb{N}$.
(c) $\mathbb{P}(\tau_n \in \cdot|\mathscr{F}_{\sigma_{n-1}}) = \mathbb{P}(\tau < \infty)$ *a.s. on* $\{\sigma_{n-1} < \infty\}$ *for each* $n \in \mathbb{N}$.
(d) $\mathbb{P}(\tau_n < \infty) = \mathbb{P}(\sigma_n < \infty) = \beta^n$ *for all* $n \in \mathbb{N}$.
(e) $\mathbb{P}(\mathbf{Z}_n \in \cdot, \tau_n < \infty|\mathscr{F}_{\sigma_{n-1}}) = \mathbb{P}(\mathbf{Z}_1 \in \cdot, \tau_1 < \infty)$ *a.s. on* $\{\sigma_{n-1} < \infty\}$ *for all* $n \in \mathbb{N}$.
(f) *Given* $\sigma_n < \infty$*, the random vectors* $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ *are conditionally iid with the same distribution as* $\mathbf{Z}_1$ *conditioned upon* $\tau_1 < \infty$.
(g) *If* $G := \mathbb{P}((\tau, S_\tau) \in \cdot|\tau < \infty)$*, then* $\mathbb{P}((\sigma_n, S_{\sigma_n}) \in \cdot|\sigma_n < \infty) = G^{*n}$ *a.s. for all* $n \in \mathbb{N}$.

In the case where τ is a.s. finite ($\beta = 1$), this implies further:

(h) $\mathbf{Z}_n$ *and* $\mathscr{F}_{\sigma_{n-1}}$ *are independent for each* $n \in \mathbb{N}$.
(i) $\mathbf{Z}_1, \mathbf{Z}_2, \ldots$ *are iid.*
(j) $(\sigma_n, S_{\sigma_n})_{n\ge 0}$ *forms a zero-delayed RW taking values in* $\mathbb{N}_0 \times \mathbb{R}^d$.

Proof. The simple proof of (a) and (b) is left to the reader. Assertion (c) and (e) follow from (2.3) when observing that, on $\{\sigma_{n-1} < \infty\}$,

$$\tau_n \;=\; \sum_{k\ge 0} \mathbf{1}_{\{\tau_n>k\}} \;=\; \sum_{k\ge 0}\prod_{j=1}^{k} \mathbf{1}_{B_j^c}(\mathbf{S}_{\sigma_{n-1},j}) \quad \text{and} \quad \mathbf{Z}_n\,\mathbf{1}_{\{\tau_n<\infty\}}$$

are measurable functions of $(S_{\sigma_{n-1},k})_{k\ge 0}$. Since $\mathbb{P}(\tau_n < \infty) = \mathbb{P}(\tau_1 < \infty, \ldots, \tau_n < \infty)$, we infer (d) by an induction over n and use of (c). Another induction in combination with (d) gives assertion (f) once we have proved that

$$\begin{aligned}&\mathbb{P}((\mathbf{Z}_1, \ldots, \mathbf{Z}_n) \in A_n, \mathbf{Z}_{n+1} \in B, \sigma_{n+1} < \infty)\\ &\qquad = \mathbb{P}((\mathbf{Z}_1, \ldots, \mathbf{Z}_n) \in A_n, \sigma_n < \infty)\,\mathbb{P}(\mathbf{Z}_1 \in B, \tau < \infty)\end{aligned}$$

for all $n \in \mathbb{N}$ and A_n, B from the σ-fields obviously to be chosen here. But with the help of (e), this is inferred as follows:

$$\begin{aligned}
&\mathbb{P}((\mathbf{Z}_1,...,\mathbf{Z}_n)\in A_n,\mathbf{Z}_{n+1}\in B,\sigma_{n+1}<\infty)\\
&\quad=\ \mathbb{P}((\mathbf{Z}_1,...,\mathbf{Z}_n)\in A_n,\mathbf{Z}_{n+1}\in B,\sigma_n<\infty,\tau_{n+1}<\infty)\\
&\quad=\ \int_{\{(\mathbf{Z}_1,...,\mathbf{Z}_n)\in A_n,\sigma_n<\infty\}}\mathbb{P}(\mathbf{Z}_{n+1}\in B,\tau_{n+1}<\infty|\mathscr{F}_{\sigma_n})\,d\mathbb{P}\\
&\quad=\ \mathbb{P}((\mathbf{Z}_1,...,\mathbf{Z}_n)\in A_n,\sigma_n<\infty)\mathbb{P}(\mathbf{Z}_1\in B,\tau<\infty).
\end{aligned}$$

Assertion (g) is a direct consequence of (f), and the remaining assertion (h),(i) and (j) in the case $\beta=1$ are just the specializations of (e),(f) and (g) to this case. □

Notice that, for a finite Markov chain, any sequence $(T_n(i))_{n\ge 0}$ of successive return times to a recurrent state i forms a sequence of copy sums, namely the one associated with $T_1(i)$, and it provides us with a cyclic decomposition of the chain as demonstrated in Section 1.3. For a RW $(S_n)_{n\ge 0}$ even more is true owing to its temporal and spatial homogeneity, namely, that *every* finite stopping time τ and its associated sequence of copy sums leads to a cyclic decomposition. This allows us to analyze intrinsic features of the RW by choosing τ in an appropriate manner.

2.2.2 Wald's identities for stopped random walks

Returning to the situation where $(S_n)_{n\ge 0}$ is a *real-valued* zero-delayed random walk, the purpose of this section is to provide two very useful identities originally due to A. WALD [28] for the first and second moment of stopped sums S_τ for finite mean stopping times τ.

Theorem 2.2.4. [Wald's identity] *Let $(S_n)_{n\ge 0}$ be a zero-delayed RW adapted to a filtration $(\mathscr{F}_n)_{n\ge 0}$ satisfying (F1) and (F2). Let further τ be an a.s. finite $(\mathscr{F}_n)$-time and $\mu:=\mathbb{E}X_1$. Then*

$$\mathbb{E}S_\tau\ =\ \mu\,\mathbb{E}\tau$$

provided that either the X_n are a.s. nonnegative, or X_1 and τ both have finite mean.

Proof. Since τ is a.s. finite, we have that

$$S_\tau\ =\ \sum_{n\ge 1}X_n\mathbf{1}_{\{\tau\ge n\}}\quad\text{a.s.}\tag{2.6}$$

Observe that, by (F2), X_n and $\{\tau\ge n\}\in\mathscr{F}_{n-1}$ are independent for all $n\in\mathbb{N}$. Hence, if the X_n are nonnegative, we infer

$$\mathbb{E}S_\tau\ =\ \sum_{n\ge 1}\mathbb{E}X_n\mathbf{1}_{\{\tau\ge n\}}\ =\ \mathbb{E}X_1\sum_{n\ge 1}\mathbb{P}(\tau\ge n)\ =\ \mu\,\mathbb{E}\tau$$

as claimed. If both, X_1 and τ, have finite mean we see that so does S_τ. Consequently, under the latter assumption we get the assertion for real-valued X_n as well when observing that $S_\tau = \sum_{n=1}^{\tau} X_n^+ - \sum_{n=1}^{\tau} X_1^-$. □

A little more difficult, though based on the same formula (2.6) for S_τ, is the derivation of Wald's second identity for $\mathbb{E}(S_\tau - \mu\tau)^2$.

Theorem 2.2.5. [Wald's second identity] *Let $(S_n, \mathscr{F}_n)_{n\ge 0}$ be as in Theorem 2.2.4 and suppose that the X_n have finite variance σ^2. Then (with μ as before)*

$$\mathbb{E}(S_\tau - \mu\tau)^2 \;=\; \sigma^2\,\mathbb{E}\tau$$

for any $(\mathscr{F}_n)$-time τ with finite mean.

Proof. W.l.o.g. let $\mu = 0$. By (2.6), we have $S_{\tau\wedge n} = \sum_{k=1}^{n} X_k \mathbf{1}_{\{\tau\ge k\}}$ and therefore

$$S_{\tau\wedge n}^2 \;=\; \left(\sum_{k=1}^{n} X_k \mathbf{1}_{\{\tau\ge k\}}\right)^2 \;=\; \sum_{k=1}^{n} X_k^2 \mathbf{1}_{\{\tau\ge k\}} + \sum_{k=1}^{n} X_k S_{k-1} \mathbf{1}_{\{\tau\ge k\}}$$

for each $n \in \mathbb{N}$. Now use the independence of X_k and $S_{k-1}, \mathbf{1}_{\{\tau\ge k\}}$ together with $\mu = 0$ to infer $\mathbb{E}X_k S_{k-1} \mathbf{1}_{\{\tau\ge k\}} = 0$ and $\mathbb{E}X_k^2 \mathbf{1}_{\{\tau\ge k\}} = \sigma^2 \mathbb{P}(\tau \ge k)$ which provides us with

$$\mathbb{E}S_{\tau\wedge n}^2 \;=\; \sigma^2\,\mathbb{E}(\tau\wedge n) \quad \text{for all } n\in\mathbb{N}.$$

As $S_{\tau\wedge n} \to S_\tau$ a.s., it follows with Fatou's lemma that

$$\mathbb{E}S_\tau^2 \;\le\; \liminf_{n\to\infty} \mathbb{E}S_{\tau\wedge n}^2 \;=\; \sigma^2\mathbb{E}\tau \;<\; \infty. \tag{2.7}$$

This proves integrability of S_τ and hence uniform integrability of the Doob-type martingale $(\mathbb{E}(S_\tau^2|\mathscr{F}_{\tau\wedge n})_{n\ge 0}$. Moreover,

$$\begin{aligned}
\mathbb{E}(S_\tau^2|\mathscr{F}_{\tau\wedge n}) - S_{\tau\wedge n}^2 \;&=\; \mathbb{E}(S_\tau^2 - S_{\tau\wedge n}^2|\mathscr{F}_{\tau\wedge n})\\
&=\; \mathbb{E}((S_\tau - S_{\tau\wedge n})(S_\tau + S_{\tau\wedge n})|\mathscr{F}_{\tau\wedge n})\\
&=\; 2S_{\tau\wedge n}\,\mathbb{E}(S_\tau - S_{\tau\wedge n}|\mathscr{F}_{\tau\wedge n}) \;+\; \mathbb{E}((S_\tau - S_{\tau\wedge n})^2|\mathscr{F}_{\tau\wedge n})\\
&\ge\; 0 \quad \text{a.s.,}
\end{aligned}$$

because $\mathbb{E}(S_\tau - S_{\tau\wedge n}|\mathscr{F}_{\tau\wedge n}) = 0$ a.s. for all $\in \mathbb{N}$ [☞ Lemma A.1.3]. But this implies that $(S_{\tau\wedge n}^2)_{n\ge 0}$ is also uniformly integrable and thus equality must hold in (2.7). □

We remark for the interested reader that the proof just given differs slightly from the one usually found in the literature [☞ e.g. GUT [16]] which shows that $(S_{\tau\wedge n})_{n\ge 0}$ forms a Cauchy sequence in the space of square integrable random variables and thus converges in L_2 to S_τ. Let us further note that Wald-type identities for higher integral moments of S_τ may also be given. We refer to [6].

2.2.3 *Ladder variables, a fundamental trichotomy, and the Chung-Fuchs theorem*

We will now define the most prominent sequences of copy sums in the theory of RW's which are obtained by looking at the record epochs and record values of a RW $(S_n)_{n\geq 0}$, or its reflection $(-S_n)_{n\geq 0}$.

Definition 2.2.6. Given a zero-delayed RW $(S_n)_{n\geq 0}$, the stopping times

$$\sigma^> := \inf\{n \geq 1 : S_n > 0\}, \quad \sigma^\geq := \inf\{n \geq 1 : S_n \geq 0\},$$
$$\sigma^< := \inf\{n \geq 1 : S_n < 0\}, \quad \sigma^\leq := \inf\{n \geq 1 : S_n \leq 0\},$$

are called *first strictly ascending, weakly ascending, strictly descending* and *weakly descending ladder epoch*, respectively, and

$$S_1^> := S_{\sigma^>}\mathbf{1}_{\{\sigma^> < \infty\}}, \quad S_1^\geq := S_{\sigma^\geq}\mathbf{1}_{\{\sigma^\geq < \infty\}},$$
$$S_1^< := S_{\sigma^<}\mathbf{1}_{\{\sigma^< < \infty\}}, \quad S_1^\leq := S_{\sigma^\leq}\mathbf{1}_{\{\sigma^\leq < \infty\}}$$

their respective *ladder heights*. The associated sequences of copy sums $(\sigma_n^>)_{n\geq 0}$, $(\sigma_n^\geq)_{n\geq 0}$, $(\sigma_n^<)_{n\geq 0}$ and $(\sigma_n^\leq)_{n\geq 0}$ are called *sequences of strictly ascending, weakly ascending, strictly descending* and *weakly descending ladder epochs*, respectively, and

$$S_n^> := S_{\sigma_n^>}\mathbf{1}_{\{\sigma_n^> < \infty\}},\ n \geq 0, \quad S_n^\geq := S_{\sigma_n^\geq}\mathbf{1}_{\{\sigma_n^\geq < \infty\}},\ n \geq 0,$$
$$S_n^< := S_{\sigma_n^<}\mathbf{1}_{\{\sigma_n^< < \infty\}},\ n \geq 0, \quad S_n^\leq := S_{\sigma_n^\leq}\mathbf{1}_{\{\sigma_n^\leq < \infty\}},\ n \geq 0$$

the respective *sequences of ladder heights*.

Plainly, if $(S_n)_{n\geq 0}$ has nonnegative (positive) increments, then $\sigma_n^\geq = n$ ($\sigma_n^> = n$) for all $n \in \mathbb{N}$. Moreover, $\sigma_n^\geq = \sigma_n^>$ and $\sigma_n^\leq = \sigma_n^<$ a.s. for all $n \in \mathbb{N}$ in the case where the increment distribution is continuous, for then $\mathbb{P}(S_m = S_n) = 0$ for all $m, n \in \mathbb{N}$.

The following proposition provides some basic information on the ladder variables and is a consequence of the SLLN and Thm. 2.2.3.

Proposition 2.2.7. *Let $(S_n)_{n\geq 0}$ be a zero-delayed RW such that $\mathbb{P}(X_1 = 0) < 1$. Then the following assertions are equivalent:*

(a) $\left(\sigma_n^\alpha, S_{\sigma_n^\alpha}\right)_{n\geq 0}$ *is a zero-delayed RW taking values in $\mathbb{N}_0 \times \mathbb{R}$ for any $\alpha \in \{>, \geq\}$ (resp. $\{<, \leq\}$).*

(b) $\sigma^\alpha < \infty$ *a.s. for $\alpha \in \{>, \geq\}$ (resp. $\{<, \leq\}$).*

(c) $\limsup_{n\to\infty} S_n = \infty$ *a.s.* *(resp.* $\liminf_{n\to\infty} S_n = -\infty$ *a.s.)*

Proof. It clearly suffices to prove equivalence of the assertions not in parentheses. The implications "(a)⇒(b)" and "(c)⇒(b)" are trivial, while "(b)⇒(a)" follows from Thm. 2.2.3(j). This leaves us with a proof of "(a),(b)⇒(c)". But $\mathbb{E}S^> > 0$ in combination with the SLLN applied to $(S_n^>)_{n\ge 0}$ implies

$$\limsup_{n\to\infty} S_n \ \ge\ \lim_{n\to\infty} S_n^> = \infty \quad \text{a.s.}$$

and thus the assertion. □

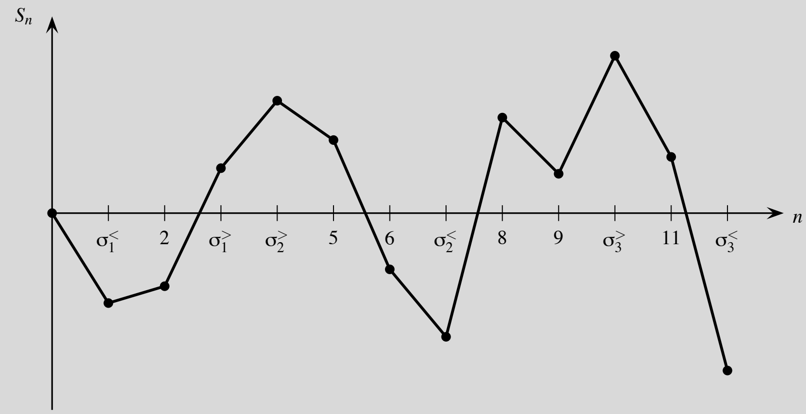

Fig. 2.1 Path of a RW with strictly ascending ladder epochs $\sigma_1^> = 3$, $\sigma_2^> = 4$ and $\sigma_3^> = 10$, and strictly descending ladder epochs $\sigma_1^< = 1$, $\sigma_2^< = 5$ and $\sigma_3^< = 12$.

If $\mathbb{E}X_1 > 0$ (resp. < 0) we thus have that $\sigma^>, \sigma^\ge$ (resp. $\sigma^<, \sigma^\le$) are a.s. finite whence the associated sequences of ladder epochs and ladder heights each constitute nondecreasing (resp. nonincreasing) zero-delayed RW's. Much deeper information, however, is provided by the next result disclosing a quite unexpected duality between ascending and descending ladder epochs that will enable us to derive a further classification of RW's as to their asymptotic behavior including the *Chung-Fuchs theorem* on the asymptotic behavior of a RW with drift zero.

Theorem 2.2.8. *Given a zero-delayed RW $(S_n)_{n\ge 0}$ with first ladder epochs $\sigma^{\ge},\sigma^{>},\sigma^{\le},\sigma^{<}$, the following assertions hold true:*

$$\mathbb{E}\sigma^{\ge} = \frac{1}{\mathbb{P}(\sigma^{<}=\infty)} \quad \textit{and} \quad \mathbb{E}\sigma^{>} = \frac{1}{\mathbb{P}(\sigma^{\le}=\infty)}, \tag{2.8}$$

$$\mathbb{P}(\sigma^{\le}=\infty) = (1-\kappa)\,\mathbb{P}(\sigma^{<}=\infty), \tag{2.9}$$

where

$$\kappa := \sum_{n\ge 1}\mathbb{P}(S_1>0,\ldots,S_{n-1}>0,S_n=0) = \sum_{n\ge 1}\mathbb{P}(\sigma^{\le}=n,S_1^{\le}=0).$$

Proof. Define

$$\begin{aligned} A_{n1} &= \{S_1\le S_2,\ldots,S_1\le S_n\},\\ A_{nk} &= \{S_1>S_k,\ldots,S_{k-1}>S_k,S_k\le S_{k+1},\ldots,S_k\le S_n\} \quad \text{for } 2\le k\le n-1\\ \text{and}\quad A_{nn} &= \{S_1>S_n,\ldots,S_{n-1}>S_n\}. \end{aligned}$$

Using $(S_k-S_i)_{1\le i<k}\stackrel{d}{=}(S_{k-i})_{1\le i<k}$ and $(S_j-S_k)_{k<j\le n}\stackrel{d}{=}(S_j)_{1\le j\le n-k}$ for each $1<k<n$, we then obtain

$$\begin{aligned} \mathbb{P}(A_{nk}) &= \mathbb{P}(S_k-S_i<0,1\le i<k,S_j-S_k\ge 0,k<j\le n)\\ &= \mathbb{P}(S_k-S_i<0,1\le i<k)\,\mathbb{P}(S_j-S_k\ge 0,k<j\le n)\\ &= \mathbb{P}(S_i<0,1\le i<k)\,\mathbb{P}(S_j\ge 0,1\le j\le n-k)\\ &= \mathbb{P}(\sigma^{\ge}\ge k)\,\mathbb{P}(\sigma^{<}>n-k), \end{aligned} \tag{2.10}$$

and a similar result for $k=1$ and $k=n$. It follows for all $n\ge 2$

$$1 = \sum_{k=1}^{n}\mathbb{P}(A_{nk}) = \sum_{k=1}^{n}\mathbb{P}(\sigma^{\ge}\ge k)\,\mathbb{P}(\sigma^{<}>n-k) \ge \mathbb{P}(\sigma^{<}=\infty)\sum_{k=1}^{n}\mathbb{P}(\sigma^{\ge}\ge k).$$

and thereupon by letting n tend to ∞ ($\infty\cdot 0=0\cdot\imath:=0$).

$$\mathbb{P}(\sigma^{<}=\infty)\,\mathbb{E}\sigma^{\ge} = \mathbb{P}(\sigma^{<}=\infty)\sum_{k\ge 1}\mathbb{P}(\sigma^{\ge}\ge k) \le 1.$$

Hence $\mathbb{P}(\sigma^{<}=\infty)>0$ entails $\mathbb{E}\sigma^{\ge}<\infty$.

Assuming conversely $\mathbb{E}\sigma^{\ge}<\infty$, we infer with the help of (2.10)

$$1 \le \sum_{k=1}^{m}\mathbb{P}(\sigma^{\ge}\ge k)\,\mathbb{P}(\sigma^{<}>n-k) + \sum_{k=m}^{n}\mathbb{P}(\sigma^{\ge}\ge k),$$

and then upon letting n tend to ∞

$$1 \le \mathbb{P}(\sigma^< = \infty) \sum_{k=1}^{m} \mathbb{P}(\sigma^\ge \ge k) + \sum_{k \ge m} \mathbb{P}(\sigma^\ge \ge k).$$

By finally taking the limit $m \to \infty$, we arrive at

$$1 \le \mathbb{P}(\sigma^< = \infty)\,\mathbb{E}\sigma^\ge$$

which proves the equivalence of $\mathbb{E}\sigma^\ge < \infty$ and $\mathbb{P}(\sigma^< = \infty) > 0$, and also the first part of (2.8). But the second part follows analogously when replacing each "$>$" with "$\ge$" and each "$\le$" with "$<$" in the definition of theA_{nk}. For the proof of (2.9), we note that

$$\begin{aligned}
\mathbb{P}(\sigma^< = \infty) - \mathbb{P}(\sigma^\le = \infty) &= \mathbb{P}(\sigma^< = \infty, \sigma^\le < \infty) \\
&= \sum_{n \ge 1} \mathbb{P}(S_1 > 0, ..., S_{n-1} > 0, S_n = 0, S_j \ge 0, j > n) \\
&= \sum_{n \ge 1} \mathbb{P}(S_1 > 0, ..., S_{n-1} > 0, S_n = 0)\,\mathbb{P}(S_j \ge 0, j \ge 1) \\
&= \mathbb{P}(\sigma^< = \infty) \sum_{n \ge 1} \mathbb{P}(S_1 > 0, ..., S_{n-1} > 0, S_n = 0) \\
&= \kappa\,\mathbb{P}(\sigma^< = \infty),
\end{aligned}$$

which obviously gives the desired result. □

By considering $(-S_n)_{n \ge 0}$ in the previous result, the relations (2.8) and (2.9) with the roles of $\sigma^>, \sigma^\le$ and $\sigma^<, \sigma^\ge$ are immediate consequences. This in turn justifies to call each of $(\sigma^>, \sigma^\le)$ and $(\sigma^<, \sigma^\ge)$ a *dual pair*.

The previous result also leads to the following trichotomy that is fundamental for a deeper analysis of RW's.

Theorem 2.2.9. *Let $(S_n)_{n \ge 0}$ be a zero-delayed RW such that $\mathbb{P}(X_1 = 0) < 1$. Then exactly one of the following three cases hold true:*

(i) $\sigma^\le, \sigma^<$ *are both defective and* $\mathbb{E}\sigma^\ge, \mathbb{E}\sigma^>$ *both finite.*
(ii) $\sigma^\ge, \sigma^>$ *are both defective and* $E\sigma^\le, E\sigma^<$ *both finite.*
(iii) $\sigma^\ge, \sigma^>, \sigma^\le, \sigma^<$ *are all a.s. finite with infinite expectation.*

In terms of the asymptotic behavior of S_n as $n \to \infty$, these three alternatives are characterized as follows:

(i) $\lim_{n\to\infty} S_n = \infty$ *a.s.*
(ii) $\lim_{n\to\infty} S_n = -\infty$ *a.s.*
(iii) $\liminf_{n\to\infty} S_n = -\infty$ *and* $\limsup_{n\to\infty} S_n = \infty$ *a.s.*

Finally, if $\mu := EX_1$ exists, thus $EX^+ < \infty$ or $EX^- < \infty$, then (i), (ii), and (iii) are equivalent to $\mu > 0$, $\mu < 0$, and $\mu = 0$, respectively.

Proof. Notice first that $\mathbb{P}(X_1 = 0) < 1$ is equivalent to $\kappa < 1$, whence (2.9) ensures that $\sigma^>, \sigma^\geq$ as well as $\sigma^<, \sigma^\leq$ are always defective simultaneously in which case the respective dual ladder epochs have finite expectation by (2.8). Hence, if neither (a) nor (b) holds true, the only remaining alternative is that all four ladder epochs are a.s. finite with infinite expectation. By combining the three alternatives for the ladder epochs just proved with Prop. 2.2.7, the respective characterizations of the behavior of S_n for $n \to \infty$ are immediate.

Suppose now that $\mu = \mathbb{E}X_1$ exists. In view of Prop. 2.2.7 it then only remains to verify that (iii) holds true in the case $\mu = 0$. But any of the alternatives (i) or (ii) would lead to the existence of a ladder epoch σ such that $\mathbb{E}\sigma < \infty$ and S_σ is a.s. positive or negative. On the hand, $\mathbb{E}S_\sigma = \mu\,\mathbb{E}\sigma = 0$ would follow by an appeal to Wald's identity 2.2.4 which is impossible. Hence $\mu = 0$ entails (iii). □

The previous result calls for a further definition that classifies a RW $(S_n)_{n\geq 0}$ with regard to the three alternatives (i), (ii) and (iii).

Definition 2.2.10. A RW $(S_n)_{n\geq 0}$ is called

- *positive divergent* if $\lim_{n\to\infty} S_n = \infty$ a.s.
- *negative divergent* if $\lim_{n\to\infty} S_n = -\infty$ a.s.
- *oscillating* if $\liminf_{n\to\infty} S_n = -\infty$ and $\limsup_{n\to\infty} S_n = \infty$ a.s.
- *trivial* if $X_1 = 0$ a.s.

While for nontrivial RW's with finite drift Theorem 2.2.9 provides a satisfactory answer in terms of a simple condition on the increment distribution as to when each of the three alternatives occurs, this is not possible for general RW's. The case of zero mean increments is the *Chung-Fuchs theorem* and stated as a corollary below. Another simple criterion for being in the oscillating case is obviously that the increment distribution is symmetric, for then $\sigma^>$ and $\sigma^<$ are identically distributed and hence both a.s. finite.

Corollary 2.2.11. (Chung-Fuchs theorem) *Any nontrivial RW $(S_n)_{n\geq 0}$ with drift zero is oscillating.*

2.3 Recurence and transience of random walks

Every nontrivial arithmetic RW $(S_n)_{n\geq 0}$ with lattice-span d is also a discrete Markov chain on $\mathbb{G}_d = d\mathbb{Z}$. It is therefore natural to ask under which condition on the increment distribution it also recurrent, i.e.

$$\mathbb{P}(S_n = x \text{ infinitely often}) = 1 \quad \text{for all } x \in \mathbb{G}_d. \tag{2.11}$$

Plainly, this is possible only in the oscillating case, but the question remains whether this is already sufficient. Moreover, the same question may be posed for the case $d = 0$, i.e. nonarithmetic $(S_n)_{n\geq 0}$, however, with the adjustment that condition (2.11) must weakened to

$$\mathbb{P}(|S_n - x| < \varepsilon \text{ infinitely often}) = 1 \quad \text{for all } x \in \mathbb{G}_d \text{ and } \varepsilon > 0, \tag{2.12}$$

because otherwise continuous RW's satisfying $\mathbb{P}(S_n = x) = 0$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ would be excluded right away. The more general condition (2.12) is called *topological recurrence* because it means that open neighborhoods of any $x \in \mathbb{G}_d$ are visited infinitely often. The purpose of this section, which follows the presentation in [4], is to show that any RW $(S_n)_{n\geq 0}$ satisfying $n^{-1}S_n \xrightarrow{\mathbb{P}} 0$ is topologically recurrent [☞ Thm. 2.3.5]. We start by defining topological recurrence for individual states.

Definition 2.3.1. Given a sequence $(S_n)_{n\geq 0}$ of real-valued random variables, a state $x \in \mathbb{R}$ is called *(topologically) recurrent* for $(S_n)_{n\geq 0}$ if

$$\mathbb{P}(|S_n - x| < \varepsilon \text{ infinitely often}) = 1 \quad \text{for all } \varepsilon > 0$$

and *transient* otherwise. If $\sup_{n\geq 0} \mathbb{P}(|S_n - x| < \varepsilon) > 0$ for all $\varepsilon > 0$, then the state x is called *possible*.

Plainly, every recurrent state is possible, and in the case where $(S_n)_{n\geq 0}$ is concentrated on a lattice $d\mathbb{Z}$ we may take $\varepsilon = 0$ in the definition of a recurrent state thus leading to the usual one also used for discrete Markov chains.

In the following, let $(S_n)_{n\geq 0}$ be a nontrivial zero-delayed RW,

$$\mathscr{R} := \{x \in \mathbb{R} : x \text{ is recurrent for } (S_n)_{n\geq 0}\}$$

its *recurrence set* and $\mathscr{E}$ its set of possible states. If $(S_n)_{n\geq 0}$ is d-arithmetic, then naturally $\mathscr{R} \subset \mathscr{E} \subset d\mathbb{Z}$ holds true. In fact, if $(S_n)_{n\geq 0}$ has lattice-span $d \geq 0$, then $\mathbb{G}_d$ is the smallest closed subgroup of $\mathbb{R}$ containing $\mathscr{E}$ as one can easily verify. Further information on the structure of $\mathscr{R}$ is provided by the next proposition.

Proposition 2.3.2. *The recurrence set $\mathscr{R}$ of a nontrivial zero-delayed RW $(S_n)_{n\ge 0}$ is either empty or a closed subgroup of $\mathbb{R}$. In the latter case, $\mathscr{R} = \mathbb{G}_d$ if $(S_n)_{n\ge 0}$ if $(S_n)_{n\ge 0}$ has lattice-span d, thus $\mathscr{R} = \mathbb{R}$ in the nonarithmetic case and $\mathscr{R} = d\mathbb{Z}$ in the d-arithmetic case.*

Proof. Suppose that $\mathscr{R} \neq \emptyset$. Let $(x_k)_{k\ge 1}$ be a sequence in $\mathscr{R}$ with limit x and pick $m \in \mathbb{N}$ such that $|x_m - x| < \varepsilon$ for any fixed $\varepsilon > 0$. Then

$$\mathbb{P}(|S_n - x| < 2\varepsilon \text{ infinitely often}) \ \ge\ \mathbb{P}|S_n - x_m| < \varepsilon \text{ infinitely often}) \ =\ 1.$$

Hence $x \in \mathscr{R}$, i.e., $\mathscr{R}$ is closed. The next step is to verify that $x - y \in \mathscr{R}$ whenever $x \in \mathscr{R}$ and $y \in \mathscr{E}$. To this end, fix any $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that $\mathbb{P}(|S_m - y| < \varepsilon) > 0$. Then

$$\begin{aligned}
&\mathbb{P}(|S_m - y| < \varepsilon)\,\mathbb{P}(|S_n - (x-y)| < 2\varepsilon \text{ finitely often})\\
&\quad=\ \mathbb{P}(|S_m - y| < \varepsilon, |S_{m+n} - S_m - (x-y)| < \varepsilon \text{ finitely often})\\
&\quad\le\ \mathbb{P}(|S_n - x| < \varepsilon \text{ finitely often})\ =\ 0
\end{aligned}$$

showing $\mathbb{P}(|S_n - (x-y)| < 2\varepsilon \text{ finitely often}) = 0$ and thus $x - y \in \mathscr{R}$. In particluar, $x - y \in \mathscr{R}$ for $x, y \in \mathscr{R}$ so that $\mathscr{R}$ is a closed additive subgroup of $\mathbb{R}$. Moreover, $\mathscr{R} = \mathscr{E}$, for $0 \in \mathscr{R}, x \in \mathscr{E}$ implies $-x = 0 - x \in \mathscr{R}$ and thus $x \in \mathscr{R}$. By what has been pointed out before this proposition, we finally conclude $\mathscr{R} = \mathscr{E} = \mathbb{G}_d$ if d is the lattice-span of $(S_n)_{n\ge 0}$. ☐

Although Prop. 2.3.2 provides complete information about the structure of the recurrence set $\mathscr{R}$ of a RW $(S_n)_{n\ge 0}$, the more difficult problem of finding sufficient conditions ensuring $\mathscr{R} \neq \emptyset$ remains. A crucial step in this direction is to relate the problem to the behavior of the renewal measure $\mathbb{U}$ of $(S_n)_{n\ge 0}$ which in the present context should be viewed as an occupation measure. Notice that $x \in \mathscr{R}$ implies $N(I) = \sum_{n\ge 0} \mathbf{1}_I(S_n) = \infty$, in particular $\mathbb{U}(I) = \mathbb{E}N(I) = \infty$ for any open interval I containing x. That the converse is also true, will be shown next.

Proposition 2.3.3. *Let $(S_n)_{n\ge 0}$ be a nontrivial zero-delayed RW. If there exists an open interval I such that*

$$0\ <\ \sum_{n\ge 0} \mathbb{P}(S_n \in I)\ =\ \mathbb{U}(I)\ <\ \infty,$$

then $(S_n)_{n\ge 0}$ is transient, that is $\mathscr{R} = \emptyset$. Conversely, if $\mathbb{U}(I) = \infty$ for some bounded interval I, then $(S_n)_{n\ge 0}$ is recurrent.

Proof. Clearly, $\mathbb{U}(I) > 0$ implies $I \neq \emptyset$, while $\mathbb{U}(I) < \infty$ in combination with the Borel-Cantelli lemma implies $\mathbb{P}(S_n \in I \text{ infintely often}) = 0$ and thus $I \subset \mathscr{R}^c$ as I is open. Consequently, $\mathscr{R} = \emptyset$ if the RW is nonarithmetic. In the d-arithmetic case, the same follows from $I \cap d\mathbb{Z} \neq \emptyset$ which in turn is a consequence of $\mathbb{U}(I) > 0$.

Now suppose that $\mathbb{U}(I) = \infty$ for some bounded and w.l.o.g. open interval I. It suffices to prove $0 \in \mathscr{R}$. Fix an arbitrary $\varepsilon > 0$. Since I may be covered by finitely many intervals of length 2ε, we have $\mathbb{U}(J) = \infty$ for some $J = (x-\varepsilon, x+\varepsilon)$. Let

$$T(J) \ := \ \sup\{n \geq 1 : S_n \in J\} \quad [\sup \emptyset := 0].$$

denote the last time ≥ 1 where the RW visits J and put $A_n := \{T(J) = n\}$ for $n \in \mathbb{N}_0$. It follows that

$$\{S_n \in J \text{ finitely often}\} \ = \ \{T(J) < \infty\} \ = \ \sum_{k \geq 0} A_k$$

and, furthermore,

$$A_k \ \supset \ \{S_k \in J, |S_{k+n} - S_k| \geq 2\varepsilon \text{ for all } n \geq 1\} \quad \text{for all } k \in \mathbb{N}_0.$$

Consequently, by using the independence of S_k and $(S_{k+n} - S_k)_{n \geq 0}$ in combination with $(S_{k+n} - S_k)_{n \geq 0} \stackrel{d}{=} (S_n)_{n \geq 0}$, we infer

$$\mathbb{P}(A_k) \ \geq \ \mathbb{P}(S_k \in J)\, \mathbb{P}(|S_n| \geq 2\varepsilon \text{ for all } n \geq 1) \quad \text{for all } k \in \mathbb{N}_0$$

and thereby via summation over $k \geq 0$

$$\mathbb{P}(S_n \in J \text{ finitely often}) \ \geq \ \mathbb{P}(|S_n| \geq 2\varepsilon \text{ for all } n \geq 1)\, \mathbb{U}(J)$$

which shows

$$\mathbb{P}(|S_n| \geq 2\varepsilon \text{ for all } n \geq 1) = 0, \tag{2.13}$$

because $\mathbb{U}(J) = \infty$. Notice that this holds for any $\varepsilon > 0$.

Defining $\widehat{J}_\delta := (-\delta, \delta)$, $\widehat{J} := \widehat{J}_\varepsilon$ and $\widehat{A}_n := \{T(\widehat{J}) = n\}$ for $n \in \mathbb{N}_0$, we have

$$\mathbb{P}(\widehat{A}_k) \ = \ \lim_{\delta \uparrow \varepsilon} \mathbb{P}(S_k \in \widehat{J}_\delta, S_{k+n} \notin \widehat{J} \text{ for all } n \geq 1) \quad \text{for all } k \in \mathbb{N}.$$

Now, (2.13) implies

$$\mathbb{P}(\widehat{A}_0) \ = \ \mathbb{P}(S_n \notin \widehat{J} \text{ for all } n \geq 1) \ = \ 0$$

and for $k \geq 1$, $\delta < \varepsilon$

$$\begin{aligned} &\mathbb{P}(S_k \in \widehat{J}_\delta, S_{k+n} \notin \widehat{J} \text{ for all } n \geq 1) \\ &\quad \leq \ \mathbb{P}(S_k \in \widehat{J}_\delta, |S_{k+n} - S_k| \geq \varepsilon - \delta \text{ for all } n \geq 1) \\ &\quad \leq \ \mathbb{P}(S_k \in \widehat{J}_\delta)\, \mathbb{P}(|S_{k+n} - S_k| \geq \varepsilon - \delta \text{ for all } n \geq 1) \ = \ 0 \end{aligned}$$

so that $\mathbb{P}(\widehat{A}_k) = 0$ for all $k \in \mathbb{N}_0$. We have thus proved that

$$\mathbb{P}(|S_n| < \varepsilon \text{ finitely often}) \;=\; \mathbb{P}(T(\widehat{J}) = \infty) \;=\; \sum_{k\geq 0} \mathbb{P}(\widehat{A}_k) \;=\; 0$$

for all $\varepsilon > 0$, that is $0 \in \mathscr{R}$. □

The following equivalences are direct consequences of the result just shown.

Corollary 2.3.4. *Given a nontrivial zero-delayed RW $(S_n)_{n\geq 0}$, the following assertions are equivalent:*

(a) $(S_n)_{n\geq 0}$ *is recurrent.*
(b) $N(I) = \infty$ *a.s. for all open intervals I such that* $I \cap \mathfrak{R} \neq \emptyset$.
(c) $\mathbb{U}(I) = \infty$ *for all open intervals I such that* $I \cap \mathfrak{R} \neq \emptyset$.
(d) $\mathbb{U}(I) = \infty$ *for some finite interval I.*

By contraposition, equivalence of

(e) $(S_n)_{n\geq 0}$ *is transient.*
(f) $N(I) < \infty$ *P-f.s. for all finite intervals I.*
(g) $\mathbb{U}(I) < \infty$ *for all finite intervals I.*
(h) $0 < \mathbb{U}(I) < \infty$ *for some finite open interval I.*

holds true.

We are now ready to prove the main result of this section.

Theorem 2.3.5. *Any nontrivial zero-delayed RW $(S_n)_{n\geq 0}$ which satisfies the WLLN $n^{-1}S_n \xrightarrow{\mathbb{P}} 0$ or, a fortiori, has drift zero is recurrent.*

Proof. In view of the previous result it suffices to show $\mathbb{U}([-1,1]) = \infty$. Define $\tau(x) := \inf\{n \geq 0 : S_n \in [x, x+1]\}$ for $x \in \mathbb{R}$. Since τ is a stopping time for $(S_n)_{n\geq 0}$, we infer with the help of Prop. 2.2.1 that

$$\begin{aligned}
\mathbb{U}([x,x+1]) \;&=\; \int_{\{\tau(x)<\infty\}} \sum_{n\geq 0} \mathbf{1}_{[x,x+1]}(S_{\tau+n})\, d\mathbb{P} \\
&\leq\; \int_{\{\tau(x)<\infty\}} \sum_{n\geq 0} \mathbf{1}_{[-1,1]}(S_{\tau+n} - S_\tau)\, d\mathbb{P} \\
&=\; \mathbb{P}(\tau(x) < \infty) \sum_{n\geq 0} \mathbb{P}(S_n \in [-1,1]) \\
&=\; \mathbb{P}(\tau(x) < \infty)\, \mathbb{U}([-1,1]).
\end{aligned} \tag{2.14}$$

Therefore

$$\mathbb{U}([-n,n]) \;\le\; \sum_{k=-n}^{n-1} \mathbb{U}([k,k+1]) \;\le\; 2n\,\mathbb{U}([-1,1]) \quad \text{for all } n \in \mathbb{N}. \tag{2.15}$$

Fix now any $\varepsilon > 0$ and choose $m \in \mathbb{N}$ so large that $\mathbb{P}(|S_k| \le \varepsilon k) \ge \frac{1}{2}$ for all $k > m$, which is possible by our assumption $n^{-1}S_n \xrightarrow{\mathbb{P}} 0$. As a consequence,

$$\mathbb{P}(|S_k| \le n) \;\ge\; \frac{1}{2} \quad \text{for all } m < k \le \frac{n}{\varepsilon}$$

and therefore

$$\mathbb{U}([-n,n]) \;\ge \sum_{m<k\le n/\varepsilon} \mathbb{P}(|S_k| \le n) \;\ge\; \frac{1}{2}\left(\frac{n}{\varepsilon} - m - 1\right).$$

This yields in combination with (2.15)

$$\mathbb{U}([-1,1]) \;\ge\; \frac{1}{2n}\,\mathbb{U}([-n,n]) \;\ge\; \frac{1}{4\varepsilon} - \frac{m+1}{4n} \quad \text{for all } n \in \mathbb{N}$$

so that, by letting n tend to infinity, we finally obtain

$$\mathbb{U}([-1,1]) \;\ge\; \limsup_{n\to\infty} \frac{1}{2n}\,\mathbb{U}([-n,n]) \;\ge\; \frac{1}{4\varepsilon}$$

which gives the desired conclusion, for $\varepsilon > 0$ was arbitrary. □

Remark 2.3.6. (a) Since any recurrent RW is oscillating, the previous result particularly provides an extension of the Chung-Fuchs theorem in that $\mathbb{E}X_1 = 0$ may be replaced with the weaker condition $n^{-1}S_n \xrightarrow{\mathbb{P}} 0$.

(b) All previous results are easily extended to the case of arbitrary delay distribution λ as long as $\lambda(\mathbb{G}_d) = 1$ in the d-arithmetic case. In fact, assuming a standard model $(\Omega, \mathfrak{A}, (\mathbb{P}_\lambda)_{\lambda\in\mathscr{P}(\mathbb{R})}, (S_n)_{n\ge 0})$ and letting $\mathscr{R}(\lambda)$ denote the recurrence set of $(S_n)_{n\ge 0}$ under $\mathbb{P}_\lambda$, it suffices to observe for $\mathscr{R}(\lambda) = \mathscr{R}(\delta_0)$ that

$$\mathbb{P}_\lambda(|S_n - x| < \varepsilon \text{ infinitely often}) = \int_{\mathbb{G}_d} \mathbb{P}_0(|S_n - (x-y)| < \varepsilon \text{ infinitely often})\;\lambda(dy)$$

holds true if $\lambda(\mathbb{G}_d) = 1$. For a transient RW the local finiteness of the renewal measure $\mathbb{U}_\lambda$ as stated in Cor. 2.3.4 for $\lambda = \delta_0$ even extends to all initial distributions λ as is readily seen.

(c) In connection with recurrence of RW's as Markov chains it is worth mentioning that Lebesgue measure $\lambda\!\!\lambda_0$ in the nonarithmetic case and d times counting measure $\lambda\!\!\lambda_d$ on $\mathbb{G}_d$ in the d-arithmetic case provides a *stationary measure* for a RW $(S_n)_{n\ge 0}$, no matter whether it is recurrent or not. This follows from

$$\begin{aligned}\mathbb{P}_{\lambda_d}(S_n \in B) \ &:= \int_{\mathbb{G}_d} \mathbb{P}_x(S_n \in B)\ \lambda\!\!\lambda_d(dx)\\ &= \int \mathbb{P}_0(S_n \in B - x)\ \lambda\!\!\lambda_d(ds)\\ &= \iint \mathbf{1}_B(x+s)\ \lambda\!\!\lambda_d(dx)\ \mathbb{P}_0(S_n \in ds)\\ &= \int \lambda\!\!\lambda_d(B-s)\ \mathbb{P}_0(S_n \in ds)\ =\ \lambda\!\!\lambda_d(B) \quad \text{for all } B \in \mathscr{B}(\mathbb{R})\end{aligned}$$

where the translation invariance of $\lambda\!\!\lambda_d$ on $\mathbb{G}_d$ has been utilized.

(d) The condition $n^{-1}S_n \xrightarrow{\mathbb{P}} 0$ in Thm. 2.3.5 is not sharp for the recurrence. Independent work by ORNSTEIN [24] and STONE [26] showed that, if $\Re(z)$ denotes the real part of a complex number z and φ is the Fourier transform of the increments of the RW, then

$$\int_{-\varepsilon}^{\varepsilon} \Re\left(\frac{1}{1-\varphi(t)}\right) dt \ = \ \text{ı}$$

provides a necessary and sufficient condition for the recurrence of $(S_n)_{n\geq 0}$. We will return to this in ????????

(e) In view of the fact that any recurrent RW is oscillating, it might be tempting to believe that these two properties are actually equivalent. However, this holds true only if $\mathbb{E}X_1$ exists [☞ final part of Thm. 2.2.9]. In the case where $\mathbb{E}X_1^- = \mathbb{E}X_1^+ = \infty$ the RW may be oscillating , so that every finite interval is crossed infinitely often, and yet be transient. KESTEN [17] further showed the following trichotomy which is stated here without proof.

Theorem 2.3.7. (Kesten) *Given a RW $(S_n)_{n\geq 0}$ with $\mathbb{E}X_1^- = \mathbb{E}X_1^+ = \infty$, exactly one of the following three cases holds true:*

(a) $\lim_{n\to\infty} n^{-1}S_n = \infty$ *a.s.*
(b) $\lim_{n\to\infty} n^{-1}S_n = -\infty$ *a.s.*
(c) $\liminf_{n\to\infty} n^{-1}S_n = -\infty$ *and* $\limsup_{n\to\infty} n^{-1}S_n = \infty$ *a.s.*

2.4 The renewal measure in the transient case: cyclic decompositions and basic properties

As shown in the previous section, the renewal measure $\mathbb{U}_\lambda$ of a transient RW $(S_n)_{n\geq 0}$ is locally finite for any initial distribution λ [☞ Cor. 2.3.4 and Rem. 2.3.6(b)]. This section is devoted to further investigations in the transient case that are also in preparation of one of the principal results in renewal theory, viz. Blackwell's renewal theorem to be derived in the next chapter.

2.4.1 Uniform local boundedness

We have seen in the Introduction that the renewal measure of a RW with exponential increments with mean μ equals exactly μ^{-1} times Lebesgue measure if the initial distribution is the same as for the increments. In view of this result and also intuitively, it is quite plausibel that for a general RW with drift μ and renewal measure $\mathbb{U}_\lambda$ the behavior of the renewal measure should at least be of a similar kind in the sense that

$$\mathbb{U}_\lambda(I) \approx \frac{\lambda\!\!\lambda_0(I)}{\mu}$$

for any bounded interval I and any initial distribution λ. What is indeed shown next is *uniform local boundedness* of $\mathbb{U}_\lambda$.

Lemma 2.4.1. *Let $(S_n)_{n\geq 0}$ be a transient RW in a standard model. Then*

$$\sup_{t\in\mathbb{R}} \mathbb{P}_\lambda(N([t,t+a]) \geq n) \leq \mathbb{P}_0(N([-a,a]) \geq n) \tag{2.16}$$

for all $a > 0$, $n \in \mathbb{N}_0$ and $\lambda \in \mathscr{P}(\mathbb{R})$. In particular,

$$\sup_{t\in\mathbb{R}} \mathbb{U}_\lambda([t,t+a]) \leq \mathbb{U}_0([-a,a]) \tag{2.17}$$

and $\{N([t,t+a]) : t \in \mathbb{R}\}$ is uniformly integrable under each $\mathbb{P}_\lambda$ for all $a > 0$.

Proof. If (2.16) holds true, then the uniform integrability of $\{N([t,t+a]) : t \in \mathbb{R}\}$ is a direct consequence, while (2.17) follows by summation over n. So (2.16) is the only assertion to be proved. Fix $t \in \mathbb{R}$, $a > 0$, and define $\tau := \inf\{n \geq 0 : S_n \in [t,t+a]\}$. Then

$$N([t,t+a]) = \begin{cases} \sum_{k\geq 0} \mathbf{1}_{[t,t+a]}(S_{\tau+k}), & \text{if } \tau < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

The desired estimate now follows by a similar argument as in (2.14), viz.

$$\begin{aligned} \mathbb{P}_\lambda(N([t,t+a]) \geq n) &= \mathbb{P}_\lambda\left(\tau < \infty, \sum_{k\geq 0} \mathbf{1}_{[t,t+a]}(S_{\tau+k}) \geq n\right) \\ &\leq \mathbb{P}_\lambda\left(\tau < \infty, \sum_{k\geq 0} \mathbf{1}_{[-a,a]}(S_{\tau+k} - S_\tau) \geq n\right) \\ &= \mathbb{P}_\lambda(\tau < \infty)\,\mathbb{P}_0(N([-a,a]) \geq n) \end{aligned}$$

for all $n \in \mathbb{N}_0$ and $\lambda \in \mathscr{P}(\mathbb{R})$. ◻

2.4.2 A useful connection with first passage times

In the case of a RP $(S_n)_{n\ge 0}$, there is a very useful connection between its renewal function $t\mapsto \mathbb{U}_\lambda(t)$ and the *first passage times*

$$\tau(t) \;:=\; \inf\{n\ge 0: S_n>t\}\quad \text{for } t\in\mathbb{R}_\ge.$$

These are clearly stopping times for $(S_n)_{n\ge 0}$, arise in many applications and will be studied in greater detail in ???????? Furthermore, as $\{\tau(t)=n\}=\{S_{n-1}\le t<S_n\}$ for each $n\in\mathbb{N}_0$ and $t\in\mathbb{R}_\ge$ (put $S_{-1}:=0$), we see that

$$\tau(t) \;=\; \sum_{n\ge 0}\mathbf{1}_{[0,t]}(S_n) \;=\; N(t)\quad \text{for all } t\in\mathbb{R}_\ge, \tag{2.18}$$

that is, $(\tau(t))_{t\ge 0}$ also equals the renewal counting process associated with $(S_n)_{n\ge 0}$.

Lemma 2.4.2. *Let $(S_n)_{n\ge 0}$ be a RP in a standard model. Then its renewal function satisfies*

$$\mathbb{U}_\lambda(t) \;=\; \mathbb{E}_\lambda\tau(t)\quad \textit{for all } t\in\mathbb{R}_{\ge 0} \textit{ and } \lambda\in\mathscr{P}(\mathbb{R}_\ge).$$

Proof. The crucial observation is that a RP has a.s. nondecreasing trajectories, for this implies

$$\{\tau(t)>n\} \;=\; \left\{\max_{0\le k\le n} S_k\le t\right\} \;=\; \{S_n\le t\}$$

and then further

$$\mathbb{E}_\lambda\tau(t) \;=\; \sum_{n\ge 0}\mathbb{P}_\lambda(\tau(t)>n) \;=\; \sum_{n\ge 0}\mathbb{P}_\lambda(S_n\le t) \;=\; \mathbb{U}_\lambda(t)$$

for all $t\in\mathbb{R}_\ge$ and $\lambda\in\mathscr{P}(\mathbb{R}_\ge)$ as claimed. □

Provided further that $(S_n)_{n\ge 0}$ has finite drift μ, the previous result may be combined with Wald's identity 2.2.4 to infer

$$\infty \;>\; \mathbb{U}_0(t) \;=\; \mathbb{E}_0\tau(t) \;=\; \frac{1}{\mu}\,\mathbb{E}_0 S_{\tau(t)} \;=\; \frac{t}{\mu} \;+\; \frac{\mathbb{E}_0(S_{\tau(t)}-t)}{\mu} \tag{2.19}$$

for all $t\in\mathbb{R}_\ge$, the finiteness of $\mathbb{U}_0(t)$ following from Prop. 1.1.1 or Cor. 2.3.4. The nonnegative random variable

$$R(t):=\; S_{\tau(t)}-t\quad \text{for } t\in\mathbb{R}_\ge$$

provides the amount by which the RP exceeds the level t at the first passage epoch $\tau(t)$ [☞ Fig. 2.2]. It plays an important role in renewal theory and has been given

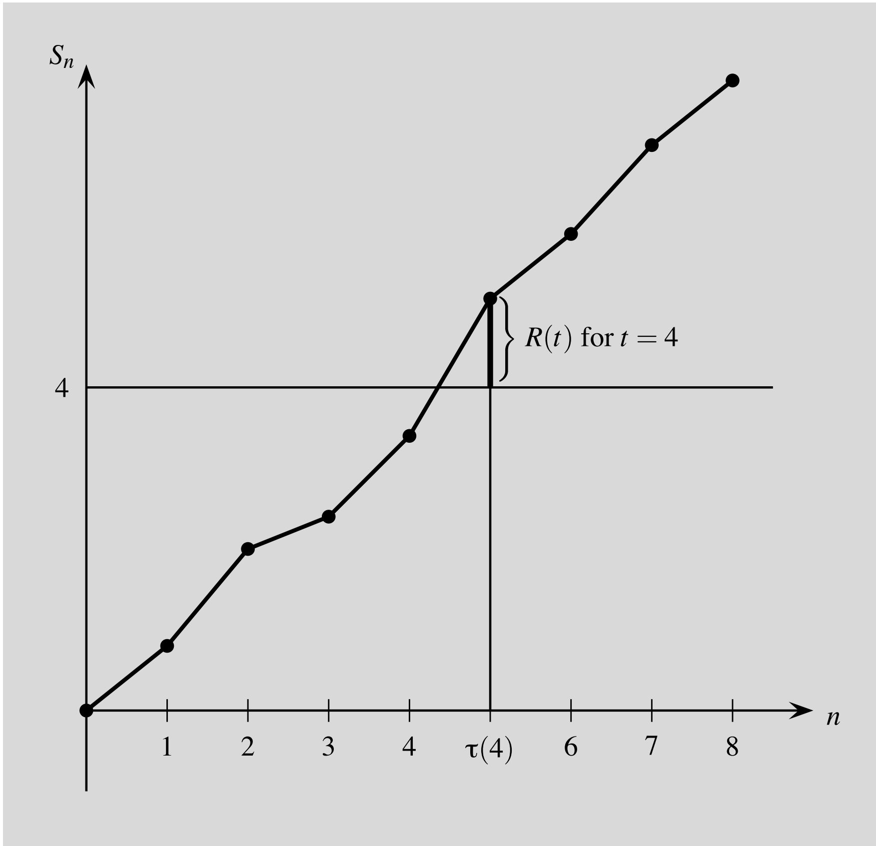

Fig. 2.2 Path of a RP with first pasage time $\tau(4) = 5$ and associated overshoot $R(4)$.

a number of different names, depending on the context in which it is discussed: *overshoot, excess (over the boundary), forward recurrence time*, or *residual waiting time*. Clearly, the first two names express the afore-mentioned exceedance property of $R(t)$, whereas the last two names refer to its meaning in a renewal scheme of being the residual time at t one has to wait until the next renewal will occur. The crucial question with regard to (2.19) is now about the asymptotic behavior of $R(t)$ under $\mathbb{P}_0$ and its expectation $\mathbb{E}_0 R(t)$ as $t \to \infty$. Since $R(t) \le X_{\tau(t)}$ and the X_n are identically distributed with finite mean, it seems plausible that $R(t)$ converges in distribution and that $\sup_{t \ge 0} \mathbb{E}_0 R(t)$ is finite. We will see already soon that the first assertion is indeed correct (modulo an adjustment in the arithmetic case), whereas the second one requires the nonobvious additional condition that the X_n have finite variances.

For general $\lambda \in \mathscr{P}(\mathbb{R}_\ge)$ use $\mathbb{U}_\lambda = \lambda * \mathbb{U}_0$ [☞ (2.2)] to obtain

$$\begin{aligned}\mathbb{U}_\lambda(t) \ &= \ \int_{[0,t]} \mathbb{U}_0(t-x)\,\lambda(dx) \ = \ \int_{[0,t]} \mathbb{E}_0\tau(t-x)\,\lambda(dx) \\ &= \ \frac{1}{\mu}\int_{[0,t]} (t-x+\mathbb{E}_0 R(t-x))\ \lambda(dx)\end{aligned} \tag{2.20}$$

A combination of (2.19) and (2.20) now leads to the following result, called *elementary renewal theorem*, which complements Prop. 1.1.2 in Section 1.1 and completes the answer to question (Q2) posed there.

Theorem 2.4.3. (Elementary renewal theorem) *Let $(S_n)_{n\ge 0}$ be a RP in a standard model. Then*

$$\lim_{t\to\infty} \frac{\mathbb{U}_\lambda(t)}{t} \ = \ \frac{1}{\mu} \tag{2.21}$$

$$\textit{and} \quad \lim_{t\to\infty} \frac{\tau(t)}{t} \ = \ \frac{1}{\mu} \quad \mathbb{P}_\lambda\textit{-a.s.} \tag{2.22}$$

for all $\lambda \in \mathscr{P}(\mathbb{R}_\ge)$.

Strictly speaking, the elementary renewal theorem is just (2.21), but we have included (2.22) because it can be viewed as its pathwise analog and also provides the extension of Prop. 1.1.2 to arbitrary initial distributions λ when recalling from (2.18) that $\tau(t) = N(t)$ for all $t \in \mathbb{R}_\ge$.

Proof. Since $(S_n)_{n\ge 0}$ satisfies the SLLN under each $\mathbb{P}_\lambda$ and $S_{\tau(t)-1} \le t < S_{\tau(t)}$, Doob's argument used in the proof of Prop. 1.1.2 works here as well to give (2.22). Left with the proof of (2.21), we infer from (2.20) with the help of Fatou's lemma

$$\liminf_{t\to\infty} \frac{\mathbb{U}_\lambda(t)}{t} \ \ge \ \frac{1}{\mu}\int \liminf_{t\to\infty}\left(\frac{t-x}{t}\mathbf{1}_{[0,t]}(x)\right)\ \lambda(dx) \ = \ \frac{1}{\mu}.$$

for any $\lambda \in \mathscr{P}(\mathbb{R}_\ge)$. For the reverse inequality we must work a little harder and consider the RP $(S_{c,n})_{n\ge 0}$ with truncated increments $X_n \wedge c$ for some $c > 0$. Denote by $\tau_c(t)$ the corresponding first passage time and note that $S_{c,n} \le S_n$ for all $n \in \mathbb{N}_0$ implies $\tau(t) \le \tau_c(t)$ and that $R_c(t) := S_{c,\tau_c(t)} - t \le c$ for all $t \in \mathbb{R}_\ge$. Consequently, by another appeal to (2.19),

$$\mathbb{E}_0\tau(t) \ \le \ \mathbb{E}_0\tau_c(t) \ = \ \frac{t+\mathbb{E}_0 R_c(t)}{\mu_c} \ \le \ \frac{t+c}{\mu_c} \quad \text{for all } t \in \mathbb{R}_\ge$$

where $\mu_c := \mathbb{E}(X_1 \wedge c)$. By using this in (2.20), the dominated convergence theorem ensures that, for any $\lambda \in \mathscr{P}(\mathbb{R}_\ge)$,

$$\limsup_{t\to\infty} \frac{\mathbb{U}_\lambda(t)}{t} \ \le \ \int \lim_{t\to\infty}\left(\frac{t-x+c}{\mu_c}\mathbf{1}_{[0,t]}(x)\right)\ \lambda(dx) \ = \ \frac{1}{\mu_c},$$

and since this is true for all $c > 0$ and $\lim_{c\to\infty}\mu_c = \mu$, we finally obtain

$$\limsup_{t\to\infty} \frac{\mathbb{U}_\lambda(t)}{t} \le \frac{1}{\mu}$$

which completes the proof. □

It is to be emphasized that all results and identities in this subsection have been obtained only for the case of RP's, that is for RW's with nonnegative positive mean increments. This may also be taken as a transitional remark so as to motivate the consideration of general RW's $(S_n)_{n\ge 0}$ with a.s. finite ladder epoch $\sigma^>$ in the next subsection, the main focus being on the case where the RW has positive drift.

2.4.3 Cyclic decomposition via ladder epochs

Here we consider a nontrivial zero-delayed RW $(S_n)_{n\ge 0}$ with renewal measure $\mathbb{U}$. Let σ be any a.s. finite stopping time for $(S_n)_{n\ge 0}$ with associated sequence $(\sigma_n)_{n\ge 0}$ of copy sums. Denote by $\mathbb{U}^{(\sigma)}$ the renewal measure of the RW $(S_{\sigma_n})_{n\ge 0}$ and define the *pre-σ occupation measure of* $(S_n)_{n\ge 0}$ by

$$\mathbb{V}^{(\sigma)}(A) := \mathbb{E}\left(\sum_{n=0}^{\sigma-1} \mathbf{1}_A(S_n)\right) \quad \text{for } A \in \mathscr{B}(\mathbb{R}), \tag{2.23}$$

which has total mass $\|V^{(\sigma)}\| = \mathbb{E}\sigma$ and is hence finite if σ has finite mean. The next lemma provides us with a useful relation between $\mathbb{U}$ and $\mathbb{U}^{(\sigma)}, \mathbb{V}^{(\sigma)}$.

Lemma 2.4.4. *Under the stated assumptions,*

$$\mathbb{U} = \mathbb{V}^{(\sigma)} * \mathbb{U}^{(\sigma)}$$

for any a.s. finite stopping time σ for $(S_n)_{n\ge 0}$.

Proof. Using cyclic decomposition with the help of the σ_n, we obtain

$$\begin{aligned}
\mathbb{U}(A) &= \mathbb{E}\left(\sum_{k\ge 0} \mathbf{1}_A(S_k\right) = \sum_{n\ge 0} \mathbb{E}\left(\sum_{k=\sigma_n}^{\sigma_{n+1}-1} \mathbf{1}_A(S_k)\right) \\
&= \sum_{n\ge 0} \int_{\mathbb{R}} E\left(\sum_{k=\sigma_n}^{\sigma_{n+1}-1} \mathbf{1}_{A-x}(S_k - S_{\sigma_n})\middle| S_{\sigma_n} = x\right) \mathbb{P}(S_{\sigma_n} \in dx) \\
&= \sum_{n\ge 0} \int_{\mathbb{R}} \mathbb{V}^{(\sigma)}(A-x)\, \mathbb{P}(S_{\sigma_n} \in dx) \\
&= \mathbb{V}^{(\sigma)} * \mathbb{U}^{(\sigma)}(A) \quad \text{for all } A \in \mathscr{B}(\mathbb{R}),
\end{aligned}$$

where (2.3) of Prop. 2.2.1 has been utilized in the penultimate line. □

Now, if $(S_n)_{n\ge 0}$ has a.s. finite ladder epoch $\sigma = \sigma^\alpha$ for some $\alpha \in \{>,\ge,<,\le\}$, we write $\mathbb{U}^\alpha$ and V^α for $\mathbb{U}^{(\sigma)}$ and $\mathbb{V}^{(\sigma)}$, respectively. Note that U^α is the renewal measure of the ladder height process $(S_n^\alpha)_{n\ge 0}$. As a trivial but important consequence of the previous lemma, we note that

$$\mathbb{U} \;=\; V^\alpha * \mathbb{U}^\alpha \quad \text{if } \mathbb{P}(\sigma^\alpha < \infty) = 1. \tag{2.24}$$

In the case where $\sigma^> < \infty$ a.s. (and thus also $\sigma^\ge < \infty$ a.s.), we thus have a convolution formula for the renewal measure $\mathbb{U}$ that involves the renewal measure of a RP, namely $\mathbb{U}^>$ or $\mathbb{U}^\ge$. This will allow us in various places to reduce a problem for RW's with positive drift to those that have a.s. nonnegative or even positive increments.

If $(S_n)_{n\ge 0}$, given in a standard model, has arbitrary initial distribution λ, then Lemma 2.4.4 in combination with $\mathbb{U}_\lambda = \lambda * \mathbb{U}_0$ immediately implies

$$\mathbb{U}_\lambda \;=\; \lambda * \mathbb{V}^{(\sigma)} * \mathbb{U}^{(\sigma)} \;=\; \mathbb{V}^{(\sigma)} * \mathbb{U}_\lambda^{(\sigma)} \tag{2.25}$$

where $\mathbb{V}^{(\sigma)}, \mathbb{U}^{(\sigma)}$ are defined as before under $\mathbb{P}_0$.

Returning to the zero-delayed situation, let us finally note that a simple computation shows that

$$\mathbb{V}^{(\sigma)} \;=\; \sum_{n\ge 0} \mathbb{P}(\sigma > n, S_n \in \cdot), \tag{2.26}$$

and that, for any real- or complex-valued function f

$$\int f \, d\mathbb{V}^{(\sigma)} \;=\; \sum_{n\ge 0} \int_{\{\sigma > n\}} f(S_n)\, d\mathbb{P} \;=\; \mathbb{E}\left(\sum_{n=0}^{\sigma-1} f(S_n)\right) \tag{2.27}$$

whenever one of the three expressions exist.

2.5 The stationary delay distribution

2.5.1 What are we looking for and why?

In order to motivate the derivation of what is called the *stationary delay distribution of a RP* $(S_n)_{n\ge 0}$, let us first dwell on some heuristic considerations that embark on the results in Section 1.2. We saw there that, if $(S_n)_{n\ge 0}$ has exponential increment distribution with parameter θ and the same holds true for S_0, then the renewal measure $\mathbb{U}$ equals $\theta\lambda\!\!\!\lambda_0$ on $\mathbb{R}_\ge$ which means that $\mathbb{U}([t,t+h])$ for any fixed h is temporally invariant. In fact, we further found that the associated renewal counting process $N := (N(t))_{t\ge 0}$ is a homogeneous Poisson process and as such a *stationary point process* in the sense that

$$\Theta_s N \;:=\; (N(s+t)-N(s))_{t\geq 0} \;\stackrel{d}{=}\; N \quad \text{for all } s\in\mathbb{R}_{\geq}. \tag{2.28}$$

This follows immediately from the fact that a Poisson process has stationary and independent increments [☞ Thm. 1.2.2]. Cleary, N is a functional of the renewal

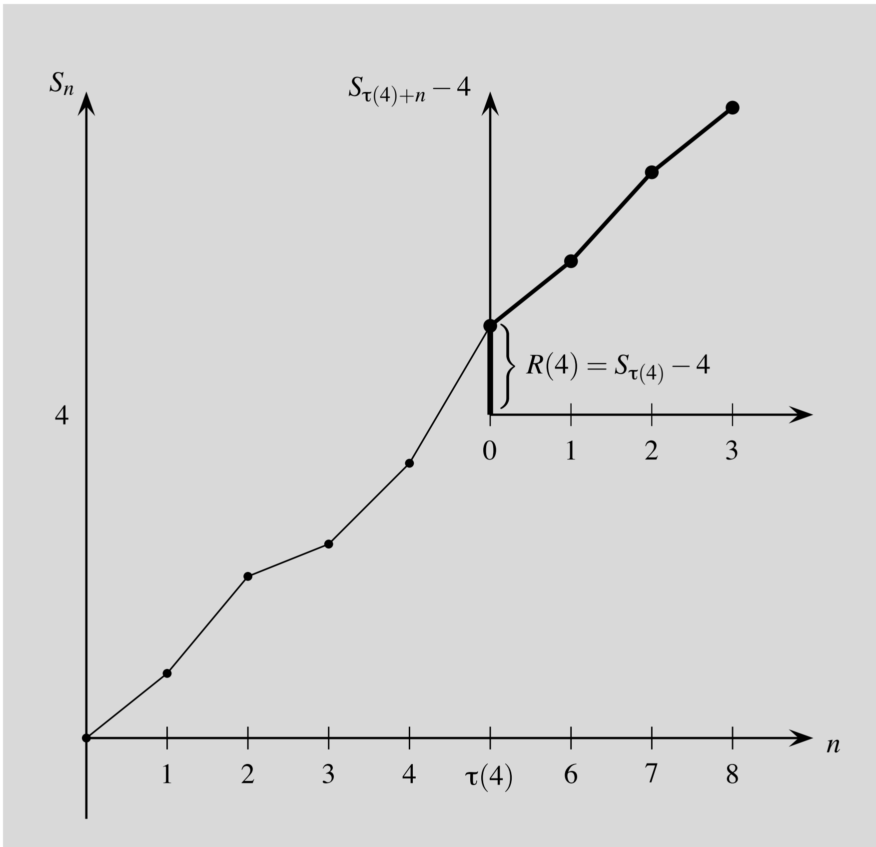

Fig. 2.3 Path of a RP with first pasage time $\tau(4)=5$ and associated overshoot $R(4)$.

process $(S_n)_{n\geq 0}$ or, equivalently, of $S_0, X_1, ...$, i.e. $N = h(S_0, X_1, ...)$ for some measurable function h. For the same reason

$$\Theta_s N \;=\; h(S_{\tau(s)}-s, X_{\tau(s)+1}, ...) \;=\; h(R(s), X_{\tau(s)+1}, ...) \quad \text{for all } s\in\mathbb{R}_{\geq},$$

where $R(s)$ is the overshoot at s [see Fig. 2.3]. After this observation, the validity of (2.28) can be explained as follows. Even without providing a rigorous argument, the memoryless property of the exponential distribution is readily seen to imply that

$R(s) \stackrel{d}{=} Exp(\theta)$ for all $s \in \mathbb{R}_\geq$. By Prop. 2.2.1, $R(s)$ is further independent of the iid sequence $(X_{\tau(s)+n})_{n\geq 1}$ with $X_{\tau(t)+1} \stackrel{d}{=} X_1$. But this is all it takes to conclude (2.28).

Now let us turn to the general case, the question being whether an appropriate choice of the delay distribution F^s of a RP $(S_n)_{n\geq 0}$ with increment distribution F renders $R(t) \stackrel{d}{=} F^s$ *for all* $t \in \mathbb{R}_\geq$. If yes then by the same reasoning as before we should obtain a stationary associated renewal counting process $N = (N(t))_{t\geq 0}$ under $\mathbb{P}_{F^s}$ in the sense of (2.28) and in particular a translation invariant renewal measure $\mathbb{U}_{F^s}$ on $\mathbb{R}_\geq$. But the translation invariance entails that $\mathbb{U}_{F^s}$ equals $\theta\lambda\!\!\lambda_0$ on $\mathbb{R}_\geq$ for some $\theta > 0$, while the elementary renewal theorem 2.4.3 gives $\theta = 1/\mathbb{E}X_1$. It will be shown in ??????? that $(R(t))_{t\geq 0}$ constitutes a continuous time Markov process for which F^s, if it exists, forms a stationary distribution. However, the way F^s is derived hereafter, does not use this fact.

The question not addressed so far is why we should strive for F^s. The short reply is that its existence provides strong evidence that for arbitrary delay distributions λ the behavior of $\Theta_t N$ under $\mathbb{P}_\lambda$ and particularly of $\mathbb{U}_\lambda([t,t+h])$ is expected to be the same as in the stationary situation if t tends to infinity. However, what we are really hoping for and in fact going to deliver later is to turn this into a rigorous result by a suitable construction of two RP's on the same probability space with the same increment distribution but different delay distributions λ and F^s, such that their renewal counting processes N and N', say, are asymptotically equal in the sense that $\lim_{t\to\infty} \rho(\Theta_t N, \Theta_t N') = 0$, where ρ denotes a suitable distance function. The technique behind this idea is well-known under the name *coupling* in the theory of stochastic processes.

2.5.2 The derivation

If λ denotes any measure on $\mathbb{R}$, let $\lambda^+ := \lambda(\cdot \cap \mathbb{R}_\geq)$ be its restriction to $\mathbb{R}_>$ hereafter in the sense that its mass on $\mathbb{R}_\leq$ has been removed. Having outlined what we are looking for, let us now turn to the derivation of the stationary delay distribution F^s for a RP $(S_n)_{n\geq 0}$ with increment distribution F, finite mean μ and given as usual in a standard model. The first thing to note is that $\mathbb{U}_\lambda = \lambda * \mathbb{U}_0$ satisfies the convolution equation

$$\mathbb{U}_\lambda \;=\; \lambda \;+\; F * \mathbb{U}_\lambda \quad \text{for all } \lambda \in \mathscr{P}(\mathbb{R}_\geq)$$

which in terms of the renewal function becomes a *renewal equation* as encountered in Section 1.4, namely

$$\mathbb{U}_\lambda(t) \;=\; \lambda(t) \;+\; \int_{[0,t]} F(t-x)\, \mathbb{U}_\lambda(dx) \quad \text{for all } \lambda \in \mathscr{P}(\mathbb{R}_\geq) \tag{2.29}$$

The goal is to find a λ such that $\mathbb{U}_\lambda(t) = \mu^{-1}t$ for all $t \in \mathbb{R}_\geq$ (thus $\mathbb{U}_\lambda = \mu^{-1}\lambda\!\!\lambda_0^+$) and we will now do so by simply plugging the result into (2.29) and solving for

$\lambda(t)$. Then, with $\overline{F} := 1 - F$

$$\begin{aligned}\lambda(t) &= \frac{t}{\mu} - \frac{1}{\mu}\int_0^t F(t-x)\,dx \\ &= \frac{1}{\mu}\int_0^t \overline{F}(t-x)\,dx = \frac{1}{\mu}\int_0^t \overline{F}(x)\,dx \quad \text{for all } t \geq 0.\end{aligned}$$

We thus see that there is only one λ, now called F^s, that gives the desired property of $\mathbb{U}_\lambda$, viz.

$$F^s(t) := \frac{1}{\mu}\int_0^t \overline{F}(x)\,dx = \frac{1}{\mu}\int_0^t \mathbb{P}(X_1 > x)\,dx \quad \text{for all } t \geq 0,$$

which is continuous and requires that μ is finite. To all those who prematurely lean back now let is be said that this is not yet the end of the story because there are questions still open like “What about the infinite mean case?” and “Is this really the answer we are looking for if the RP is arithmetic?”

If $(S_n)_{n\geq 0}$ is d-arithmetic, w.l.o.g. suppose $d = 1$, then indeed a continuous delay distribution gives a continuous renewal measure which is in sharp contrast to the zero-delayed situation where the renewal measure is concentrated on $\mathbb{N}_0$. The point to be made here is that an appropriate definition of stationarity of the associated renewal counting process N must be restricted to those times at which renewals naturally occur in the 1-arithmetic case, namely integer epochs. In other words, the stationary delay distribution F^s to be defined must now be concentrated on $\mathbb{N}$, but only give (2.28) for $t \in \mathbb{N}_0$ under $\mathbb{P}_{F^s}$. This particularly implies $\mathbb{U}_{F^s}^+ = \mu^{-1}\lambda\!\!\lambda_1^+$ and thus $\mathbb{U}_{F^s}^+(n) = \mathbb{U}_{F^s}(\{1,...,n\}) = \mu^{-1}n$ for $n \in \mathbb{N}$, where $\lambda\!\!\lambda_1$ is counting measure on the set of integers $\mathbb{Z}$. By pursuing the same argument as above, but for $t \in \mathbb{N}_0$ only, we then find that F^s must satisfy

$$F^s(n) = \frac{n}{\mu} - \frac{1}{\mu}\sum_{k=1}^{n} F(n-k) \quad \text{for all } n \in \mathbb{N}$$

and therefore

$$F^s(n) = \frac{1}{\mu}\sum_{k=0}^{n-1}\overline{F}(k)\,dx = \frac{1}{\mu}\sum_{k=1}^{n}\mathbb{P}(X_1 \geq k) \quad \text{for all } n \in \mathbb{N}$$

as the unique solution among all distributions concentrated on $\mathbb{N}$. We summarize our findings as follows.

Proposition 2.5.1. *Let $(S_n)_{n\geq 0}$ be a RP in a standard model with finite drift μ and lattice-span $d \in \{0,1\}$. Define its* ***stationary delay distribution*** *F^s on $\mathbb{R}_\geq$ by*

$$F^s(t) := \begin{cases} \frac{1}{\mu}\int_0^t \mathbb{P}(X_1 > x)\,dx, & \text{if } d = 0, \\ \frac{1}{\mu}\sum_{k=1}^n \mathbb{P}(X_1 \geq k)\,\mathbf{1}_{[n,n+1)}(t), & \text{if } d = 1 \end{cases} \tag{2.30}$$

for $t \in \mathbb{R}_{\geq}$. *Then* $\mathbb{U}_{F^s} = \mu^{-1}\lambda\!\!\lambda_d^+$.

Now observe that the integral equation (2.29) remains valid if λ is any locally finite measure on $\mathbb{R}_{\geq}$ and $\mathbb{U}_\lambda$ is still defined as $\lambda * \mathbb{U}_0$. This follows because (2.29) is a linear in λ. Hence, if we drop the normalization μ^{-1} in the definition of F^s, we obtain without further ado the following extension of the previous proposition.

Corollary 2.5.2. *Let* $(S_n)_{n\geq 0}$ *be a RP in a standard model with lattice-span* $d \in \{0,1\}$. *Define the locally finite measure* ξ *on* $\mathbb{R}_{\geq}$ *by*

$$\xi(t) \; := \; \begin{cases} \int_0^t \mathbb{P}(X_1 > x)\, dx, & \text{if } d = 0, \\ \sum_{k=1}^n \mathbb{P}(X_1 \geq k)\, \mathbf{1}_{[n,n+1)}(t), & \text{if } d = 1 \end{cases} \tag{2.31}$$

for $t \in \mathbb{R}_{\geq}$. *Then* $\mathbb{U}_\xi = \lambda\!\!\lambda_d^+$.

2.5.3 The infinite mean case: restricting to finite horizons

There is no stationary delay distribution if $(S_n)_{n\geq 0}$ has infinite mean μ, but Cor. 2.5.2 helps us to provide a family of delay distributions for which stationarity still yields when restricting to finite horizons, that is to time sets $[0,a]$ for $a \in \mathbb{R}_>$. As a further ingredient we need the observation that the renewal epochs in $[0,a]$ of $(S_n)_{n\geq 0}$ and $(S_{a,n})_{n\geq 0}$, where $S_{a,n} := S_0 + \sum_{k=1}^n (X_k \wedge a)$, are the same. As a trivial consequence they also have the same renewal measure on $[0,a]$, whatever the delay distribution is. But by choosing the latter appropriately, we also have a domination result on (a,∞) as the next result shows.

Proposition 2.5.3. *Let* $(S_n)_{n\geq 0}$ *be a RP in a standard model with drift* $\mu = \infty$ *and lattice-span* $d \in \{0,1\}$. *With* ξ *given by* (2.31) *and for* $a > 0$, *define distributions* F_a^s *on* $\mathbb{R}_{\geq}$ *by*

$$F_a^s(t) \; := \; \frac{\xi(t \wedge a)}{\xi(a)} \quad \text{for } t \in \mathbb{R}_{\geq}. \tag{2.32}$$

Then, for all $a \in \mathbb{R}_>$, $\mathbb{U}_{F_a^s} \leq \xi(a)^{-1}\lambda\!\!\lambda_d^+$ *with equality holding on* $[0,a]$.

Proof. Noting that F_a^s can be written as $F_a^s = \xi(a)^{-1}\xi - \lambda_a$, where $\lambda_a \in \mathscr{P}(\mathbb{R}_{\geq})$ is given by

$$\lambda_a(t) := \frac{\xi(t)-\xi(a\wedge t)}{\xi(a)} = \mathbf{1}_{(a,\infty)}(t)\frac{\xi(t)-\xi(a)}{\xi(a)} \quad \text{for all } t\in\mathbb{R}_\geq,$$

we infer with the help of Cor. 2.5.2 that

$$\mathbb{U}_{F_a^s} = \xi(a)^{-1}\mathbb{U}_\xi - \lambda_a * \mathbb{U}_0 \le \xi(a)^{-1}\mathbb{U}_\xi = \xi(a)^{-1}\lambda\!\!\lambda_d \quad \text{on } \mathbb{R}_\geq$$

as claimed. □

2.5.4 And finally random walks with positive drift via ladder epochs

A combination of the previous results with a cyclic decomposition via ladder epochs allows us to further extend these results to the situation where $(S_n)_{n\ge 0}$ is a RW with positive drift μ and lattice-span $d\in\{0,1\}$ and thus may also take on negative values. This is accomplished by considering F^s, F_a^s and ξ as defined before, but for the associated ladder height RP $(S_n^>)_{n\ge 0}$. Hence we put

$$\xi(t) := \begin{cases} \int_0^t \mathbb{P}(S_1^> > x)\,dx, & \text{if } d=0,\\ \sum_{k=1}^n \mathbb{P}(S_1^> \ge k)\,\mathbf{1}_{[n,n+1)}(t), & \text{if } d=1 \end{cases} \tag{2.33}$$

for $t\in\mathbb{R}_\geq$ and then again F_a^s by (2.32) for $a\in\mathbb{R}_>$. If $S_1^>$ has finite mean $\mu^>$ and hence ξ is finite, then let F^s be its normalization, i.e. $F^s=(\mu^>)^{-1}\xi$. In order to be able to use the results from the previous section we must first verify that $S_1^>$ and X_1 are of the same lattice-type.

Lemma 2.5.4. *Let $(S_n)_{n\ge 0}$ be a nontrivial zero-delayed RW and σ an a.s. finite first ladder epoch. Then $d(X_1)=d(S_\sigma)$.*

Proof. If X_1 is nonarithmetic the assertion follows directly from the obvious inequality $d(S_\sigma)\le d(X_1)=0$. Hence it suffices to consider the case when $d(X_1)>0$. W.l.o.g. suppose $d(S_\sigma)=1$ and $\sigma=\sigma^>$, so that $d(X_1)=1$ must be verified. If $d(X_1)<1$, then $p:=\mathbb{P}(X_1=c)>0$ for some $c\notin\mathbb{Z}$. Define $W_n:=S_{n+1}-S_1$ for $n\in\mathbb{N}_0$, clearly a copy of $(S_n)_{n\ge 0}$ and independent of X_1. Then, with $\tau(t):=\inf\{n\ge 0: W_n>t\}$, we have

$$S_1^> = X_1\mathbf{1}_{\{X_1>0\}} + (X_1+W_{\tau(-X_1)})\mathbf{1}_{\{X_1\le 0\}}.$$

Consequently, if $c>0$, then $\mathbb{P}(S_1^>=c)\ge p>0$ which is clearly impossible as $c\notin\mathbb{Z}$. If $c<0$, then use that $W_{\tau(t)}$ is a strictly ascending ladder height for $(W_n)_{n\ge 0}$ and thus integer-valued for any $t\in\mathbb{R}_\geq$ to infer that

$$\mathbb{P}(S_1^>\in c+\mathbb{Z}) \ge p\,\mathbb{P}(c+W_{\tau(-c)}\in c+\mathbb{Z}) = p\,\mathbb{P}(W_{\tau(-c)}\in\mathbb{Z}) > 0$$

which again contradicts our assumption $d(S_1^>) = 1$, for $(c+\mathbb{Z})\cap\mathbb{Z} = \emptyset$. □

Proposition 2.5.5. *Let $(S_n)_{n\ge0}$ be a RW in a standard model with positive drift μ and lattice-span $d \in \{0,1\}$. Then the following assertions hold with ξ, F_a^s and F^s as defined in* (2.33) *and thereafter.*

(a) $\mathbb{U}_\xi^+ = \mathbb{E}\sigma^> \lambda\!\!\!\lambda_d^+$.

(b) $\mathbb{U}_{F_a^s}^+ \le \xi(a)^{-1}\mathbb{E}\sigma^> \lambda\!\!\!\lambda_d^+$ *for all* $a \in \mathbb{R}_>$.

(c) *If μ is finite, then* $\mathbb{U}_{F^s}^+ = \mu^{-1}\lambda\!\!\!\lambda_d^+$.

Proof. First note that $\mu > 0$ implies $\mathbb{E}\sigma^> < \infty$ [☞ Thm. 2.2.9] and $\mu^> = \mathbb{E}S_1^> = \mu\,\mathbb{E}\sigma^>$ by Wald's identity. By (2.25), $\mathbb{U}_\lambda = \mathbb{V}^> * \mathbb{U}_\lambda^>$ for any distribution λ, where

$$\mathbb{V}^>(A) \;=\; \mathbb{E}_0\left(\sum_{n=0}^{\sigma^>-1}\mathbf{1}_A(S_n)\right) \quad \text{for } A \in \mathscr{B}(\mathbb{R})$$

is the pre-$\sigma^>$ occupation measure and thus concentrated on $\mathbb{R}_\le$ with $\|\mathbb{V}^>\| = \mathbb{E}\sigma^>$. Of course, (2.25) extends to arbitrary locally finite measures λ. Therefore,

$$\begin{aligned}\mathbb{U}_\xi(A) \;&=\; \mathbb{V}^> * \mathbb{U}_\xi^>(A) \;=\; \int_{\mathbb{R}_\le}\mathbb{U}_\xi^>(A-x)\,\mathbb{V}^>(dx)\\ &=\; \int_{\mathbb{R}_\le}\lambda\!\!\!\lambda_d^+(A-x)\,\mathbb{V}^>(dx) \;=\; \mathbb{V}^>(\mathbb{R}_\le)\,\lambda\!\!\!\lambda_d^+(A)\\ &=\; \mathbb{E}\sigma^>\,\lambda\!\!\!\lambda_d^+(A) \quad \text{for all } A \in \mathscr{B}(\mathbb{R}_\ge)\end{aligned}$$

where Cor. 2.5.2, the translation invariance of $\lambda\!\!\!\lambda_d$ and $A-x \subset \mathbb{R}_\ge$ for all $x \in \mathbb{R}_\le$ have been utilized. Hence assertion (a) is proved. As (b) and (c) are shown in a similar manner, we omit supplying the details again and only note for (c) that, if $\mu < \infty$, $\mathbb{U}_{F^s}^+ = (\mu^>)^{-1}\,\mathbb{E}\sigma^>\,\lambda\!\!\!\lambda_d^+$ really equals $\mu^{-1}\lambda\!\!\!\lambda_d^+$ because $\mu^> = \mu\,\mathbb{E}\sigma^>$ as mentioned above. □

Remark 2.5.6. (a) Replacing the strictly ascending ladder height $S_1^>$ with the weakly ascending one $S_1^\ge$ in the definition (2.33) of ξ leads to $\mathbb{U}_\xi = \mathbb{E}\sigma^\ge\,\lambda\!\!\!\lambda_d$ and thus to a proportional, but different result if $S_1^>$ and $S_1^\ge$ are different. On the other hand, it is clear that, for some $\beta \in (0,1]$,

$$\mathbb{P}(S_1^\ge \in \cdot) \;=\; (1-\beta)\delta_0 \,+\, \beta\,\mathbb{P}(S_1^> \in \cdot) \tag{2.34}$$

where $\mu^\ge = \beta\mu^>$ gives $\beta = \mu^\ge/\mu^>$. Consequently, a substitution of $S_1^>$ for $S_1^\ge$ in (2.33) changes ξ merely by a factor and thus has the same normalization in the case $\mu < \infty$, namely F^s. In other words, the stationary delay distributions of $(S_n^>)_{n\ge0}$ and $(S_n^\ge)_{n\ge0}$ are identical.

(b) Given a RP $(S_n)_{n\ge 0}$ with finite drift μ, we have already mentioned that the forward recurrence times $R(t)$ form a continuous time Markov process with stationary distribution $\pi(dt) := \mu^{-1}\mathbb{P}(X_1 > t)\,dt$, and we have found that $\mathbb{U}_\pi = \mu^{-1}\lambda\!\!\lambda_0^+$, no matter what the lattice-type of $(S_n)_{n\ge 0}$ is. On the other hand, if the latter is 1-arithmetic, then the stationary delay distribution F^s that gives $\mathbb{U}_{F^s} = \mu^{-1}\lambda\!\!\lambda_1^+$ is discrete and thus differs from π. In terms of the forward recurrence times this can be explained as follows: The Markov process $(R(t))_{t\ge 0}$ has continuous state space $\mathbb{R}_>$ regardless of the lattice-type of the underlying RP and as such the continuous stationary distribution π. As a consequence, its subsequence $(R(n))_{n\ge 0}$ forms a Markov chain with the same state space and the same stationary distribution. However, in the 1-arithmetic case $\mathbb{N}$ forms a closed subclass of states in the sense that $\mathbb{P}(R_n \in \mathbb{N} \text{ for all } n \ge 0) = 1$ if $\mathbb{P}(R_0 \in \mathbb{N}) = 1$. Hence $(R(n))_{n\ge 0}$ may also be considered as a *discrete* Markov chain on $\mathbb{N}$ and as such has stationary distribution F^s.

(c) The *positive* forward recurrence time $R(t) = S_{\tau(t)} - t$, $\tau(t) = \inf\{n : S_n > t\}$ has been used in the previous derivations so as to motivate the stationary delay distribution. As an alternative, one may also consider the *nonnegative* variant $V(t) := S_{\nu(t)} - t$ with $\nu(t) := \inf\{n : S_n \ge t\}$. In the nonarithmetic case, this does not lead to anything different because stopping on the boundary, i.e. $V(t) = 0$, has asymptotic probability zero. However, if $(S_n)_{n\ge 0}$ is 1-arithmetic, then the stationary delay distribution on $\mathbb{R}_\ge$ instead of $\mathbb{R}_>$ is obtained, namely

$$\widehat{F}^s(t) \;:=\; \frac{1}{\mu^>}\sum_{k=0}^{n}\mathbb{P}(S_1^> > k)\mathbf{1}_{[n,n+1)}(t) \quad \text{for } t \in \mathbb{R}_\ge.$$

As a consequence, $\mathbb{U}_{\widehat{F}^s}$ equals $\mu^{-1}\lambda\!\!\lambda_1$ on $\mathbb{R}_\ge$ instead of $\mathbb{R}_>$ only. Similar adjustments apply to the definitions of ξ and F_a^s when dealing with $V(t)$, but we refrain from a further discussion.