How to generalize reflection groups

Multiple ways to define Coxeter groups

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30.04.2021

Reflections and their generalizations

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- Example: Symmetry groups of platonic solids.

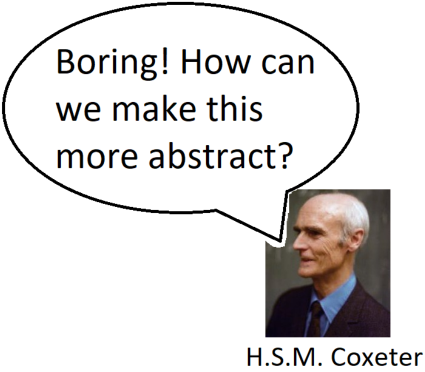

Figure: Wikipedia

Overview

Definition of Dihedral Groups

Definition

A group generated by two **involutions**, i.e. elements of order two, is called a **dihedral group**.

(All) The dihedral groups

Finite dihedral groups

Given $m \geq 2$, let $L, L' \subseteq \mathbb{R}^2$ be two lines through the origin in the Euclidean plane with angle $\frac{2\pi}{m}$ between them. Furthermore, let r_L, $r_{L'} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflections along those lines. We define $D_m :=< r_L, r_{L'} >\leq O(2) \leq Isom(\mathbb{R}^2)$.

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Infinite dihedral group

Let $r_0, r_1 : \mathbb{R} \to \mathbb{R}$ be the reflections about 0, 1 resp. We define $D_\infty :=< r_0, r_1 > \leq Isom(\mathbb{R})$.

The dihedral groups as semidirect products

Reminder: Semi-direct product

Let G, H be groups and $\phi : G \times H \to H$ an action of G on H. The set $H \times G$ carries a group structure via $(h_1, g_1) \cdot (h_2, g_2) := (h_1\phi(g_1, h_2), g_1 g_2)$, called the **semi-direct product** and denoted by $H \rtimes_\phi G$.

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Construction via semi-direct product

Denote by C_m the cyclic group of order m (including ∞). Write $C_2 = \{\pm 1\}$. Then C_2 acts on C_m via $\epsilon g = g^\epsilon$. Then $C_m \rtimes C_2$ is generated by $(0, -1), (d, -1)$ where 0 is the neutral and d the generating element of C_m.

Finite presentations of dihedral groups

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Let S be any set and R be a set of words over $S \cup S^{-1}$. We denote by $< S|R >$ the quotient of the free group over S by its normal subgroup generated by R.

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Construction via (finite) presentations

The groups $< s, t|s^2, t^2, (st)^m >$ for $m > 1$ and $< s, t|s^2, t^2 >$ are clearly generated by the involutions s, t.

Equivalence of the definitions

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Let W be a dihedral group generated by the involutions s, t. Then $P :=< st >$ is normal in W, $W = P \rtimes C_2$ and $[W : P] = 2$.

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Corollary

The following are isomorphisms:

$$\begin{array}{ccc} D_m \to C_m \rtimes C_2 & \to < s, t | s^2, t^2, (st)^m > \\ r_L, r_{L'} \mapsto (0, -1), (d, -1) \mapsto s, t \end{array}$$

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- The relation $sp = t = p^{-1}s$ allows swapping s and p^k around, so every $w \in W$ can uniquely we written as $s^m p^n$.
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- $W = P \cup sP \Rightarrow [W : P] \leq 2$.
  Suppose $[W : P] = 1$, i.e. $W = P$.

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  - Contradiction to $1, s, t \in W$ being mutually distinct.

Geometric properties of dihedral groups

Pre-reflection Systems

Definition

Let W be a group, $R \subseteq W$ a generating set, Ω a connected simplicial graph which is acted on by W, and $v_0 \in Vert(\Omega)$ a base point. Then (R, Ω, v_0) is called a **prereflection system** for W, if

1. All elements of R have order 2,
2. R is closed in W under conjugation,
3. For each edge of Ω there is a unique element of R which flips it (i.e. swaps its endpoints).
4. Each element of R flips at least one edge of Ω.

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Let (R, Ω, v_0) be a prereflection system for a group W. Then W acts transitively on Ω.

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- Each edge $\{v_i, v_{i+1}\}$ is flipped by some $r_i \in R$.
- $\Rightarrow r_n ... r_1 v = w$

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Deletion property

Lemma

Let (R, Ω, v_0) be a prereflection system for a group W and $S = S(v_0)$. If $s = (s_0, \ldots, s_k)$ is a word over S and $r_i = r_j$ for some $i < j$ where $\Phi(s) = (r_1, \ldots, r_k)$, then $s_1 \ldots s_k = s_1 \ldots \widehat{s_i} \ldots \widehat{s_j} \ldots s_k$.

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- Multiply from right with $s_1 \dots s_j$, from left with $s_i \dots s_1$.
- $\Rightarrow s_i \dots s_j = s_{i+1} \dots s_{j-1}$.
- $\Rightarrow$ Replace subword $(s_i, \dots, s_j)$ by $(s_{i+1}, \dots, s_{j-1})$.

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Lemma

For each $r \in R$, $\Omega \setminus \Omega^r$ has either one or two connected components. If it has two, they are interchanged by r.

The proof that walls separate the world

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- For a vertex v there is either a path in $\Omega \setminus \Omega^s$ to v_0 or to sv_0.
  - Let $t = (s_1, ..., s_k)$ be the word corresponding to a minimal edge path in Ω from v_0 to v.

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  - Deletion lemma: path from v_0 to sv_0 not crossing Ω^s.

Definition

Let (R, Ω, v_0) be a prereflection system for a group W. Then it is called a **reflection system**, if for each $s \in S(v_0)$ the graph $\Omega \setminus \Omega^s$ has two components. The elements of R are called **reflections** and the elements of S are called **fundamental reflections**.

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- Each wall Ω^{s_i} is crossed an even number of times.
- $\Rightarrow$ Apply deletion lemma $\Rightarrow$ ↯ to minimal length!

Road to Coxeter Systems

Definition

A group W together with a generating set S of elements of order two is called a **pre-Coxeter system**.

Combinatorial conditions for a pre-Coxeter systems (W, S)

(D) - Deletion

If $s = (s_1, ..., s_k)$ is a word in S with $k > l(w(s))$, then there are indices $i < j$ so that the subword $s' = (s_1, ..., \widehat{s_i}, ..., \widehat{s_j}, ..., s_k)$ is also an expression for $w(s)$.

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Given a reduced expression $s = (s_1, ..., s_k)$ for $w \in W$ and an element $s \in S$, either $l(sw) = k + 1$ or else there is an index i such that $w = ss_1 ... \widehat{s_i} ... s_k$.

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(F) - Folding

Suppose $w \in W$ and $s, t \in S$ are such that $l(sw) = l(w) + 1$ and $l(wt) = l(w) + 1$. Then either $l(swt) = l(w) + 2$ or $swt = w$.

Equivalence of the conditions

Theorem

Given a pre-Coxeter system (W, S), the conditions (D), (E) and (F) are equivalent.

Why all that?

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Suppose $(R, Cay(W, S), 1)$ is a reflection system for a pre-Coxeter system (W, S). Then (D), (E) and (F) hold.

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- $k > l(w) \geq \# R(1,w)$ - every wall has to be crossed by w.
- $\Rightarrow r_i = r_j$ for some $i < j$.
- Apply deletion lemma for prereflection systems.

Coxeter Systems

Definition

Given a set S, a **Coxeter matrix** on S is a symmetric matrix $(m_{s,t})_{s,t\in S}$ where $m_{s,t} \in \mathbb{N} \cup \{\infty\}$ such that $m_{s,t} = 1$ iff $s = t$.

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A pre-Coxeter system (W, S) is a **Coxeter system**, if the map $\tilde{W} \to W$ defined by $s \mapsto s$ is an isomorphism. In this case, we call W a **Coxeter group** and S a **fundamental set of generators**.

Coxeter Diagrams

Definition

Let $M = (m_{s,t})_{s,t}$ a Coxeter matrix on a set S. The **Coxeter graph** for M consists of a vertex for each element of S and edges s, t wherever $m_{s,t} \geq 3$. The edges where $m_{s,t} \geq 4$ are labelled with $m_{s,t}$. The labelled graph is called a **Coxeter diagram**.

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Definition

A Coxeter system is called **irreducible** if its Coxeter graph is connected.

Our next goal

Theorem

Let (W, S) be a pre-Coxeter system. The following are equivalent:

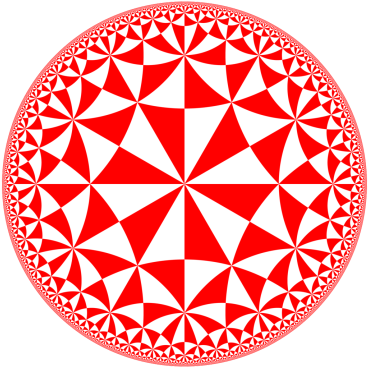
- (W, S) is a Coxeter system.
- $Cay(W, S)$ is a reflection system.
- (W, S) satisfies the exchange condition (E).

A look in the rear view mirror

Proposition

Dihedral groups are Coxeter groups.

Math inspires Art inspires Math



(a) Hyperbolic domain construction

(b) Circle Limit I (M.C. Escher)

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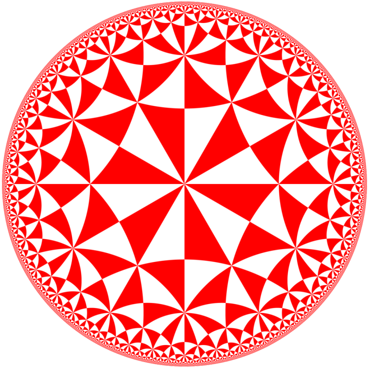

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Thanks for your attention. Any questions?