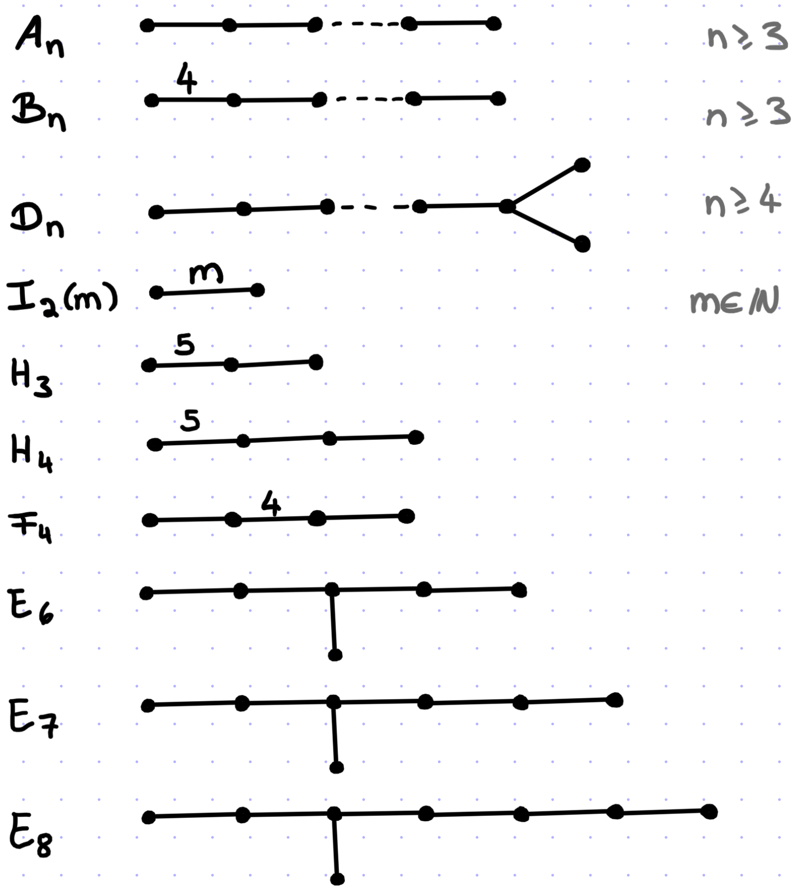
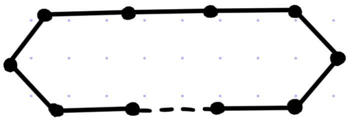
# Classification of spherical and euclidean Coxeter groups

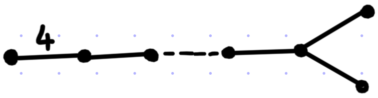
[based on Appendix C in Davis' book]

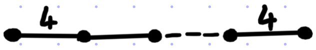
**Theorem C.1.2:** (Classification of finite Coxeter groups) The connected, positive definite Coxeter diagrams are those listed below.
In other words, this is the complete list of diagrams of the irreducible Coxeter systems $(W,S)$ with $W$ finite.

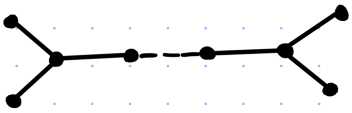


**Theorem C.1.3:** (Classification of euclidean Coxeter groups) The connected positive semi-definite Coxeter diagrams, which are not positive definite, are those listed below. In other words, this is the complete list of diagrams of the irreducible Euclidean reflection groups.

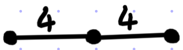
$\tilde{A}_n$  $n \geq 1$

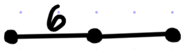
$\tilde{B}_n$  $n \geq 2$

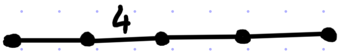

$\tilde{C}_n$  $n \geq 2$

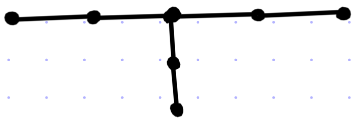

$\tilde{D}_n$  $n \geq 4$

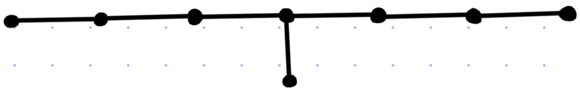
$\tilde{A}_1$ 

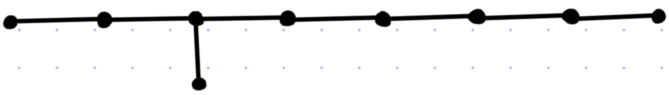

$\tilde{B}_2$ 


$\tilde{G}_2$ 


$\tilde{F}_4$ 


$\tilde{E}_6$ 

$\tilde{E}_7$ 

$\tilde{E}_8$ 

Recall: **Theorem 6.8.12 (ii):** An irreducible Coxeter group can be represented as a euclidean reflection group if and only if it's cosine matrix is positive semidefinite with one-dimensional kernel.

To any Coxeter matrix $M = (m_{ij})$ we define the cosine matrix $C = (c_{ij})$ by

$$c_{ij} = -\cos\left(\frac{\pi}{m_{ij}}\right).$$

For $m_{ij} = \infty$, we say $\frac{\pi}{m_{ij}} = 0$ and $c_{ij} = -1$.

Further note that $c_{ii} = 1$ for any $i$ and $c_{ij} < 0$ for $i \neq j$.

An $n \times n$ matrix $(a_{ij})$ is decomposable, if there is a non-trivial partition of the index set $\{1, \dots, n\}$ as $I \cup J$ such that $a_{ij} = a_{ji} = 0$ for $i \in I$ and $j \in J$.

$A$ - $n \times n$ symmetric matrix

$A_{(n)}$ - deleting last row and column of $A$ (principal submatrix)

Remark:

1) $A$ positive definite $\Rightarrow$ $A_{(n)}$ positive definite
2) $A_{(n)}$ positive definite $\Rightarrow$ $A$ is either positive definite (if $\det A > 0$) or positive semi-definite with corank 1 (if $\det A = 0$) or non-degenerate of type $(n,1)$ (if $\det A < 0$)

A principal minor $d_k$ of $A$ is defined to be the determinant of the matrix obtained by deleting the last $k$ rows and columns of $A$, for $0 \leq k \leq n$.

$A$ positive definite $\Leftrightarrow$ each principal minor positive

Goal: All spherical diagrams listed above are positive definite.

(I) rank 2, i.e. $I_2(m)$ $\bullet\overset{m}{\text{——}}\bullet$

$$M = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \rightsquigarrow \det(C) = \det\begin{pmatrix} 1 & -\cos(\frac{\pi}{m}) \\ -\cos(\frac{\pi}{m}) & 1 \end{pmatrix} = \sin^2\left(\frac{\pi}{m}\right) > 0$$

$\Rightarrow$ $I_2(m)$ positive definite

(II) rank $\geq 3$

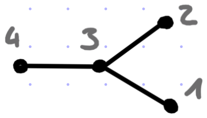
Observation: greatest possible label is 5, i.e. the cosine matrices only contain the negatives of the following values

$$\cos\left(\frac{\pi}{2}\right) = 0, \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$$

Number vertices such that

→ $n^{th}$ vertex is connected to only one other vertex

→ this edge is labeled by 3 or 4

e.g. $D_4$ 



$H_3$ 



Let $r := -2\cos\left(\frac{\pi}{m}\right)$ for $m = 3$ or $4$.

$$\det(2C) = \left(\begin{array}{ccc|c} & *_{n-1} & & \begin{matrix}0\\ \vdots\\ 0\\ r\end{matrix} \\ \hline 0 & \text{—} & 0\ r & 2 \end{array}\right) = -r\det\left(\begin{array}{c|c} *_{n-2} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline * & r \end{array}\right) + 2\det(*_{n-1})$$

$$= -r^2 \det(*_{n-2}) + 2\det(*_{n-1})$$

$$= -r^2 d_{n-2} + 2 d_{n-1}$$

$$= \begin{cases} 2d_{n-1} - d_{n-2} & \text{if } m = 3 \\ 2d_{n-1} - 2d_{n-2} & \text{if } m = 4 \end{cases}$$

Inductively we can obtain $\det(2C)$ for any spherical diagram with rank $\geq 3$.

| | $A_n$ | $B_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $H_3$ | $H_4$ |
|---|---|---|---|---|---|---|---|---|---|
| $\det(2C)$ | $n+1$ | 2 | 4 | 3 | 2 | 1 | 1 | $3-\sqrt{5}$ | $\frac{(7-3\sqrt{5})}{2}$ |

e.g. $A_n$ 

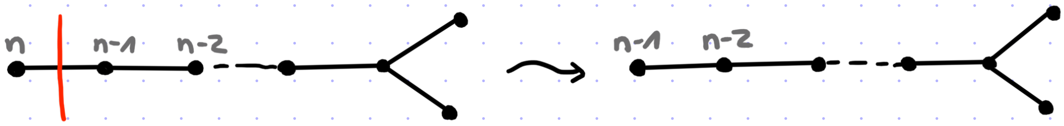
$$2C_n = \begin{pmatrix} 2 & 1 & 0 & & & \\ 1 & 2 & 1 & 0 & & \\ 0 & 1 & 2 & 1 & 0 & \\ \vdots & & & \ddots & & \\ \vdots & & & & & \\ 0 & \cdots & & 0 & 1 & 2 \end{pmatrix} = \left(\begin{array}{ccc|c} & 2C_{n-1} & & \begin{matrix}0\\ \vdots\\ 0\\ 1\end{matrix} \\ \hline 0 & \text{—} & 0\ 1 & 2 \end{array}\right)$$

<u>Lemma C.2.1:</u> Each of the spherical diagrams listed before is positive definite.

<u>proof:</u> $C_{(n)}$ positive definite $\xRightarrow{\det C > 0}$ $C$ positive definite

$C_{(n)}$ is the cosine matrix of another spherical diagram with lesser rank ∎

e.g. $D_n$




$D_{n-1}$

Goal: Every euclidean diagram is positive semidefinite with one-dimensional kernel.

Lemma 6.3.7 (i): Suppose that $A=(a_{ij})$ is an indecomposable symmetric positive semi-definite degenerate matrix with $a_{ij} \leq 0$ for all $1 \leq i \neq j \leq n$.
Then $A$ has a one-dimensional kernel. Moreover, it's kernel is spanned by a vector with positive coordinates.

proof: Let $N$ be the kernel of $A$ and $0 \neq \sum_j c_j e_j \in N$, i.e. $\sum_j a_{ij} c_j = 0$ for all $1 \leq i \leq n$.
WLOG $c_j \geq 0 \ \forall j$ (by Lemma 6.3.5(i))
Assume $I := \{ j \in \{1, \dots, n\} \mid c_j \neq 0 \} \neq \{1, \dots, n\}$ and fix $i \notin I$.

$$\left.\begin{array}{ll} j \in I \Rightarrow a_{ij} c_j \leq 0 & \text{since } i \neq j \\ j \notin I \Rightarrow a_{ij} c_j = 0 & \end{array}\right\} \Rightarrow a_{ij} c_j = 0 \ \forall 1 \leq j \leq n$$

Thus $a_{ij} = 0$ for all $i \notin I$ and $j \in I$, and $A$ is decomposable. ↯
In particular, all coordinates of a non-zero vector in $N$ are non-zero.
$\Longrightarrow \dim N = 1$ ■

Lemma C.2.2: Each of the euclidean diagrams is positive semidefinite with one-dimensional kernel.

proof:
Observation from last time: Every proper subdiagram is positive definite.
Thus it only remains to show that all cosine matrices have determinant 0.

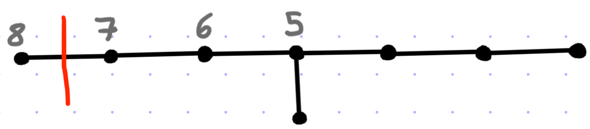
(I) $\tilde{A}_n$

$$2C_n = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$

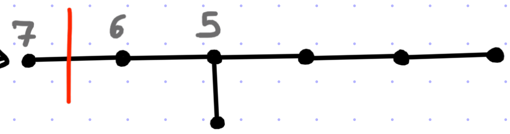
all rows sum to $0 \Rightarrow \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ eigenvector to eigenvalue $0 \Rightarrow \det(2C_n) = 0$

(II) All other cases are trees.

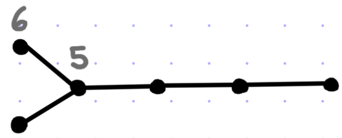
Removing a terminal vertex leads to a spherical diagram. Using the results about spherical diagrams and $\det(2C) = \begin{cases} 2d_{n-1} - d_{n-2} & \text{if } m=3 \\ 2d_{n-1} - 2d_{n-2} & \text{if } m=4 \end{cases}$ finishes the proof. ∎

e.g. 


$\tilde{E}_7$

$\rightsquigarrow$ 


$E_7$

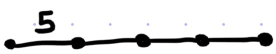
$\rightsquigarrow$ 


$D_6$

Thus, $\det(2C_{\tilde{E}_7}) = 2\det(2C_{E_7}) - \det(C_{D_6}) = 2\cdot 2 - 4 = 0$.

**Lemma C.2.3:** The cosine matrices of the diagrams

$Z_4$: 

and $Z_5$: 


have negative determinants. (hyperbolic diagrams)

**Goal:** There are no spherical or euclidean diagrams other than the ones listed before.

$\Gamma, \Gamma'$ – Coxeter diagrams $\qquad$ $\Gamma$ connected

$\Gamma$ dominates $\Gamma'$ ($\Gamma \succsim \Gamma'$), if $\Gamma'$ is a subgraph of $\Gamma$ and the label on each edge of $\Gamma'$ is $\leq$ the label on the corresponding edge of $\Gamma$.

If, in addition, $\Gamma \neq \Gamma'$, then we say $\Gamma$ strictly dominates $\Gamma'$ ($\Gamma \succ \Gamma'$).

**Lemma C.3.1:** Suppose an irreducible Coxeter diagram $\Gamma$ is positive semidefinite. If $\Gamma \succ \Gamma'$, then $\Gamma'$ is positive definite.

proof: $C = (c_{ij})$ cosine matrix of $\Gamma$

$C' = (c'_{ij})$ cosine matrix of $\Gamma'$

WLOG: Assume that $C'$ is the $k \times k$ matrix in the upper left corner of $C$ for some $k \leq n$

$$\Gamma \gtrsim \Gamma' \xRightarrow[\text{on } [0,\pi]]{\phantom{x}}_{-\cos \text{ increasing}} \quad c_{ij} \leq c'_{ij} \leq 0 \text{ for all } 1 \leq i \neq j \leq k$$

Assume that $C'$ is not positive definite, i.e. $\exists\, x \in \mathbb{R}^k \setminus \{0\}$ s.t. $x^t C x \leq 0$.

Then: $$0 \underset{\substack{\downarrow\\ A \text{ positive}\\ \text{semidefinite}}}{\leq} \sum_{i,j=1}^{k} c_{ij} |x_i||x_j| \underset{\substack{\downarrow\\ \Gamma < \Gamma'}}{\leq} \sum_{i,j=1}^{k} c'_{ij} |x_i||x_j| \underset{\substack{\downarrow\\ c'_{ij} \leq 0 \text{ for } i \neq j}}{\leq} \sum_{i,j=1}^{k} c'_{ij} x_i x_j \underset{\substack{\downarrow\\ \text{by assumption}\\ \text{on } C'}}{\leq} 0,$$

so we have equality.

With $y = (|x_1|, \dots, |x_k|) \in \mathbb{R}^k$ and $z = (|x_1|, \dots, |x_k|, 0, \dots, 0) \in \mathbb{R}^n$ this reads

$$0 = z^t C z = y^t C' y = x^t C' x = 0.$$

$\Rightarrow z \in \ker C$

Hence, all coordinates of $z$ are $\neq 0$ by Lemma 6.3.7 and $k = n$.

$\Gamma > \Gamma' \Rightarrow \exists$ pair $\{i,j\}$ such that $c_{ij} < c'_{ij}$

$\Rightarrow z^t C z < y^t C' y$ $\lightning$

$\blacksquare$

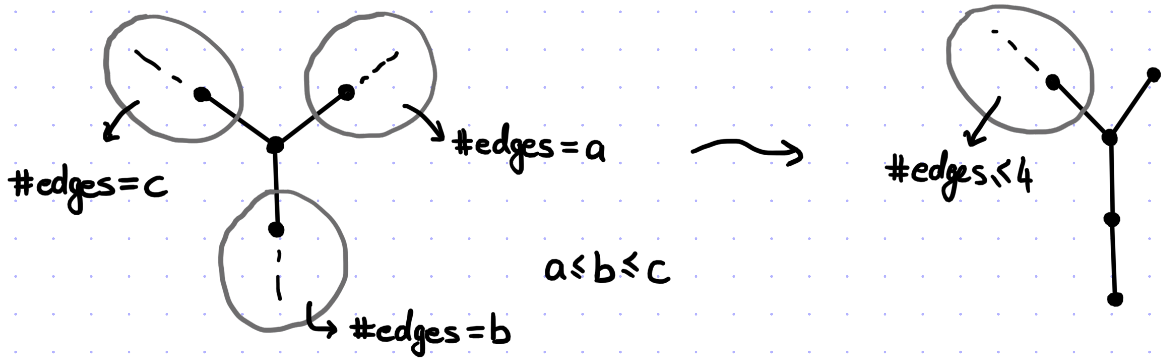
Suppose that $\Gamma$ is a connected positive semidefinite Coxeter diagram not on the list. Let $n$ be the rank of $\Gamma$ and $m$ be the maximum label on any edge.

Observation: $\Gamma$ cannot (strictly) dominate any euclidean diagram.

(I)
- $n \geq 3$ (all Coxeter diagrams of rank $\leq 2$ are positive semidefinite or $I_2(m)$)
- $m \neq \infty$ (since $\Gamma \not\geq \tilde{A}_1$)
- $\Gamma$ is a tree (since $\Gamma \not\geq \tilde{A}_n$)

(II) Suppose $m = 3$.

- $\Gamma$ must have a branch vertex (since $\Gamma \neq A_n$)
- the branch vertex is unique (since $\Gamma \not\geq \tilde{D}_n, n > 4$)
- each branch vertex has degree 3 (since $\Gamma \not\geq D_4$)




- $a=1$ (since $\Gamma \not\geq \tilde{E}_6$)
- $b\leq 2$ (since $\Gamma \not\geq \tilde{E}_7$)
- $c \leq 4$ (since $\Gamma \not\geq \tilde{E}_8$)
- $b \neq 1$ (since $\Gamma \neq D_n$)

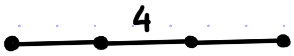
But $\Gamma \neq E_6, E_7, E_8$, so $m=3$ is not possible.

III • $m \geq 4$

- only one edge has a label $>3$ (since $\Gamma \not\geq \tilde{C}_n$)
- $\Gamma$ has no branch vertices (since $\Gamma \not\geq \tilde{B}_n$)

IV Suppose $m=4$.

- The two extreme edges of $\Gamma$ are labelled 3 (since $\Gamma \neq B_n$, $\Gamma \not\geq \tilde{C}_n$).
- $n=4$ (since $\Gamma \not\geq \tilde{F}_4$)




, so $m=4$ is impossible since $\Gamma \neq F_4$.

$m \geq 6$ is impossible since $\Gamma \not\geq \tilde{G}_2$.

V • $m=5$

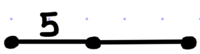
- The edge labelled 5 must be an extreme edge (since $\Gamma \not\geq Z_4$).

Assume $\Gamma \geq Z_4$. Then

$$0 \leq \underbrace{\sum c_{ij}|x_i||x_j|}_{\Gamma \text{ positive semidefinite}} \underbrace{\leq}_{\Gamma \geq Z_4} \underbrace{\sum c'_{ij}|x_i||x_j| \leq 0}_{Z_4 \text{ negative semidefinite}} \Rightarrow \Gamma = Z_4 \ \lightning$$

- $n=4$ (since $\Gamma \not\geq Z_5$)

But then $\Gamma$ is




$H_3$ or


$H_4$, which is not possible.