The Basic construction $\mathcal{U}$

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Introduction

- For a Coxeter system (W, S), a space X and a family of subspaces $(X_s)_{s\in S}$, we want to construct a space $\mathcal{U}(W, X)$
- The idea of the construction is to paste together copies of X, one for each element of W
- Our construction will be slightly more general than needed, we will construct our space $\mathcal{U}$ for an arbitrary group G
- This can be useful in the discussion of geometric realizations of buildings

Mirror structures

- A mirror structure on a space X consists of an index set and a family of closed subspaces $(X_s)_{s\in S}$ (the mirrors) of X
- We assume, that each $x \in X$ has a neighborhood, that intersects only finitely many of the X_s
- We set

$$S(x) := \{s \in S : x \in X_s\}$$

For $T \subseteq X$ nonempty, we set

$$X_T := \bigcap_{t\in T} X_t \quad \text{and} \quad X^T := \bigcup_{t\in T} X_t$$

and $X_\emptyset = X$ and $X^\emptyset = \emptyset$

Definition

A family of groups over a set S of a group G consists of subgroups $B \subseteq G$ and $(G_s)_{s \in S}$ s.t. each G_s contains B

- We will assume, that G is a topological group and B is an open subgroup s.t. G/B has the discrete topology
- For G discrete, we will just consider the discrete topology
- In our case, we will always assume, that $B = \{id\}$

Definition ($\mathcal{U}(W, X)$)

- Suppose X is a mirrored space over S and $(G_s)_{s \in S}$ is a family of subgroups of G over S
- Define an equivalence relation $\sim$ on $G \times X$ by

$$(h, x) \sim (g, y) \Longleftrightarrow x = y \text{ and } h^{-1}g \in G_{S(x)}$$

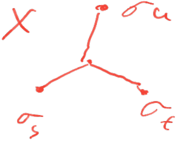
- Consider $G/B \times X$ endowed with the product topology and define

$$(G/B \times X)/\sim$$

Example

Let $G = \langle s, t, u : s^2 = t^2 = u^2 = (st)^3 = (ut)^3 = (us)^3 = 1 \rangle$
and $X = Cone\{\sigma_s, \sigma_t, \sigma_u\}$

$$S(x) = \begin{cases} \emptyset & x \in \{\sigma_s, \sigma_t, \sigma_u\} \, X \\ \{s\} & x = \sigma_s \\ \{t\} & x = \sigma_t \\ \{u\} & x = \sigma_u \end{cases}$$



$\rightsquigarrow G_{S(x)}$ is either $\{1\}$, $\{1,s\}$, $\{1,t\}$ or $\{1,u\}$

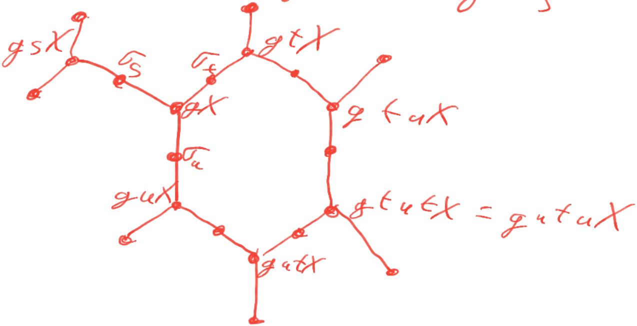
$x = \sigma_s \rightsquigarrow (g, \sigma_s) \sim (a, \sigma_s) \Leftrightarrow g^{-1}g' \in \{1, s\}$

$\Leftrightarrow g = g'$ or $g' = gs$

here, $[g, \sigma_s] = \{(g, \sigma_s), (gs, \sigma_s)\}$

$x \notin \{\sigma_s, \sigma_t, \sigma_u\} \Rightarrow [g, x] = \{(g, x)\}$

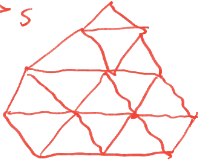
$\rightsquigarrow$ glue gX and gsX along σ_s



other example: $G = D_6$, $X = 2$-simplex

$X_s = \Delta_s$

codim-1 faces



Important remarks

- Suppose, $X = Cone\{\sigma_s : s \in S\}$, $X_s = \sigma_s$, then, the space $\mathcal{U}(W, X)$ is the Caley graph of (W, S) up to subdivision
- We denote the image of (gB, x) in $\mathcal{U}(G, X)$ by $[g, x]$
- For $g \in G$, gX denotes the image of $gB \times X$ in $\mathcal{U}(G, X)$ and is called a chamber
- G acts on $G/B \times X$ via $g(hB, x) = (ghB, x)$
- This G-action on $G/B \times X$ preserves the equivalence relation, hence, it descends to an action on $\mathcal{U}(G, X)$
- The orbit space of the G-action on $G/B \times X$ is X
- $\mathcal{U}(G, X)/G$ and X are homeomorphic

set of chambers is identified with G/B

orbit proj (proj onto the 2^{nd} factor) descends to proj $p : \mathcal{U}(G,X) \to X$

Since
$$\begin{tikzcd}
\mathcal{U}(G,X) \arrow[r, "p"] \arrow[rr, bend right, "id"'] & X \arrow[r, "i"] & \mathcal{U}(G,X)
\end{tikzcd}$$

p is a retraction $(p \circ i) = id$

p is an open mapping (because of the def of $\sim$, an open set in $\mathcal{U}$ is open in the 2^{nd} coordinate)

p induces a cont. bij $\tilde{p} : \mathcal{U}(G,X)/G \to X$
(since the orbit relation is coarser than $\sim$)

p open $\Rightarrow \tilde{p}$ open
$\Rightarrow \mathcal{U}(G,X)/G \cong X$

Definition (Fundamental domain)

- Suppose, a group G acts on a space Y, A closed subset $C \subseteq Y$ is called a fundamental domain for G on Y if each G-orbit intersects C and if for each x in the interior of C, $Gx \cap C = \{x\}$
- C is called a strict fundamental domain if it intersects each G-orbit in exactly one point.

- X is a strict fundamental domain for G on $\mathcal{U}(G, X)$

1. It's clear, that each G-orbit Gy intersects X since $\mathcal{U} = G \times X/_\sim$

2. Each G-orbit Gy intersects X in at most one point:

remember: $X \overset{i}{\hookrightarrow} U(G,X)/_G \xrightarrow{\tilde{p}} X$

suppose $\exists\, x, x' \in X : x \neq x'$, $x, x' \in Gy$

then $x \mapsto Gy \mapsto y$

$x' \mapsto Gy \mapsto y$

since $i \circ p = id \Rightarrow x = x'$

$\Rightarrow X$ is strict fundamental domain

Lemma

$\mathcal{U}(G, X)$ is connected if

1. *The family of subgroups $(G_s)_{s \in S}$ generates G*
2. *X is connected*
3. *$X_s \neq \emptyset$ for all $s \in S$*

Conversely, if $\mathcal{U}(G, X)$ is connected, then 1. and 2. hold

"$\Rightarrow$" $\mathcal{U}(G, X)$ is endowed with the quotient topology

a subset of $\mathcal{U}(G, X)$ is open (closed) iff it's intersection with each chamber is open (closed)

X connected $\Rightarrow$ any subset, which is open and closed is a union of chambers AX

for some $A \subset G/B$

suppose $A \subset G/B$ nonempty proper set, s.t. AX open and closed in $\mathcal{U}(G, X)$. Let H be the inverse image of A in G.

If $X_1 \neq \emptyset$, $x \in X_s$, then for $g_s \in G_s$, $hB \in A$ any open neighborhood of $[hg_s, x]$ must intersect hX and $hg_s X$.

$\Rightarrow H G_s \subset H \Rightarrow H$ is the subgroup $\hat{G}$ of G generated by the G_s, $s \in S$

Hence, if $\hat{G} = G \Rightarrow AX = \mathcal{U}(G,X)$

i.e., $\mathcal{U}(G,X)$ is connected

"$\Leftarrow$" Suppose $\mathcal{U}(G,X)$ is connected. Since the orbit map is a retraction, X is connected (hence 2. holds)

$\hat{G}$ contains all isotropy subgroups $G_{s(x)}$ $x \in X$, it follows that $\hat{G}X$ is open in $\mathcal{U}(G,X)$. Clearly, $\hat{G}X$ is closed.

Hence, $\hat{G} = G$ (hence 1. holds)

Definition (Properly discontinuous action)

Suppose G is discrete. A G-action on a Hausdorff space Y is called properly discontinuous, if

1. Y/G is Hausdorff
2. For each $y \in Y$, $G_y := \{g \in G : gy = y\}$ is finite
3. Each $y \in Y$ has a neighborhood U_y, s.t. $gU_y \cap U_y = \emptyset$ for all $g \in G_y$

Definition

A mirror structure on X is called G- finite, if $X_T = \emptyset$ for any $T \subseteq S$ such that G_T/B is infinite

Lemma

Suppose G is discrete. The G-action on $\mathcal{U}(G, X)$ is properly discontinuous if and only if

1. *X is Hausdorff*
2. *The mirror structure is G-finite*

"$\Rightarrow$" X Hausdorff and the fact that the mirror structure is G-finite follows immediately from the aforementioned def.

"$\Leftarrow$" It suffices to establish that each $[1, x] \in \mathcal{U}(G, X)$ (for an $x \in X$ arbitrary) has a $G_{S(x)}$-stable neighborhood U_x s.t. $gU_x \cap U_x = \emptyset \ \forall g \in G \setminus G_{S(x)}$:

Let $V_x := X \setminus \bigcup_{s \notin S(x)} X_s$. and $U_x = G_{S(x)} V_x$

U_x is an open $G_{S(x)}$-stable neighborhood of $[1, x]$ in $\mathcal{U}(G, X)$ and clearly $gU_x \cap U_x = \emptyset \quad \forall g \in G \setminus G_{S(x)}$

- Suppose (W, S) is a pre-Coxeter system
- This gives us a family of subgroups if for each $s \in S$, we define W_s as the subgroup generated by s
- For any subset A of W/B, define

$$AX := \bigcup_{a \in A} aX$$

Lemma

Suppose, X is connected (resp. path connected) and $X_s \neq \emptyset$ for each $s \in S$. Given a subset $A \subseteq W$, AX is connected (resp. path connected)

We proof the statement for X connected:

"$\Rightarrow$" a subset of AX is both open and closed has the form BX for some $B \subset A$ (as seen before)

Let B be a proper nonempty subset of A s.t. BX open and closed in AX. Set $B^c = A \setminus B$.

Suppose A connected. wlog, suppose $b \in B$ and $b' \in B^c$ are connected by an edge (with label s) in the Cayley graph. Then, $bX_s = b'X_s$ lies in $BX \cap B^c X$.

Since $X_s \neq \emptyset$, BX and $B^c X$ cannot be disjoint

$\Rightarrow$ AX is connected

"$\Leftarrow$" Suppose AX is connected, the argument above shows that A cannot be partioned into disjoint subsets B and B^c s.t. no element of B can be connected by an edge to an element of B^c by an edge.

$\Rightarrow$ A is connected

Corollary

$\mathcal{U}(W, x)$ is connected (resp. path connected) if the following two conditions hold:

1. *X is connected (resp. path connected)*
2. *$X_s \neq \emptyset$ for each $s \in S$*

- This is just the special case $A = W$ of the aforementioned lemma

Example

If (W, S) is only required to be a pre-Coxeter system, then it's not true, that 2. is necessary for $\mathcal{U}(W, x)$ to be path connected. Take $W = C_2 \times C_2$ and $S = \{s, t, st\}$ the set of it's nontrivial elements

Lemma (Vinberg)

Suppose Y is a space and let W be a group acting on Y. Let Y^s denote the fixed pint set of s on Y. Let $f : X \to Y$ be continuous, s.t. $f(X_s) \subseteq Y^s$. Then, there exists an unique extension of f to a W-equivariant continuous map $\hat{f} : \mathcal{U} \to Y$ given by
$\hat{f}([w, x]) = wf(x)$

Continuity of $\hat{f}$: Given $V \subseteq Y$ open

$\hat{f}^{-1}(V)$ open $\Leftrightarrow$ it's preimage in $W \times X$ open (by quotient topology). Since W has the discrete top., $U = \{w\} \times A$ for $A \subseteq X$ open. To show, that U is open, find open neighborhood for (w, x) in U $\Leftrightarrow$ find open neighborhood of x in $f^{-1}(w^{-1}V)$ (clear by cont. of f)

Definition

The Action of a discrete group $\widehat{W}$ on a space Y is a reflection group if there is a Coxeter system (W, S) and a subspace $X \subseteq Y$ s.t.

1. $\widehat{W} = W$
2. If a mirror structure on X is defined by setting X_s equal to the intersection of X with the fixed set of s on Y, then the map $\mathcal{U}(W, X) \to Y$, induced by the inclusion of x in Y is a homeomorphism

Example (The Coxeter complex)

- Let Δ be a simplex of dimension $Card(S) - 1$ and that the faces of codimension 1 $\{\Delta_s\}_{s\in S}$ are indexed by the Elements of S
- $\{\Delta_s\}_{s\in S}$ is a mirror structure on Δ
- $\mathcal{U}(W, \Delta)$ is a simplicial complex, called the Coxeter complex
- We will see, that, if W is finite, $\mathcal{U}(W, \Delta)$is homeomorphic to a sphere and if W is infinite, $\mathcal{U}(W, \Delta)$ is contractible