1. Übungszettel zur Vorlesung „Räume nichtpositiver Krümmung“

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WWU Münster

Prof. Dr. Linus Kramer
Dr. Bakul Sathaye
Philip Möller

Each question is worth 4 points.

Aufgabe 1.1
Show that $\mathbb{R}^n$ is a normed vector space with

a) p-norm $||x||_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $1 \leq p < \infty$.
   Hint: Use Minkowski's inequality.

b) the ∞-norm $||x||_\infty = \sup_{1 \leq i \leq n} |x_i|$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Aufgabe 1.2
Prove that

a) $(\mathbb{R}^2, ||\cdot||_1)$ and $(\mathbb{R}^2, ||\cdot||_\infty)$ are isometric.

b) $(\mathbb{R}^3, ||\cdot||_1)$ and $(\mathbb{R}^3, ||\cdot||_\infty)$ are not isometric.

Aufgabe 1.3

a) Let X be an arbitrary set. The space $l_\infty(X)$ is the set of all bounded real functions on X. This is a vector space with respect to pointwise addition and multiplicaiton by scalars. Let $||f||_\infty = \sup\{|f(x)| : x \in X\}$. Show that $(l_\infty(X), ||\cdot||_\infty)$ is a Banach space.

b) Show that $(\mathbb{R}^n, ||\cdot||_\infty) \cong (l_\infty(X), ||\cdot||_\infty)$ for a suitable set X.

Aufgabe 1.4
Let (X, d) be a metric space. Define the distance between two subsets A and B of X by $\mathrm{dist}(A, B) = \inf \{d(x, y)|\ x \in A, y \in B\}$.

a) Is “dist” a metric on the space of subsets of X? Explain.

b) If A and B are compact, then there exists $a \in A$ and $b \in B$ such that $d(a, b) = \mathrm{dist}(A, B)$.

***-Aufgabe (Bonus Question)** (New metrics from old ones)
Suppose that $0 < \epsilon \leq 1$. Prove that $|a+b|^\epsilon \leq |a|^\epsilon + |b|^\epsilon$ holds for all $a, b \in \mathbb{R}$. Conclude that if (X, d) is a metric space, then (X, d^ϵ) is also a metric space, with $d^\epsilon(u, v) = d(u, v)^\epsilon$.
What can you say about the completeness of (X, d^ϵ) if (X, d) is complete?

Abgabe bis: Donnerstag, den 12.11.2020, 8 Uhr online im Learnwebkurs