10. Übungszettel zur Vorlesung „Räume nichtpositiver Krümmung“

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Each question is worth 4 points.

Aufgabe 10.1 (Stabilizers and transitive actions)
Let G be a group acting transitively on a set X and let $K \subset G$ be the stabilizer of $p \in X$. Let $L \subset G$ be a subset. Show that the following are equivalent

a) $G = L \cdot K$.

b) For each $q \in X$, there is $l \in L$ with $l(p) = q$.

Aufgabe 10.2 (Stabilizer is compact)
Let M be a connected, complete Riemannian manifold and let $p \in M$. Show that the stabilizer of p, $\mathrm{Stab}(p)$ is compact in $\mathrm{Isom}(M)$.

In particular, for a sequence $\{g_n\}$ of isometries of M such that $g_n p = p$ for all n, show that there is a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ that converges uniformly on compact subsets of M.

Hint: This is a special case of the Arzela-Ascoli Theorem. Let C be a compact set. For each $c \in C$, the sequence $\{g_n c\}$ is bounded, and therefore has a convergent subsequence. By Cantor diagonalization, there is a subsequence that works for all c in a countable, dense subset of C.

Aufgabe 10.3 (Connected components of $\mathrm{Isom}(X)$)
Show that if X is a connected complete Riemannian manifold, then $\mathrm{Isom}(X)$ has only finitely many connected components.

Hint: Every component of $G = \mathrm{Isom}(X)$ intersects the compact group $\mathrm{Stab}_G(x)$.

Aufgabe 10.4 (Exp and adjoint)
Let $A, B \in M(n, \mathbb{R})$, the algebra of $n \times n$ matrices with real entries. Show that

$$\exp(A)\; B\; \exp(-A) = \exp(\mathrm{ad}_A)(B),$$

where $\mathrm{ad}_A(B) = AB - BA$.

Hint: Express ad_A as $L_A - R_A$, where where L_A (resp. R_A) is the endomorphism of $M(n, \mathbb{R})$ mapping C to AC (resp. CA). Note that $L_A R_A = R_A L_A$, hence

$$(L_A - R_A)^k = \sum_{p+q=k} \frac{k!}{p!q!} L_A^p R_{-A}^q$$

10.*-Aufgabe (Self-adjoint)
Given $X \in S(n, \mathbb{R})$, consider the endomorphism of $M(n, \mathbb{R})$ defined by the formula

$$\tau_X(Y) = \frac{d}{dt} \exp(-X/2) \exp(X + tY) \exp(-X/2) \; |_{t=0}$$

Prove that τ_X is self-adjoint, i.e., for any $Y, Z \in M(n, \mathbb{R})$, we have $(\tau_X Y|Z) = (Y|\tau_X Z)$. Recall that for $A, B \in M(n, \mathbb{R})$, $(A|B) := \mathrm{Tr}(AB^t)$ defines a scalar product.

Hint: First show that

$$\tau_X(Y) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{p+q=k-1} \exp(-X/2) X^p Y X^q \exp(-X/2).$$

Then use the fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$).

Abgabe bis: Donnerstag, den 28.1.2021, 8 Uhr online im Learnwebkurs.