3. Übungszettel zur Vorlesung „Räume nichtpositiver Krümmung“

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Each question is worth 4 points.

Aufgabe 3.1 (Products)
Given nonempty metric spaces $(X_1, d_1), \ldots, (X_m, d_m)$, define the metric product as $X = X_1 \times \ldots \times X_m$ with metric $d((x_1, ..., x_m), (y_1, ..., y_m)) = \sqrt{\sum_{1 \leq j \leq m} d_j(x_j, y_j)^2}$.

Prove that

a) (X, d) is a metric space.
b) the product is complete if and only if all the X_j are complete.
c) the product is CAT(0) if and only if all the X_j are CAT(0).

Aufgabe 3.2 (Convexity of distance function)
Let (X, d) be a CAT(0) space.

a) Fix $x_0 \in X$. Prove that the function $x \mapsto d(x, x_0)$ is convex.

b) Let C be a complete convex subset of X. Show that the function $d_C : x \mapsto d(x, C)$ is convex. Further, for all $x, y \in X$, $|d_C(x) - d_C(y)| \leq d(x, y)$.

Aufgabe 3.3 (Embeddings of finite metric spaces)

a) Show that every metric space X consisting of at most three points admits an isometric embedding into $(\mathbb{R}^2, || - ||_2)$.

b) Can every metric space consisting of four points be isometrically embedded into $(\mathbb{R}^n, || - ||_2)$, for some $n \geq 1$?

c) Can every metric space consisting of finitely many points be isometrically embedded into $(\mathbb{R}^n, || - ||_p)$, for some $n \geq 1$ and some $1 \leq p \leq \infty$?

c) Does there exist a metric space in which every finite metric space can be isometrically embedded?

Aufgabe 3.4 (Local geodesics)
Let (X, d) be a metric space. A *local geodesic* in X is a map c from an interval $I \subseteq \mathbb{R}$ to X such that for every $t \in I$ there is $\epsilon > 0$ such that $d(c(t'), c(t'')) = |t' - t''|$ for all $t', t'' \in I$ with $|t - t'| + |t - t''| \leq \epsilon$.

Show that if X is a CAT(0) space then a local geodesic is a geodesic in X.

3.*-Aufgabe ($\mathbb{R}$-trees)
An $\mathbb{R}$-tree is a metric space (T, d) such that

i) given any two distinct points $x, y \in T$, there is a unique geodesic segment from x to y, and

ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$, where for a pair of points $a, b \in T$, $[a, b]$ is a geodesic segment from a to b in T.

Prove that every $\mathbb{R}$-tree is a CAT(0) space.

Abgabe bis: Donnerstag, den 26.11.2020, 8 Uhr online im Learnwebkurs.