4. Übungszettel zur Vorlesung „Räume nichtpositiver Krümmung“

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Each question is worth 4 points.

Aufgabe 4.1 (Completeness)
Let (X, d) be a nonempty metric space.

a) Show that if X is spherically complete, then it is complete.

b) Show that if X is complete, then every nested sequence of closed balls, $\bar{B}_{r_i}(x_i)$ with $\lim\limits_{i\to\infty} r_i = 0$, has a common point.

Aufgabe 4.2 (Fixed points)
Let G be a group acting isometrically on a metric space X.

a) Show that the fixed point set X^G is closed.

b) Assume further that X is a CAT(0) space. Show that the fixed point set X^G is convex (assuming that it is not empty).

Aufgabe 4.3 (Groups of Isometries)
Recall that $\mathrm{GL}(n, \mathbb{R})$ is the group of $n \times n$ invertible matrices. It acts on $\mathbb{R}^n$ via the usual linear transformations: for $A = (a_{ij}) \in \mathrm{GL}(n, \mathbb{R})$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $A(x) = (y_1, \ldots, y_n)$ where $y_i = \sum_j a_{ij}x_j$.

a) Show that the action of A preserves the Euclidean norm $|| - ||_2$ on $\mathbb{R}^n$ if and only if $A \in O(n)$, where $O(n)$ is the group of orthogonal matrices.

b) Use part (a) to show that $O(n)$ acts on $S^{n-1} \subset \mathbb{R}^n$ by isometries.

c) Consider the usual linear action of $\mathrm{GL}(n+1, \mathbb{R})$ on $\mathbb{R}^{n+1}$. Let $O(1, n)$ be the subgroup formed by the matrices that leave invariant the bilinear form β on $\mathbb{R} \oplus \mathbb{R}^n$, $\beta(u, v) = u_0v_0 - \sum u_iv_i$. Show that $O(1, n)$ consists of those matrices A such that $A^tJA = J$, where

$$J = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 \end{pmatrix}$$

d) Let $O^+(n, 1)$ be the subgroup of matrices that preserve the upper sheet $\mathbb{H}^n$ of the hyperboloid $\{x \in R^{n+1} \mid \beta(x, x) = 1\}$. Show that this is an index two subgroup in $O(1, n)$.

Aufgabe 4.4 (Models of the hyperbolic plane)
Consider the hyperbolic space $\mathbb{H}^n$ defined as a subspace of $\mathbb{R} \oplus \mathbb{R}^n$

$$\mathbb{H}^n = \{u = (u_0, u_1) \in \mathbb{R} \oplus \mathbb{R}^n \mid u_0 > 0,\ \beta(u, u) = 1\}.$$

Let $B^n = \{(u_0, u_1) \in \mathbb{R} \oplus \mathbb{R}^n \mid u_0 = 0,\ -\beta(u, u) < 1\}$ be the unit n-ball in $\{0\} \oplus \mathbb{R}^n$. Let $P = (-1, 0, \dots, 0) \in \mathbb{R} \oplus \mathbb{R}^n$.

Define the *stereographic projection* of $\mathbb{H}^n$ from the point P to the unit ball as the map $s : \mathbb{H}^n \to B^n$ such that for every $x \in \mathbb{H}^n$, the points $x, s(x)$, and P are collinear.

a) Give a formula for the map s.

b) Describe a metric $\tilde{d}$ on B^n so that s becomes an isometry.

The metric space $(B^n, \tilde{d})$ gives another model for the hyperbolic space, called the *Poincaré ball model*.

4.*-Aufgabe (Spherical completeness)
Consider the space of natural numbers $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 1\}$ with the metric

$$d(m, n) = \begin{cases} 0 & \text{if} \quad m = n \\ 1 + \max\{\frac{1}{m}, \frac{1}{n}\} & \text{if} \quad m \neq n \end{cases}$$

Show that d is a metric and that $(\mathbb{N}, d)$ is complete but not spherically complete.

Abgabe bis: Donnerstag, den 3.12.2020, 8 Uhr online im Learnwebkurs.