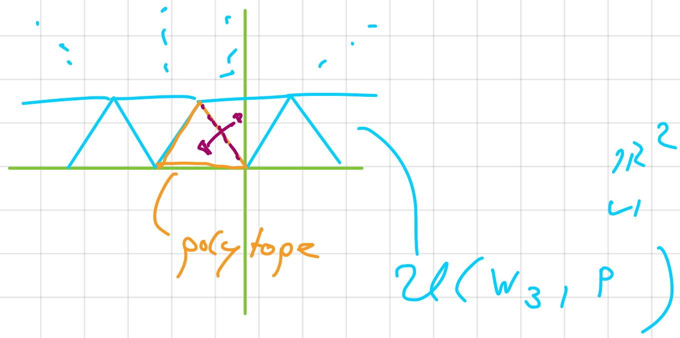
# Geometric reflection groups

## Motivation:

Generalize : $W_3 \curvearrowright \mathbb{R}^2$

$W_3$ = $(3,3,3)$ - group

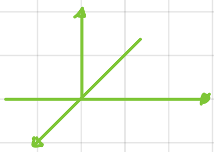



Question : $\mathcal{U}(W_3, P) \underset{\text{homeo}}{\cong} \mathbb{R}^2$ ?

$\rightsquigarrow$ Can $\mathbb{R}^2$ be tiled by $P$ ?

## Spaces of constant curvature

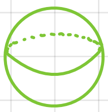
- Euclidean space : $\mathbb{E}^n$



$\mathbb{E}^n$ := " affine space associated to $\mathbb{R}^n$ "

$\rightsquigarrow$ constant (sectional) curvature $0$.

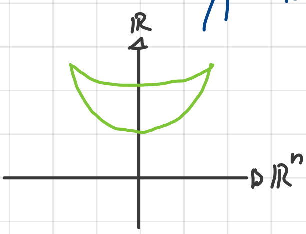
- n - Sphere : $\mathbb{S}^n$



$\mathbb{S}^n := \{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1 \}$

$\rightsquigarrow$ constant sectional curvature $= 1$.

- Hyperbolic space : $\mathbb{H}^n$ (hyperboloid model)




$\mathbb{H}^n := \{ \underset{(x_1,\dots,x_n,x_{n+1})}{x} \in \mathbb{R}^n \oplus \mathbb{R} \mid \beta(x,x) = -1,\ x_{n+1} > 0 \}$

with $\beta : \mathbb{R}^n \oplus \mathbb{R} \times \mathbb{R}^n \oplus \mathbb{R} \to \mathbb{R}$ bilinear

$(x, y) \mapsto \langle x, y \rangle_{std} - x_{n+1} y_{n+1}$

$\rightsquigarrow$ constant sectional curvature $= (-1)$.

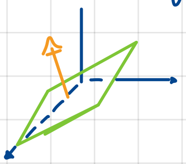
## Properties:

- $n \geq 2$: $\mathbb{H}^n, \mathbb{E}^n, \mathbb{S}^n$ are simply connected.
- After a suitable rescaling any (smooth) simply connected Riemannian manifold with constant sectional curvature is given by $\mathbb{H}^n, \mathbb{E}^n$ or $\mathbb{S}^n$.

Summarize $\mathbb{H}^n, \mathbb{E}^n, \mathbb{S}^n$ as $\underline{\mathbb{X}^n}$ in dependancy of the geometry.

## Polytopes

### Definitions

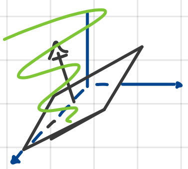


A hyperplane is a (affine) subspace $H \leq X$ with codimension 1.

v.s.

$\hookrightarrow$ induces a map $h: X \longrightarrow \mathbb{R}$ (p.ex inner product
$x \longmapsto \langle u, x \rangle_X$ via orthogonal

$\leadsto$ $H$ hyperplane $\Leftrightarrow$ $h(H) = \{0\}$ complement $u \in X$)

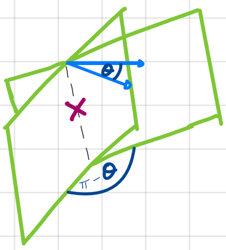


A set $H^+ \leq X$ is a halfspace if $\forall x \in H^+$: $h(x) \geq 0$.

$\hookrightarrow$ Describe $H^+$ via $u \in X$. ($u$ is (normal) inward-pointing vector)

### Definition: dihedral angle

Suppose $H_1$ and $H_2$ are hyperplanes in $\mathbb{X}^n$ bounding $E_1$ and $E_2$ half-spaces with $E_1 \cap E_2 \neq \emptyset$.



Let $x \in H_1 \cap H_2$ and $u_1, u_2$ the inward-pointing unit normals at $x$. Then

$$\theta := \arccos(\langle u_1, u_2 \rangle_{\mathbb{X}^n})$$

is the exterior dihedral angle and

$$\boxed{\pi - \theta}$$

is the (interior) dihedral angle.

**Definition:** <u>non-obtuse dihedral angle</u>

- In the above setting the half-spaces $E_1$ and $E_2$ have <u>non-obtuse dihedral angle</u> if

    a.) $H_1 \cap H_2 = \emptyset$

  or b.) $H_1 \cap H_2 \neq \emptyset$ and

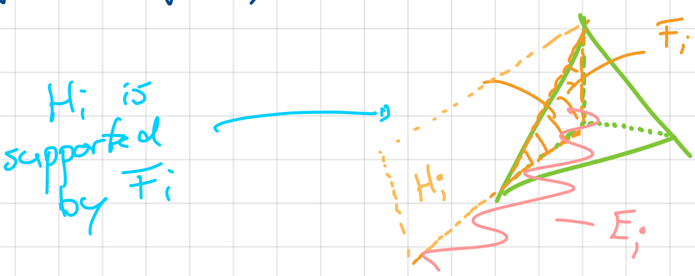
    the dihedral angle along $H_1 \cap H_2$ is $\leq \frac{\pi}{2}$

- A <u>family of half-spaces</u> $\{E_1, \dots, E_k\} \subseteq \mathbb{X}^n$ has non-obtuse angle, if $E_i$ and $E_j$ have non-obtuse angle for every $i,j = 1, \dots, k$.

- Let $P^n \subseteq \mathbb{X}^n$ be a convex polytope and $F_1, \dots F_k$ its codimension-1 faces.

  Let $H_i$ be the hyperplane determined by $F_i$ and and let $E_i$ be the half-space bounded by $H_i$ which contains $P^n$ for every $i = 1, \dots, k$.

  The <u>polytope</u> $P^n$ has non-obtuse dihedral angle if the family $\{E_1, \dots, E_k\}$ has this property.

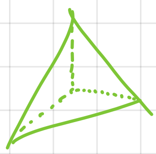


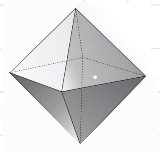

**Definition:** <u>simple</u> polytope

A $n$-dimensional polytope $P^n$ ($\subseteq \mathbb{X}^n$) is <u>simple</u> if exactly $n$ codimensional-one faces meet at each vertex.

<u>Examples:</u>

- $n$-simplex $\Delta^n$



- octahedron 

## Proposition:

Suppose $P^n \subseteq \mathbb{X}^n$ is a convex polytope with non-obtuse dihedral angles. Then $P^n$ is simple.

proof strategy:

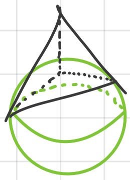
1.) Let $P^n \subseteq S^n$ convex polytope with non-obtuse angles. Then $P^n$ is an $n$-simplex.

analyze $f(v) := \langle u, v \rangle$ to deduce linear independancy.

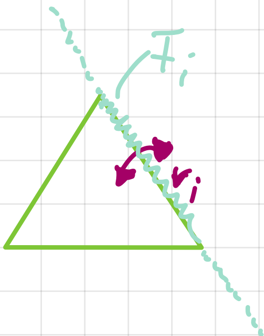
2.) $v$ vertex of $P^n$, $S(v)$ sphere with midpoint $v$

$\Rightarrow$ apply 1.) to $P^n \cap S(v)$

$\Rightarrow$ $P^n$ simple. ▨



## Setting for the universal construction




Let

- $P^n \subseteq \mathbb{X}^n$ be a convex polytope
- $(F_i)_{i \in I}$ its codim-1 faces.
- $(r_i)_{i \in I}$ be the isometric reflections of $\mathbb{X}$ across $F_i$.
- $W = \langle (r_i)_{i \in I} \rangle \leq \mathrm{Isom}(\mathbb{X}^{n+1})$

Furthermore, $P^n$ has to be simple!

Why?

- all dihedral angles have to be integral submultipicals of $\pi$ (i.e. $\pi / m_{ij}$ between $F_i$ and $F_j$)

$\Rightarrow$ all dihedral angles are non-obtuse ($m_{ij} \geq 2$)

$\overset{\text{Proposition}}{\Rightarrow}$ $P^n$ is simple.

"$P^n$ generates $W$"

If $F_i \cap F_j = \emptyset$, then $m_{ij} := \infty$, $m_{ii} = 1$

Thus $(m_{ij})_{i,j \in I}$ is the Coxeter matrix of the pre-Coxeter-system $(\overline{W}, (r_i)_{i \in I})$.

Let $(W,S)$ $S = \{(s_i)_{i \in I}\}$ the corresponding Coxeter system.

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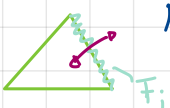


mirror structure on $P^n$ : ?

$\hookrightarrow$ mirror corresponding to $i$ is $F_i$

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Define the mapping

$$\begin{aligned} \phi : W &\longrightarrow \overline{W} \\ s_i &\longmapsto r_i \end{aligned}.$$

is a homomorphism(!) and surjective.

<u>Reminder</u>: Vinberg Theorem:

$\overline{W} \curvearrowright \mathbb{X}^n$ and consider $P^n \xhookrightarrow{i} \mathbb{X}^n$

$i((P^n)^S) \subseteq (\mathbb{X}^n)^S$ ✓. Then there ex. a (unique) extension

$$\begin{aligned} \mathcal{U}(W, P^n) &\xhookrightarrow{\tilde{i}} \mathbb{X}^n \\ [w, x] &\longmapsto \varphi(w)\, x \end{aligned}$$

## <u>Main Theorem</u>

Suppose $P^n$ is a simple convex polytope in $\mathbb{X}^n$ for $n \geq 2$ and let $W$ be generated by $P^n$.

Then the mapping

$$\tilde{i} : \mathcal{U}(W, P^n) \longrightarrow \mathbb{X}^n$$

is a homeomorphism.

## Main Corollary

This implies:

a) $\overline{W} \curvearrowright \mathbb{X}^n$ properly $\rightsquigarrow \overline{W} \leq \mathrm{Isom}(\mathbb{X}^n)$ discrete subgroup

b) $P^n$ is the (strict) fundamental domain of the $\overline{W}$-action. $\rightsquigarrow \mathbb{X}^n$ can be tiled by congruent copies of $P^n$.

### proof of the corollary:

The action is $\overline{W}$-finite.

$\Rightarrow$ statements follow from the talk last week. #

## Definition: Geometric reflection group

A geometric reflection group is a group $W$ with:

- $W \curvearrowright \mathbb{X}^n$
- $W$ is generated by a convex, simple polytope.

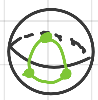
### to the proof of the main theorem:

- $P^n$ simple convex polytope
- $W$ is generated by $P^n$.

To show: $\tilde{i}: \mathcal{U}(W, P^n) \xrightarrow{\sim} \mathbb{X}^n$ is a homeomorphism.

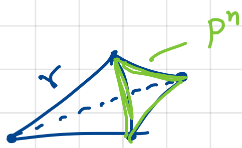
By induction on the the dimension $n$.

Some notation:



- $(S_n)$ is the claim when $\mathbb{X}^n = \mathbb{S}^n \rightsquigarrow P^n = \sigma^n$ spherical simplex
- $(E_n)$ is the claim when $\mathbb{X}^n$ is replaced by $B_r(x)$, $x \in \mathbb{X}^n$, $r > 0$

  $P^n$ is replaced by the open simplicial cone $C_r(x)$




Definition: simplicial cone: $C_r(x) = B_r(x) \cap P^n$.

• $(t_n)$ is the claim in dimension $n$.

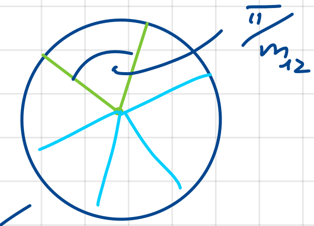
We show: $(c_n) \overset{2.)}{\Rightarrow} (t_n) \overset{\checkmark}{\Rightarrow} (s_n) \overset{1.)}{\Rightarrow} (c_{n+1})$

## Induction beginning $n=2$:

We start with $(c_2)$ i.e. in $\mathbb{X}^2$ we consider

$$W := \langle s_1, s_2 \mid s_i^2 = 1 \;\; i=1,2, \;\; (s_1 s_2)^{m_{12}} = 1 \rangle$$

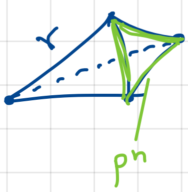
$$\Rightarrow \quad W = D_{2m_{12}}$$




Basic construction of $W, C_r(x)$:

Homeomorphism $\mathcal{U}(W, C_r(x)) \xrightarrow{\sim} B_r(x)$. ✓

## Step 1 $(s_n) \Rightarrow (c_{n+1})$



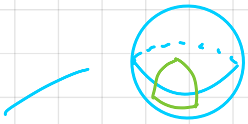

Suppose $C_r^{n+1} \subseteq \mathbb{X}^n$ is a simplicial cone of radius $r$ with nonobtuse dihedral angles $\pi/m_{ij}$.

Then, the Coxeter group associated to $C^{n+1}$ is the same as the Coxeter group associated to $\sigma^n$.

Why?

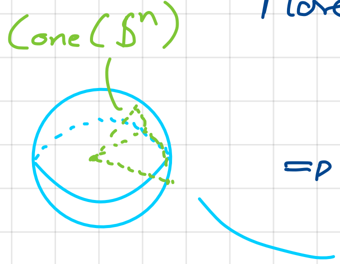
The dihedral angles coincide!

Moreover, there holds:



(1) $C^{n+1}$ is a cone on $\sigma^n$

$\Rightarrow$ (2) $\mathcal{U}(W, C^{n+1})$ is a cone on $\mathcal{U}(W, \sigma^n)$




(3) an open ball in $\mathbb{X}^{n+1}$ is a cone on $S^n$

$$\Rightarrow \quad \tilde{i}: \mathcal{U}(W, C^{n+1}) \xrightarrow{\sim} B_r(x) \subseteq \mathbb{X}^{n+1}$$

$$B_r(x) = \operatorname{Cone}(S^n) \cong \operatorname{Cone}(\mathcal{U}(W, \sigma^n)) = \mathcal{U}(W, C^{n+1})$$

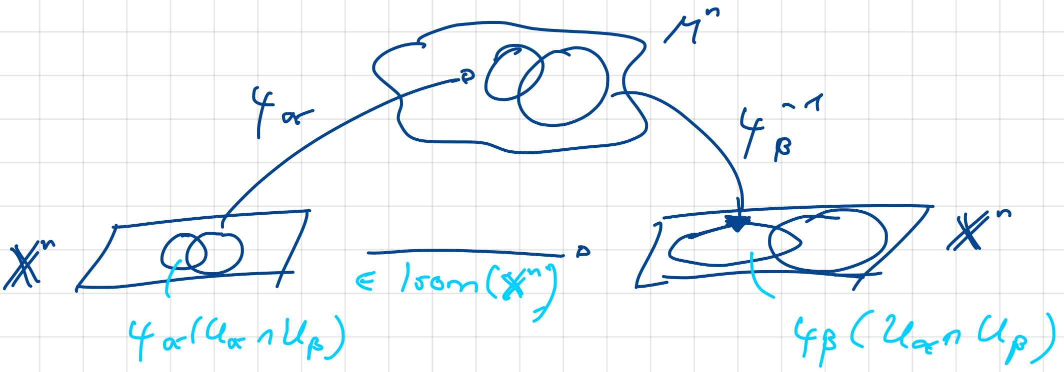
Step 2: $(c_{n+1}) \Rightarrow (t_{n+1})$

Before we start with this part, we need to introduce the definition of an $\mathbb{X}^{n+1}$-structure

Definition: A n-dim. topo. manifold $M^{n+1}$ has an $\mathbb{X}^{n+1}$-structure if it has an atlas $\{\varphi_\alpha : U_\alpha \longrightarrow \mathbb{X}^n\}$, where $(U_\alpha)_{\alpha\in A}$ is an open cover of $M^n$, $\varphi_\alpha$ homeomorphic onto its image and $\forall \alpha, \beta \in A$:

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is the restriction of an element of $\mathrm{Isom}(\mathbb{X}^n)$.

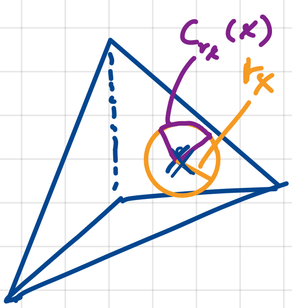



Claim: $\mathcal{U}(W, P^{n+1})$ has an $\mathbb{X}^{n+1}$-structure!

$\Rightarrow \tilde{\iota} : \mathcal{U}(W, P^{n+1}) \longrightarrow \mathbb{X}^{n+1}$ is a local isometry.

proof:

Let $x \in P^{n+1}$, let $S(x)$ denote the set of reflections $s_i$ across the codim-1 faces $F_i$ which contain $x$.




Moreover let $r_x$ denote the distance from $x$ to the nearest face which doesn't contain $x$.

Let $C_{r_x}(x)$ be an open simplicial cone.

By Jonas his Talk, $(W_{S(x)}, S(x))$ is a Coxeter system, so let's consider an open neighborhood of $[1,x]$ in $\mathcal{U}(W, P^n)$. This can be given by $\mathcal{U}(W_{S(x)}, C_{r_x}(x))$.

By step 1:

$$\mathcal{U}(W_{S(x)}, C_{r_x}(x)) \xrightarrow{\tilde{\iota}} B_{r_x}(x) \subseteq \mathbb{X}^n$$

is a homeomorphism. Since $\tilde{\iota}$ is $w$-equivariant we have for each $w \in W$

$$\varphi_{w,x} : w\,\underset{\ni [1,x]}{\mathcal{U}(W_{S(x)}, C_{r_x}(x))} \longrightarrow \underbrace{\underset{\in \mathrm{Isom}(\mathbb{X}^n)}{\varphi(w)}\, B_{r_x}(x)}_{\subseteq \mathbb{X}^n}$$

is also a homeomorphism.

With other words

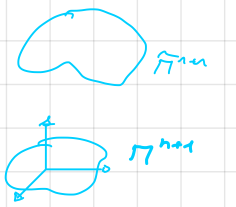
$$\left(w\,\mathcal{U}(W_{S(x)}, C_{r_x}(x))\right)_{x \in P^n,\, w \in W}$$

is an open cover for $\mathcal{U}(W, P^n)$ and $(\varphi_{x,w})_{x \in P^n,\, w \in W}$ is an atlas. One can calculate the chart change (!)

Thus, $\mathcal{U}(W, P^{n+1})$ has an $\mathbb{X}^{n+1}$-structure. #

Facts

- An $\mathbb{X}^{n+1}$-structure on $M^{n+1}$ $(= \mathcal{U}(W, P^{n+1}))$ induces one on its universal cover $\tilde{M}^{n+1}$.
- If $M^{n+1}$ is metrically complete then the developing map $D : \tilde{M}^{n+1} \longrightarrow \mathbb{X}^{n+1}$ is a universal covering map. $\rightsquigarrow$ local homeomorphism.



(*) Assume for a moment that $\cancel{\mathcal{U}(W, P^{n+1})}$ $P^{n+1}$ is metrically complete.

Since $\mathcal{U}(W, P^{n+1})$ is connected the developing map

$D: \widetilde{\mathcal{U}(W, P^{n+1})} \longrightarrow \mathbb{X}^{n+1}$ is locally given by

$$\hat{i} : \mathcal{U}(W, P^{n+1}) \longrightarrow \mathbb{X}^{n+1}.$$

Moreover, $\hat{i}$ is globally defined so $\hat{i}$ is a covering map.

Since $\mathbb{X}^n$ is simply connected we have

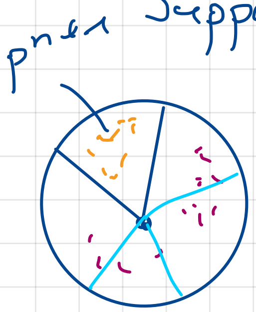
$$\mathcal{U}(\widetilde{W, P^{n+1}}) = \mathcal{U}(W, P^{n+1})$$

and $D = \tilde{i}$.

Hence $\tilde{i}$ is a global homeomorphism //

to (*)

Suppose $(x_k)_{k \in \mathbb{N}} \subseteq \cancel{\mathcal{U}(W, P^{n+1})}$ $P^{n+1}$ is a Cauchy-sequence



Since $\mathcal{U}(W, P^{n+1})/W = P^{n+1}$, for every $x_k$ there ex. $g_k \in W$ such that $g_k x_k \in P^{n+1}$.

Since $P^{n+1}$ is compact there ex. a convergent subsequence of $(g_k x_k)_{k \in \mathbb{N}}$, say $g_{k_j} x_{k_j} \xrightarrow[k_j \to \infty]{} gx$

We note that $W \curvearrowright \mathcal{U}(W, P^{n+1})$ isometrically and proper (talk last week)

Find some $(g_{k_0})^{-1}$ s.t. $x_{k_j}$ is convergent (!) #

## Selected examples:

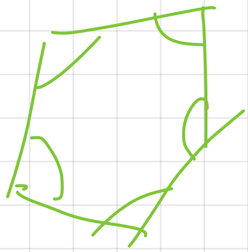
Let $P^2 \subseteq \mathbb{X}^2$ be an $m$-gon.

Now, the (local) Gauß-Bonnet can be applied, i.e

$$\Rightarrow \quad \varepsilon \cdot \mathrm{Area}(P^2) \quad + \quad 0 \quad + \sum_{i \le m} (\pi - \alpha_i) = 2\pi(m - m + 1) = 1 \cdot 2\pi$$

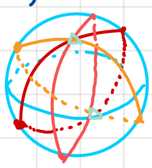
with $\varepsilon \in \{-1, 0, 1\}$ in dependancy of the choice of $\mathbb{X}^n$.

$$\Rightarrow \quad \sum_{i \le m} \alpha_i \overset{<}{\underset{>}{=}} (m-2)\pi \qquad (*)$$



## Example 1: spherical case

Let $P^2 \subseteq \mathbb{S}^2$ be a spherical polygon. Therefore $\alpha_i \le \frac{\pi}{2}$. Why?



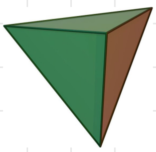
Otherwise, two great-circle would meet on the other side and there they would have an intersecting angle $< \frac{\pi}{2}$ ↯

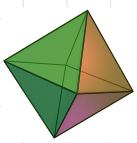
By $(*)$ it follows $m < 4$.

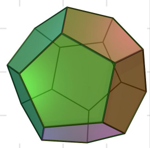
Therefore $\frac{\pi}{m_1} + \frac{\pi}{m_2} + \frac{\pi}{m_3} > \pi$.

// Assume that $m_i$ $i = 1, 2, 3$ are integers

Some calculations shows that the triplets

$(2, 3, 3)$ $\updownarrow$ 

$(2, 3, 4)$ $\updownarrow$ 

$(2, 3, 5)$ $\updownarrow$ 

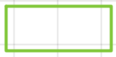
$(2, 2, n)$

solve $(*)$.

## 2.) Euclidian case

Therefore we consider the equation $\sum_{i \le m} \alpha_i = (m-2)\pi$

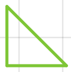
Since $\alpha_i \le \frac{\pi}{2} \Rightarrow m \le 4 \Rightarrow m_1 = m_2 = m_3 = m_4 = 2$ ✓

$\rightsquigarrow$ standard rectangular tiling of $E^2$. 

So let $m = 3$. An analogue calculation as above shows that the equation

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 1$$

is solved by

$(2,4,4)$ 

$(2,3,6)$ 

$(3,3,3)$

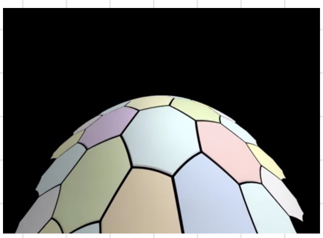
$\rightsquigarrow$ Corresponding reflection groups are called Euclidean reflection groups.

## 3.) Hyperbolic case:

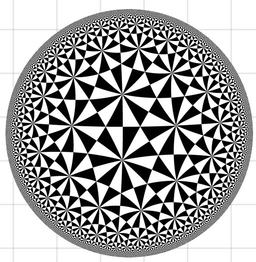
We have to solve

$$\frac{1}{m_1} + \dots + \frac{1}{m_k} < 1 .$$

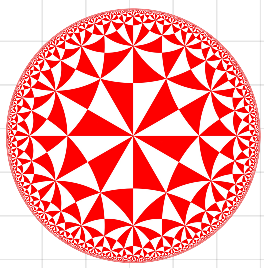
Thus, there ex. infinitly many tupels that solve the inequality above.



$(7,3)$ – tiling



Disk model with $(2,3,7)$ – tiling



$(6,4,2)$ tiling $\rightsquigarrow$ M.C. Escher.