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WS 2012/13

10. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of December 17 2012 before the lecture.

Aufgabe 10.1 (Proper actions of subgroups of $(\mathbb{R}^m, +)$)

(3 marks) Suppose that $H \subseteq (\mathbb{R}^m, +)$ is a subgroup. If the action of H on $\mathbb{R}^m$ by translations is proper, show that H is free abelian and of $\mathbb{Q}$-rank at most m.

Aufgabe 10.2 (Normalizers and fixed points)

(a) (1 mark) Suppose that a group G acts on a set X and that $N \trianglelefteq G$ is normal. If N has a nonempty set of fixed points $F \subseteq X$, show that G leaves F invariant.

(b) (2 marks) Let G be a group and $H \subseteq G$ a subgroup of finite index. Show that there is a finite index normal subgroup $N \subseteq G$ which is contained in H.
Hint: Consider the action of G on the finite set G/H.

(c) (2 marks) Suppose that G acts as a group of isometries on a complete CAT(0) space X, and that $H \subseteq G$ is a subgroup of finite index. If H has a fixed point, show that G has also a fixed point.

Aufgabe 10.3 (Group actions by elliptic isometries without a common fixed point)

(a) (3 marks) Let T be a $\mathbb{Z}$-tree (as in Problem 9.2) in which every branch point has exactly three geodesic segments emerging from it, with another branch point on each geodesic segment at distance one from the first branch point. Such a tree is called a regular ternary tree.

Show that there exists a countably infinite family of isometries of T with the property that:

(i) Each isometry is an involution; that is, has order 2;
(ii) The group G of isometries generated by the family of involutions consists entirely of elliptic isometries;
(iii) Every finitely generated subgroup of G has a common fixed point;
(iv) The group G has no common fixed point.

Hint: Consider one geodesic line l in T, with denumerably many branch points on it, and a subtree of T emerging from each branch point. Choose your denumerable family of involutions in such a way that there is a natural one-to-one correspondence between the involutions and the branch points on l.

(b) (2 marks) Show further that it is possible to arrange for G to be abelian.