Prof. Dr. Linus Kramer
Dr. Rupert McCallum
Dr. Daniel Skodlerack
Antoine Beljean

WS 2012/13

5. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of November 12 2012 before the lecture.

Aufgabe 5.1 (Existence of geodesics)

(4 marks) (a) Let X be a complete metric space. Suppose that for all $x, y \in X$, there exists a midpoint, that is, a point $m \in X$ such that

$$d(x,m) = d(m,y) = \frac{1}{2}d(x,y).$$

Show that X is a geodesic space.

(2 marks) (b) Show that the metric completion of a CAT(0) space is geodesic.

(2 marks) (c) A metric space X is said to be locally geodesic if, for every $p \in X$, there exists an open neighborhood of p which is a geodesic space. Can you give an example of a path-connected locally geodesic space which is not a geodesic space?

Aufgabe 5.2 (Products) Given metric spaces $(X_1, d_1), \ldots, (X_m, d_m)$, define the metric product as $X = X_1 \times \cdots \times X_m$ with metric

$$d((u_1, \ldots, u_m), (v_1, \ldots, v_m)) = \sqrt{\sum_j d_j(u_j, v_j)^2}$$

(1 mark) (a) Prove that this is a metric.

(1 mark) (b) The product is complete if and only if all the X_j are complete.

A metric space X is said to be proper if, given any $p \in X$ and any real number $r > 0$, the closed ball $\bar{B}_r(p)$ is compact.

(1 mark) (c) The product is proper if and only if all the X_j are proper.

(2 marks) (d) The product is CAT(0) if and only if all the X_j are CAT(0).

Aufgabe 5.3 (Embeddings of finite metric spaces)

(1 mark) (a) Show that every metric space X consisting of at most three points admits an isometric embedding into $(\mathbb{R}^2, \|-\|_2)$.

(2 marks) (b) Can every metric space consisting of four points be isometrically embedded into $(\mathbb{R}^n, \|-\|_2)$, for some $n \geq 1$?

(1 mark) (c) Can every metric space consisting of finitely many points be isometrically embedded into $(\mathbb{R}^n, \|-\|_p)$, for some $\infty \geq p \geq 1$ and $n \geq 1$?

(2 marks) (d) Does there exist a metric space in which every finite metric space can be isometrically embedded?

Aufgabe 5.4 (Fixed points)

(1 mark) (a) Let G be a group acting isometrically on a metric space X. Show that the fixed point set X^G is closed.

(1 mark) (b) Assume in addition that X is CAT(0). Show that the fixed point set X^G is convex (assuming it is not empty).

(2 marks) (c) Let X be a complete CAT(0) space and $f : X \longrightarrow X$ a 1-Lipschitz map. The f-orbit of x is the set $E = \{f(x), f(f(x)), f(f(f(x))), \ldots\}$. If some f-orbit is bounded, show that f has a fixed point.
Hint: put $E_0 = E$ and $E_{n+1} = f(E_n)$. Consider the centers z_n and radii r_n of the E_n.