Prof. Dr. Linus Kramer
Dr. Rupert McCallum
Dr. Daniel Skodlerack
Antoine Beljean

WS 2012/13

6. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of November 19 2012 before the lecture.

Aufgabe 6.1 ($\mathbb{R}$-trees)

A metric space (T, d) is said to be an $\mathbb{R}$-tree if, given any two distinct points $x, y \in T$, there exists a unique geodesic from x to y, and if a concatenation of two geodesics is again a geodesic, provided that it is injective. (This means: if $\gamma : [a, b] \longrightarrow T$ is injective and if the restrictions of γ to $[a, z]$ and $[z, b]$ are geodesics, for some $a < z < b$, then γ is already a geodesic.)

(a) (2 marks) Prove that every $\mathbb{R}$-tree is CAT(0).

(b) (2 marks) Consider the metric space $T = \mathbb{R}^2$ with the metric

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2| + |y_1| + |y_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$

Prove that this is a complete $\mathbb{R}$-tree.

(c) (2 marks) Prove that the completion of an $\mathbb{R}$-tree is an $\mathbb{R}$-tree.

Aufgabe 6.2 (The Lorentz group)

(3 marks) Let $\mathrm{O}_{1,n}\mathbb{R}$ be the group of matrices which preserve the form β on $\mathbb{R} \oplus \mathbb{R}^n$ such that

$$\beta(u, v) = u_0 v_0 - \langle \tilde{u}, \tilde{v} \rangle = u_0 v_0 - \sum_{j=1}^{m} u_j v_j.$$

Consider the subgroup of $\mathrm{O}_{1,n}\mathbb{R}$ consisting of those elements $g \in \mathrm{O}_{1,n}\mathbb{R}$ which have positive determinant, and furthermore if $u_0 > 0$, $\beta(u, u) > 0$, then $g(u) = (v_0, \ldots, v_m)$ has $v_0 > 0$. Prove that this is indeed a subgroup. Show that this subgroup has index four in the group $\mathrm{O}_{1,n}\mathbb{R}$ and is equal to the group $\Omega_{1,n}$ described in the lectures.
Hint: Consider the stabilizer of the subspace $\mathbb{R}(1, 0, \ldots, 0)$. The following observation may be useful. If a group G acts transitively on a set X and if $H \subseteq G$ is a transitive subgroup, then $G = H$ holds if and only if $G_x = H_x$ holds for some $x \in X$.

Aufgabe 6.3 (Nested sequences of bounded convex sets)

Suppose that $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n \supseteq \ldots$ is a nested sequence of nonempty bounded closed convex sets in a complete CAT(0) space X.

(a) (2 marks) Prove that the center of C_i is a member of C_i.

(b) (2 marks) Prove that $\bigcap_{j=1}^{\infty} C_j \neq \emptyset$.