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WS 2012/13

8. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of December 3 2012 before the lecture.

Aufgabe 8.1 (The flat rank of hyperbolic space)

(3 marks) Let $D_r = \{v \in \mathbb{R}^2 \mid \|v\|_2 \leq r\}$. Show that there is no isometric embedding $D_r \longrightarrow H^m$ for any $r > 0$. Conclude that the hyperbolic space H^m has flat rank 1, for all $m \geq 1$.

Aufgabe 8.2 (Parallel 1-flats)

(a) (2 marks) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function. Show that f is constant if it is bounded.

(b) (2 marks) Let E, E' be 1-flats in a CAT(0) space X. Show that E and E' are parallel if and only if there exists an $r > 0$ such that $E' \subseteq \bigcup\{B_r(v) \mid v \in E\}$.

(c) (1 mark) Show that parallelism is an equivalence relation for 1-flats in a CAT(0) space.

Aufgabe 8.3 (Angles)

(a) (3 marks) Let γ, γ' be nonconstant geodesics in a CAT(0) space, with $\gamma(0) = \gamma'(0) = p$. If $\measuredangle_p(\gamma, \gamma') < \pi/2$, show that there exists $s, t > 0$ with $d(\gamma(s), \gamma'(t)) < s$.

(a) (1 mark) Let C be a complete convex set in a CAT(0) space. Suppose that $q = \mathrm{proj}_C(p) \neq p$ and that $q \neq c \in C$. Prove that

$$\measuredangle_q(p, c) \geq \pi/2.$$

Aufgabe 8.4 (Flat triangles)

(3 marks) Let $\Delta(a, b, c, \alpha, \beta, \gamma)$ be a geodesic triangle in a CAT(0) space X, with euclidean comparison triangle $\Delta(\bar{a}, \bar{b}, \bar{c}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$. Suppose that

$$d(a, \alpha(t)) = \|\bar{a} - \bar{\alpha}(t)\|_2$$

holds for some t with $0 < t < d(b, c)$. Show that the triangle is flat.