PROJECTIVITIES OF GENERALIZED POLYGONS

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In this paper we determine the groups of projectivities of all finite Moufang polygons. Generalized polygons have been introduced by J. Tits in [9], where he also defined perspectivities and projectivities of these structures. We start with the definition of projectivities of generalized polygons and derive some of their basic properties. The groups of projectivities of generalized polygons turn out to be doubly transitive permutation groups. These groups are not in general triply transitive; simple counterexamples are provided by the finite generalized quadrangles associated to orthogonal polarities in four-dimensional projective spaces of odd characteristic. In section 2 we study Moufang polygons and show that all their even projectivities are induced by collineations. For the finite Moufang polygons, this information allows to determine explicitly the groups of projectivities.

1. GENERALIZED POLYGONS AND THEIR PROJECTIVITIES.

Let $\mathcal{P}=(P,L,I)$ be an incidence structure, i.e. P and L are disjoint sets and I is a subset of $P\times L$. Instead of $(x,y)\in I$ we usually write xIy. Put $V=P\cup L$. The elements of V are called vertices. Let s be any natural number. An s-path in $\mathcal{P}$ is a sequence $\Sigma=(x_0,\dots,x_s)$ of vertices x_i such that $x_i I x_{i+1}$ and $x_i\neq x_{i+2}$ for $i=0,\dots,s-1$. We say that Σ joins x_0 and x_s. Two vertices x and y are at distance s, denoted as $d(x,y)=s$, if there is an s-path joining x and y and s is minimal w.r.t. this property. We set $\Gamma_i(x):=\{y\in V\,|\,d(x,y)=i\}$ for $x\in V$ and $i\in\mathbb{N}$. Instead of $\Gamma_1(x)$ we usually write $\Gamma(x)$.

$\mathcal{P}$ is a generalized n-gon if the following hold:

(1) For all $x,y\in V$ we have $d(x,y)\leq n$.

(2) If $d(x,y)=s<n$, then the s-path joining x and y is unique.

(3) We have $|\Gamma(x)|\geq 3$ for all $x\in V$.

Sometimes these structures are called thick generalized n-gons where (3) is referred to as thickness. The definition of generalized n-gon is obviously selfdual; the dual of a generalized n-gon $\mathcal{P}$ will be denoted by $\mathcal{P}^*$. Let $x,y\in V$ with $d(x,y)=n$. For all $z\in\Gamma(x)$ there is a unique (n-1)-path $(z=x_0,\dots,x_{n-2},x_{n-1}=y)$ joining z and y. Define $[x,y](z)=x_{n-2}$. The mapping

ARS COMBINATORIA 25B(1988), pp. 265-275.

$$[x,y]: \Gamma(x) \to \Gamma(y)$$

defined in this way is called the perspectivity from x to y. Given a sequence $x_0,\dots,x_k$ of vertices with $d(x_i,x_{i+1})=n$ for $i=0,\dots,k-1$ one can form the product of the perspectivities $[x_0,x_1],\dots,[x_{k-1},x_k]$ in this order. We denote this product by $[x_0,x_1,\dots,x_k]$ and call it a projectivity. For $x_0=x_k=x$ such a projectivity is a bijection of $\Gamma(x)$. The set of all these bijections forms a group $\Pi(x)$ which we call the group of projectivities of x. A projectivity is called even if it is the product of an even number of perspectivities. The even projectivities in $\Pi(x)$ form a subgroup $\Pi^+(x) \leq \Pi(x)$. For odd n we always have $\Pi^+(x) = \Pi(x)$, whereas $\Pi^+(x)$ is a normal subgroup of $\Pi(x)$ of index at most 2 for even n.

The following Lemma is obvious.

LEMMA 1.1: Let $x,y \in V$ and let $\sigma: \Gamma(x) \to \Gamma(y)$ be a projectivity. Then we have $\Pi(y) = \Pi(x)^\sigma$ and $\Pi^+(y) = \Pi^+(x)^\sigma$.

Thus for odd n there is, up to isomorphism of permutation groups, just one group of projectivities; this group will be denoted by $\Pi(\mathcal{P})$. For even n there are four groups to be considered. $\Pi(\mathcal{P})$ resp. $\Pi^+(\mathcal{P})$ denotes the abstract permutation group which is isomorphic to $\Pi(g)$ resp. $\Pi^+(g)$ for some line g of $\mathcal{P}$. As the points of $\mathcal{P}$ are the lines of $\mathcal{P}^*$, the abstract permutation groups $\Pi(p)$ and $\Pi^+(p)$ for a point p of $\mathcal{P}$ are denoted by $\Pi(\mathcal{P}^*)$ and $\Pi^+(\mathcal{P}^*)$.

LEMMA 1.2: The groups $\Pi(x)$ and $\Pi^+(x)$ are doubly transitive permutation groups for all $x \in V$.

PROOF: Certainly $\Pi^+(x)$ has no fixed points. Hence it suffices to show that for all $p \in \Gamma(x)$ the stabilizer $\Pi^+(x)_p$ is transitive on $\Gamma(x)\setminus\{p\}$. Choose $y \in \Gamma(p)\setminus\{x\}$ and $z \in V$ with $d(x,z)=d(y,z)=n$. Let $a,b \in \Gamma(x)\setminus\{p\}$, and define $c:=[x,z,y](a)$. There is a $(2n-4)$-path $(b=a_0,a_1,\dots,a_{2n-4}=c)$ joining b and c. Choose $w \in \Gamma(a_{n-2})\setminus\{a_{n-3},a_{n-1}\}$. Then $[x,z,y,w,x]$ fixes p and maps a to b.

In general neither $\Pi^+(x)$ nor $\Pi(x)$ will be triply transitive, cp. Th.3.1.

Let $\sigma: V \to V$ be a collineation of $\mathcal{P}$. A line $g \in L$ is called an axis of σ if σ fixes every point incident with g. A center of a collineation is defined dually.

LEMMA 1.3: Let $\sigma: V \to V$ be a collineation of $\mathcal{P}$. Assume that σ possesses an axis $g \in L$. Then the restriction of σ to $\Gamma(h)$ is an even projectivity for all lines $h \in L$.

PROOF: Let $x \in V$ with $d(x,g)=d(x,h)=n$. Then $\sigma|_{\Gamma(x)}$ coincides with the projectivity $[x,g,\sigma(x)]$, and $\sigma|_{\Gamma(h)}$ coincides with $[h,x,g,\sigma(x),\sigma(h)]$.

We will now derive some conditions which assure that $\Pi^+(x)$ and $\Pi(x)$ coincide. Let $\mathcal{P}$ be a generalized n-gon with $n=2m$ even. Let $x,y \in V$ with $d(x,y)=2m$. The span of x and y is the set

$$sp(x,y):=\{z \in V \mid \Gamma_m(x)\cap\Gamma_m(z)=\Gamma_m(x)\cap\Gamma_m(y)\} \cup \{x,y\}.$$

The pair (x,y) is called regular if the following holds:

$$sp(x,y)=\Gamma_m(a)\cap\Gamma_m(b) \quad \text{for all} \quad a,b\in\Gamma_m(x)\cap\Gamma_m(y)\ ,\ a\neq b.$$

$\mathcal{P}$ is called regular if any pair of lines of maximum distance is regular.

LEMMA 1.4: Let $x,y \in V$ with $d(x,y)=n$. Assume that $|sp(x,y)|\geq 3$. Then we have $\Pi^+(x)=\Pi(x)$.

PROOF: Let $z \in sp(x,y)\setminus\{x,y\}$. It follows from the definition of $sp(x,y)$ that $[x,y]=[x,z,y]$. Thus the identity in $\Pi(x)$ can be written as a product of three perspectivities, and the result follows.

COROLLARY 1.5: Assume that $\mathcal{P}$ is regular. Then we have $\Pi(\mathcal{P}) = \Pi^+(\mathcal{P})$.

2. MOUFANG POLYGONS.

The notations and definitions in this section are taken from [13] and [14]. Let $\mathcal{P}$ be a generalized n-gon. The n-paths of $\mathcal{P}$ are called roots, and the closed (2n-1)-paths, i.e. paths $(x_0,\dots,x_{2n-1})$ with $x_0 \, I \, x_{2n-1}$, are called apartments. Let $\Phi=(x_0,\dots,x_n)$ be a root of $\mathcal{P}$. A collineation τ of $\mathcal{P}$ is called a root automorphism w.r.t. Φ if τ fixes all vertices in $\Gamma(x_1)\cup\dots\cup\Gamma(x_{n-1})$. The group of all root automorphisms w.r.t. Φ is denoted by $U(\Phi)$. This group acts semiregularly on the apartments containing Φ; if the action is even regular $U(\Phi)$ is said to be a transitive root group. $\mathcal{P}$ is called a Moufang n-gon if all root groups of $\mathcal{P}$ are transitive. If $\mathcal{P}$ is a Moufang polygon, then the group generated by all root automorphisms is denoted by $S(\mathcal{P})$; we call it the little projective group of $\mathcal{P}$.

Let $\mathcal{P}$ be a Moufang n-gon and let $\Sigma=(x_0,\dots,x_{2n-1})$ be a fixed apartment of $\mathcal{P}$. Put $\Phi_i:=(x_i,\dots,x_{i+n})$ for $i\in\mathbb{N}$, where indices are taken mod 2n. Define $U_i:=U(\Phi_i)$ and $U_{[i,j]}=\langle U_k \mid i\leq k\leq j\rangle$.

LEMMA 2.1: $U_{[1,n-1]}$ acts transitively on $\Gamma_n(x_n)$.

PROOF: Let $y \in \Gamma_n(x_n)$. There is an n-path $(x_n, y_1, \dots, y_n = y)$ joining x_n and y. Define inductively a sequence of mappings $\tau_i \in U_i$, $i = 1, \dots, n$, as follows: τ_1 is the unique element in U_1 which maps y_1 to x_{n+1}. If τ_i has been constructed, then τ_{i+1} is the unique element in U_{i+1} which maps $(\tau_1 \circ \dots \circ \tau_i)(y_{i+1})$ to x_{n+1}. Now $\tau = \tau_1 \circ \dots \circ \tau_n$ maps $y_n = y$ to $x_{2n} = x_0$.

LEMMA 2.2: Let $\mathcal{P}$ be a Moufang n-gon, let $x, y \in V$ and let $\sigma : \Gamma(x) \to \Gamma(y)$ be an even projectivity. Then there exists a collineation $\tau \in S(\mathcal{P})$ such that $\sigma = \tau|_{\Gamma(x)}$.

PROOF: It suffices to prove the assertion for products of two perspectivities. Let $\sigma = [x,a,y]$, where $d(x,a) = d(y,a) = n$. From Lemma 2.1 we infer that there exists a collineation $\tau \in S(\mathcal{P})$ which maps x to y and which fixes all vertices incident with a. (Notice that all elements of $U_{[1,n-1]}$ fix $\Gamma(x_n)$ elementwise.) Hence $\sigma = \tau|_{\Gamma(x)}$.

The next result follows immediately from Lemma 1.3 and Lemma 2.2.

PROPOSITION 2.3: Let $\mathcal{P}$ be a Moufang polygon. Then $\Pi^+(\mathcal{P})$ is permutation isomorphic to the group which is induced by the stabilizer of a line in $S(\mathcal{P})$ on the points incident with that line.

3. FINITE MOUFANG POLYGONS.

All finite Moufang polygons have been determined by Fong and Seitz [3]. They are all associated with groups of Lie type.

Let $\mathcal{P}$ be a finite Moufang n-gon and denote by S its little projective group. Let $\Sigma = (x_0, \dots, x_{2n-1})$ be a fixed apartment of $\mathcal{P}$. Assume that x_1 is a line. It follows from Lemma 2.1 that S_{x_1} and $S_{x_1, x_{n+1}}$ induce the same permutation group on $\Gamma(x_1)$. By [3: § 7] the group $H = S_{x_1, x_{n+1}, x_0, x_2}$ is abelian. It follows immediately that the two point stabilizer in $\Pi^+(\mathcal{P})$ is abelian. (This result may be viewed as geometric version of an Artin-Zorn Theorem for generalized polygons.) From [3: 7F] we infer that $\Pi^+(\mathcal{P})$ is one of the groups PSL(2,q), PGL(2,q), PSU(3,q), PGU(3,q) or Sz(q).

A finite generalized polygon has order (s,t), if each line is incident with s+1 points and each point is incident with t+1 lines.

THEOREM 3.1: The groups of projectivities of the finite Moufang polygons are given by the following table:

$_n, y_1, \ldots, y_n = y)$ joining x_n and
s $\tau_i \in U_i$, $i=1,\ldots,n$, as fol-
n maps y_1 to x_{n+1}. If τ_i has
e element in U_{i+1} which maps
n maps $y_n = y$ to $x_{2n} = x_0$.
$x, y \in V$ and let $\sigma : \Gamma(x) \to \Gamma(y)$
a collineation $\tau \in S(\mathcal{P})$ such

for products of two perspec-
a)=n. From Lemma 2.1 we infer
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of the finite Moufang polygons

n	$\mathcal{P}$	$S(\mathcal{P})$	(s,t)	$\Pi^+(\mathcal{P})$	$\Pi(\mathcal{P})$
3	PG(2,q)	PSL(3,q)	(q,q)	PGL(2,q)	PGL(2,q)
4	W(q)	PSP(4,q)	(q,q)	PGL(2,q)	PGL(2,q)
	Q(4,q)	$P\Omega(5,q)$	(q,q)	PSL(2,q)	PSL(2,q)
	H(3,q)	PSU(4,q)	(q^2,q)	$PSL(2,q^2)$	$PSL(2,q^2) \rtimes \langle\sigma\rangle$
	Q(5,q)	$P\Omega(6,q)$	(q,q^2)	PGL(2,q)	PGL(2,q)
	H(4,q)	PSU(5,q)	(q^2,q^3)	$PGL(2,q^2)$	$PGL(2,q^2) \rtimes \langle\sigma\rangle$
	H(4,q)*	PSU(5,q)	(q^3,q^2)	PGU(3,q)	PGU(3,q)
6	H(q)	$G_2(q)$	(q,q)	PGL(2,q)	PGL(2,q)
	H(q)*	$G_2(q)$	(q,q)	PGL(2,q)	PGL(2,q)
		${}^3D_4(q)$	(q^3,q)	$PGL(2,q^3)$	$PGL(2,q^3)$
		${}^3D_4(q)$	(q,q^3)	PGL(2,q)	PGL(2,q)
8		${}^2F_4(q)$	(q,q^2)	PGL(2,q)	PGL(2,q)
		${}^2F_4(q)$	(q^2,q)	Sz(q)	Sz(q)

The names for the generalized quadrangles are taken from [5]. There seem to be no special names for the generalized polygons associated with the groups ${}^3D_4(q)$ and ${}^2F_4(q)$. The table consists of dual pairs, with the exception of the first line. The number q denotes an arbitrary prime power; only in the n=8 case q is restricted to an odd power of 2. All groups act in their natural doubly transitive permutation representation. σ denotes the unique involutorial automorphism of a field with q^2 elements.

PROOF: a) Generalized octagons.

Let $\mathcal{P}$ be the Moufang octagon of order (q,q^2) which is associated with ${}^2F_4(q)$. Let $\Sigma = (x_0, \ldots, x_{15})$ be an apartment of $\mathcal{P}$ and assume that x_1 is a line. The group $\langle U_1, U_9 \rangle$ is isomorphic to SL(2,q) and acts effectively on $\Gamma(x_1)$. Similarly $\langle U_0, U_8 \rangle$ is isomorphic to Sz(q) and acts effectively on $\Gamma(x_0)$. This is proved in [10]. Because q is a power of 2, the groups SL(2,q) and PGL(2,q) coincide, and we conclude that $\Pi^+(\mathcal{P}) = PGL(2,q)$ and $\Pi^+(\mathcal{P}^*) = Sz(q)$.

The group of outer automorphisms of PGL(2,q) as well as Sz(q) is isomorphic to the automorphism group of $\mathbb{F}_q$. As q is an odd power of 2, this group has odd order. Thus neither $\Pi^+(\mathcal{P})$ nor $\Pi^+(\mathcal{P}^*)$ are subgroups of index 2 in larger permutation groups, and we have $\Pi(\mathcal{P}) = \Pi^+(\mathcal{P})$ and $\Pi(\mathcal{P}^*) = \Pi^+(\mathcal{P}^*)$.

b) Generalized hexagons.

Let $\mathcal{P}=(P,L,I)$ be a finite Moufang hexagon of order (s,t), and let S be its little projective group. Up to duality, we have either $(s,t)=(q,q)$ and $S=G_2(q)$, or $(s,t)=(q^3,q)$ and $S={}^3D_4(q)$. Let $x,y\in P$ with $d(x,y)=6$ and let $a,b\in\Gamma(x)$ with $a\neq b$. We want to show that both $\Pi^+(x)$ and $\Pi^+(a)$ are triply transitive. The following equations are well-known:

$$|S|=s^3t^3(s^2-1)(t^2-1)(1+st+s^2t^2)$$
$$|P|=(s+1)(1+st+s^2t^2)$$
$$|\Gamma_6(x)|=s^3t^2.$$

We conclude that the group $H=S_{x,y,a,b}$ has order $(s-1)(t-1)$. Let z be the unique point in $\Gamma(a)\cap\Gamma_4(y)$. Assume that $\Pi^+(x)$ or $\Pi^+(a)$ is not triply transitive. Then we can find $w\in\Gamma(a)\setminus\{x,z\}$ and $c\in\Gamma(x)\setminus\{a,b\}$ such that $H_{w,c}$ contains an element $\psi\neq id$. The action of ψ on $\Gamma(a)$ resp. $\Gamma(x)$ is equivalent to the action of an element of $PGL(2,s)$ resp. $PGL(2,t)$, hence ψ induces the identity on both sets. But then ψ is the identity on P and on L, and we arrive at a contradiction. Thus we have $\Pi^+(\mathcal{P})=PGL(2,s)$ and $\Pi^+(\mathcal{P}^*)=PGL(2,t)$. By a result of Ronan [7: 5.9] all Moufang hexagons are regular, thus it follows from Corollary 1.5 that $\Pi(\mathcal{P})=\Pi^+(\mathcal{P})$ and $\Pi(\mathcal{P}^*)=\Pi^+(\mathcal{P}^*)$.

c) Generalized quadrangles.

Generalized quadrangles are not so easy to handle as generalized hexagons and octagons. This is mainly due to the fact that the Schur multiplier of the little projective group of a finite Moufang quadrangle is not trivial. This leads to smaller groups H; indeed, the table above shows that H does not always act transitively on $\Gamma(x)\setminus\{a,b\}$.

Thus we approach generalized quadrangles via their embeddings into polarities of projective spaces. Fortunately, for finite quadrangles we only have to consider symplectic and unitary polarities.

Let V be a vector space of dimension $d\geq 4$ over a field F and let $f:V\times V\to F$ be a nondegenerate sesquilinear form of index 2. Define an incidence structure $\mathcal{P}=(P,L,I)$ by taking as points and lines all 1-dimensional resp. 2-dimensional totally isotropic subspaces of V with the natural incidence. Then $\mathcal{P}$ is a Moufang quadrangle, and, up to duality, all finite Moufang quadrangles arise in this way. There are three cases to be considered:

(1) $d=4$ and f symplectic

(2) $d=4$ and f unitary

exagon of order (s,t), and let S be
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$a)\backslash\{x,z\}$ and $c \in \Gamma(x)\backslash\{a,b\}$ such that
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(3) d=5 and f unitary

In case (1) we put $|F|=q$, and in the other two cases we put $|F|=q^2$. Denote by G the group $Sp(4,q)$ in case (1), $SU(4,q)$ in case (2) and $SU(5,q)$ in case (3).

First we determine the groups $\Pi^+(\mathcal{P})$ and $\Pi(\mathcal{P})$. Let g and h be lines of $\mathcal{P}$ with $d(g,h)=4$. It follows from Lemma 2.1 that both G_g and $G_{g,h}$ induce $\Pi^+(g)$ on $\Gamma(g)$. We can choose a basis $(e_1,\dots,e_d)$ of V such that the following hold:

$$g=\langle e_1,e_2\rangle,\quad h=\langle e_3,e_4\rangle,\quad (g\oplus h)^\perp=\langle e_5\rangle \text{ in case (3)},$$
$$f(e_1,e_3)=f(e_2,e_4)=1, \quad\text{and}\quad f(e_1,e_4)=f(e_2,e_3)=0.$$

The matrix of f w.r.t. this basis looks as follows:

$$\begin{pmatrix} & I & \\ \varepsilon I & & \\ & & J \end{pmatrix},$$

where I denotes the 2×2 identity matrix, and $\varepsilon=+1$ if f is unitary and $\varepsilon=-1$ if f is symplectic. In case (3) we can choose J=1, whereas in the other two cases J is considered to be absent. Now every element of $G_{g,h}$ is described by a matrix of the form

$$M = \begin{pmatrix} A & & \\ & B & \\ & & c \end{pmatrix},$$

where $A,B \in GL(2,F)$ and $c \in F^*$ in case (3); again c is considered to be absent in the other cases.

If f is symplectic, then the condition for M to lie in $G_{g,h}$ reads

$$\begin{pmatrix} A & \\ & B \end{pmatrix}\begin{pmatrix} & I \\ -I & \end{pmatrix}\begin{pmatrix} A^t & \\ & B^t \end{pmatrix}=\begin{pmatrix} & I \\ -I & \end{pmatrix},$$

i.e. $AB^t=I$. It follows at once that we have $\Pi^+(g)=PGL(2,F)$ in this case. Let $\lambda,\mu \in F$, then

$$(\lambda e_1+\mu e_2)^\perp=\langle e_1,e_2,\mu e_3-\lambda e_4\rangle \quad\text{and}$$
$$[g,h](\langle\lambda e_1+\mu e_2\rangle)=\langle\mu e_3-\lambda e_4\rangle.$$

The linear mapping described by the symplectic matrix

$$\begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

maps g to h and induces the perspectivity $[g,h]$, hence $\Pi(g)$ and $\Pi^+(g)$ coincide.

If f is unitary, then M lies in $G_{g,h}$ if

$$\begin{pmatrix} A & & \\ & B & \\ & & c \end{pmatrix}\begin{pmatrix} & I & \\ I & & \\ & & J \end{pmatrix}\begin{pmatrix} \bar{A}^t & & \\ & \bar{B}^t & \\ & & \bar{c} \end{pmatrix} = \begin{pmatrix} & I & \\ I & & \\ & & J \end{pmatrix} \tag{1}$$

and

$$\det A \cdot \det B \cdot c = 1 \ , \tag{2}$$

where in case d=4 we put c=1. Now (1) is equivalent to

$$A\bar{B}^t = I \tag{3}$$

and

$$c\bar{c} = 1 \ . \tag{4}$$

From (3) we get

$$\det(A\bar{B}^t) = \det A \cdot \overline{\det B} = 1 \ , \tag{5}$$

und substituting (5) into (2) leads to

$$\det A \cdot \overline{\det A}^{-1} \cdot c = 1 \ . \tag{6}$$

Assume that F has order q^2. In case d=4 we obtain from (6) that detA is always contained in the subfield of F of order q. Every element in this subfield is a sqare in F, and this fact implies that $\Pi^+(g) = PSL(2,q^2)$ in case (2). Now let d=5. Then for any $A \in GL(2,F)$ we can find $c \in F$ such that (4) and (6) hold. Thus we have $\Pi^+(g) = PGL(2,q^2)$ in case (3).

Let $\lambda, \mu \in F$, then we have

$$[g,h](\langle \lambda e_1 + \mu e_2 \rangle) = \langle \bar{\mu} e_3 - \bar{\lambda} e_4 \rangle \ .$$

This perspectivity is induced by the mapping

$$\psi : V \to V : (x_1,\dots,x_d) \to (-\bar{x}_4, \bar{x}_3, \bar{x}_2, -\bar{x}_1, \dots, \bar{x}_d) \ .$$

The matrix

$$\begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

has determinant +1, therefore the group generated by SU(d,q) and ψ contains the mapping

$$\tau : V \to V : (x_1,\dots,x_d) \to (\bar{x}_1,\dots,\bar{x}_d) \ .$$

It follows that $\Pi(g)$ is the semidirect product of $\Pi^+(g)$ by the group generated by the automorphism $\sigma : F \to F : x \to \bar{x}$.

Now we determine the groups $\Pi^+(\mathcal{P}^*)$ and $\Pi(\mathcal{P}^*)$. Let x and y be points of $\mathcal{P}$ with d(x,y)=4. The group $G_{x,y}$ induces $\Pi^+(x)$ on $\Gamma(x)$. It is possible to choose a basis $(e_1,\dots,e_d)$ of V such that the following hold:

$$x = \langle e_1 \rangle, \quad y = \langle e_2 \rangle, \quad (x \oplus y)^{\perp} = \langle e_3,\dots,e_d \rangle, \quad \text{and } f(e_1,e_2) = 1.$$

if

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(2)

is equivalent to (3)

(4)

(5)

to (6)

d=4 we obtain from (6) that detA is F of order q. Every element in this fact implies that $\Pi^+(g)=PSL(2,q^2)$ in $A\in GL(2,F)$ we can find $c\in F$ such that $g)=PGL(2,q^2)$ in case (3).

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$\mathcal{P}$*) and $\Pi(\mathcal{P}^*)$. Let x and y be points induces $\Pi^+(x)$ on $\Gamma(x)$. It is possi- ,y of V such that the following hold: $e_3,\ldots,e_d\rangle$, and $f(e_1,e_2)=1$.

Denote the restriction of f to $(x\oplus y)^{\perp}$ by f'. There is a natural correspondence between $\Gamma(x)$ and the 1-dimensional isotropic subspaces of the metric vector space $((x\oplus y)^{\perp}, f')$. The matrix of f w.r.t. the basis $(e_1,\ldots,e_d)$ looks as follows:

$$\begin{pmatrix} & 1 & \\ \varepsilon & & \\ & & J \end{pmatrix},$$

where $\varepsilon=+1$ if f is unitary and $\varepsilon=-1$ if f is symplectic, and J describes f'.

If f is symplectic, then every linear mapping which preserves f' can be extended to a linear mapping which fixes x and y and preserves f. Hence we conclude that $\Pi^+(x)=PSp(2,F)=PSL(2,F)$ in this case.

Now let f be unitary. Then every element of SU(d,F) which fixes x and y is described by a matrix of the form

$$M=\begin{pmatrix} a & & \\ & b & \\ & & C \end{pmatrix},$$

where $a,b\in F$, $C\in GL(d-2,F)$, and

$$a\bar{b}=1\ , \qquad (1)$$

$$CJ\bar{C}^t=J\ , \qquad (2)$$

$$a\cdot b\cdot \det C=1\ . \qquad (3)$$

From (1) and (3) we infer that

$$\det C=\bar{a}\cdot a^{-1}\ . \qquad (4)$$

Now $\det C\cdot\overline{\det C}=1$, and it follows from Hilbert's theorem 90 that for every $C\in U(d-2,F)$ there are $a,b\in F$ such that M is in SU(d,F). Hence we have $\Pi^+(x)=PU(d-2,F)=PGU(d-2,F)$. Now PGU(2,q) is well-known to be isomorphic to PGL(2,q). In case (1) and in case (2), $\mathcal{P}^*$ is known to be regular, and in case (3) we have $|sp(x,y)|=q+1$ ([5: 3.3.1]). Thus it follows from Lemma 1.4 that $\Pi(\mathcal{P}^*)=\Pi^+(\mathcal{P}^*)$ in all three cases.

d) Projective planes.

The first line of the table has been taken up for the sake of completeness. Everything is well-known in this case.

REMARKS:

1. In view of Witt's theorem it is remarkable that there exist generalized quadrangles which can be embedded into projective spaces and have

proper subgroups of PGL(2,F) as groups of even projectivities.

2. The hexagons and the octagons admit a treatment similar to the one given for the generalized quadrangles. The finite Moufang hexagons are embedded into trialities of D_4-geometries just by construction [9]. Recently, Sarli [8] has announced that the finite Moufang octagons can be embedded into polarities of metasymplectic spaces.

3. The classical von Staudt theorem states that a projective plane is pappian if and only if its group of projectivities is sharply triply transitive. A result pointing in the same direction has been obtained by Tits in [11: 9.6]; he characterizes in terms of projectivities those generalized quadrangles whose duals can be embedded into orthogonal polarities.

4. In view of the results obtained in [4] it seems plausible to conjecture that the group of projectivities of a finite generalized non-Moufang polygon always contains the alternating group.

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