ACTIONS OF SAut(F_n)

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Abstract. We study actions of $\mathrm{SAut}(F_n)$, the unique subgroup of index two in the automorphism group of a free group of rank n and obtain rigidity results for its representations. In particular, we show that every smooth action of $\mathrm{SAut}(F_n)$ on a low dimensional torus is trivial.

1. Introduction

In this work we study smooth actions of $\mathrm{SAut}(F_n)$ as a part of the generalized 'Zimmer program' which aims at understanding actions of large groups on compact manifolds. Given a group whose low dimensional linear representations are trivial, one wants to know whether all its representations into $\mathrm{Diff}(M)$, with M a low dimensional compact manifold, are trivial as well.

Here we study low dimensional linear representations of the group $\mathrm{SAut}(F_n)$ and we obtain strong rigidity results for these. To be precise, let $\mathbb{Z}^n$ be the free abelian group and F_n the free group of rank n. The abelianization map $F_n \twoheadrightarrow \mathbb{Z}^n$ induces a natural epimorphism $\pi : \mathrm{Aut}(F_n) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z})$. The group $\mathrm{SAut}(F_n)$ is defined as the preimage of $\mathrm{SL}_n(\mathbb{Z})$ under this map. It acts naturally by linear transformations on $\mathbb{R}^n$ via the compasition $\mathrm{SAut}(F_n) \twoheadrightarrow \mathrm{SL}_n(\mathbb{Z}) \subseteq \mathrm{SL}_n(\mathbb{R})$ and we shall show that this representation is minimal in the following sense.

Theorem A. *Let $n \geq 3$ and $\rho : \mathrm{SAut}(F_n) \to \mathrm{GL}_d(K)$ be a linear representation of degree d over a field K with $\mathrm{char}(K) \neq 2$. If $d < n$, then ρ is trivial.*

We would like to point out that certain cases of Theorem A are well known to the experts, see Proposition 2.2. However, our proof is different and works in a larger generality.

It was proven by Bridson and Vogtmann in [BV11] that if $n \geq 3$ and $d < n$, then $\mathrm{SAut}(F_n)$ cannot act non-trivially by homeomorphisms on any contractible manifold of dimension d. Using their techniques we prove in a purely group theoretical way the above theorem.

Let us further mention two results about the low dimensional linear representation theory of $\mathrm{Out}(F_n) := \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$ by Potapchik-Rapinchuk [PR00] and Kielak [Kil13]. One can deduce the following statement directly from [PR00, 3.1].

Theorem 1.1. *([PR00, 3.1]) Let $n \geq 3$ and $\rho : \mathrm{Out}(F_n) \to \mathrm{GL}_d(\mathbb{C})$ be a linear representation of degree d. If $d \leq 2n-2$, then ρ factors through the natural projection $\pi : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$.*

For lange n, this result can be strengthened as shown by Kielak in [Kil13, 3.13].

Theorem 1.2. *([Kil13, 3.13]) Let $n \geq 6$ and $\rho : \mathrm{Out}(F_n) \to \mathrm{GL}_d(K)$ be a linear representation of degree d over a field K with characteristic either equal to zero or greater than $n+1$. If $d < \binom{n+1}{2}$, then ρ factors through the natural projection $\pi : \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$.*

Much of the work on $\mathrm{SAut}(F_n)$ is motivated by the idea that $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SAut}(F_n)$ should have many properties in common. Here we follow this idea and present analogies between these groups with respect to smooth actions on low dimensional tori.

It was shown by Weinberger in [Wei93] that any smooth action of $\mathrm{SL}_n(\mathbb{Z})$ on the d-dimensional torus $\mathbb{T}^d = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is trivial if $d < n$. This bound on the dimension of the torus is sharp, as $\mathrm{SL}_n(\mathbb{Z})$ admits a linear faithful action on the flat torus $\mathbb{R}^n/\mathbb{Z}^n$. We prove an analogue of Weinberger's result for smooth actions of $\mathrm{SAut}(F_n)$.

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Theorem B. *Let $n \geq 3$ and $\Phi : \mathrm{SAut}(F_n) \to \mathrm{Diff}(\mathbb{T}^d)$ be a smooth action. If $d < n$, then Φ is trivial.*

This obtained by combining Theorem A with a result of Weinberger concerning smooth actions of non-abelian finite groups on tori and results of Bridson and Vogtmann about strong constrains on homomorphisms from $\mathrm{SAut}(F_n)$.

2. Linear representations

2.1. **The automorphism group of a free group.** The main protagonist in this work is the group $\mathrm{SAut}(F_n)$. We start by recalling the definition of this group, and establish some notation to be used throughout.

We begin with the definition of the automorphism group of the free group of rank n. Let F_n be the free group of rank n with a fixed basis $X := \{x_1, \dots, x_n\}$. We denote by $\mathrm{Aut}(F_n)$ the automorphism group of F_n and by $\mathrm{SAut}(F_n)$ the unique subgroup of index two in $\mathrm{Aut}(F_n)$.

Let us introduce notations for some specific elements of $\mathrm{Aut}(F_n)$. We define involutions e_i for $i = 1, \dots, n$, $i \neq j$ as follows:

$$e_i(x_k) := \begin{cases} x_i^{-1} & \text{if } k = i, \\ x_k & \text{if } k \neq i. \end{cases}$$

2.2. **Some finite subgroups of $\mathrm{Aut}(F_n)$.** Consider the following finite subgroups of $\mathrm{Aut}(F_n)$ and $\mathrm{SAut}(F_n)$:

$$\begin{aligned} \Sigma_n &:= \left\{\alpha \in \mathrm{Aut}(F_n) \mid \alpha_{|X} \in \mathrm{Sym}(X)\right\} \cong \mathrm{Sym}(n), \\ \mathrm{A}_n &:= \Sigma_n \cap \mathrm{SAut}(F_n) \cong \mathrm{Alt}(n), \\ \mathrm{N}_n &:= \langle \{e_i \mid i = 1, \dots, n\}\rangle \cong \mathbb{Z}_2^n, \\ \mathrm{SN}_n &:= \mathrm{N}_n \cap \mathrm{SAut}(F_n) = \langle \{e_1 e_i \mid i = 2, \dots, n\}\rangle \cong \mathbb{Z}_2^{n-1}, \\ \mathrm{W}_n &:= \langle \mathrm{N}_n \cup \Sigma_n \rangle = \mathrm{N}_n \rtimes \Sigma_n, \\ \mathrm{SW}_n &:= \mathrm{W}_n \cap \mathrm{SAut}(F_n). \end{aligned}$$

The following variant of a result by Bridson and Vogtmann [BV11, 3.1] will be used here to prove triviality of certain actions of $\mathrm{SAut}(F_n)$.

Proposition 2.1. *Let $n \geq 3$, G be a group and $\phi : \mathrm{SAut}(F_n) \to G$ a group homomorphism.*

- *(i) If there exists $\alpha \in \mathrm{SW}_n - \{\mathrm{id}_{F_n}, e_1 e_2 \dots e_n\}$ with $\phi(\alpha) = 1$, then ϕ factors through $\mathrm{SL}_n(\mathbb{Z}_2)$.*
- *(ii) If n is even and $\phi(e_1 e_2 \dots e_n) = 1$, then ϕ factors through $\mathrm{PSL}_n(\mathbb{Z})$.*
- *(iii) If there exists $\alpha \in \mathrm{SW}_n - \mathrm{SN}_n$ with $\phi(\alpha) = 1$, then ϕ is trivial.*

The following fact concerning linear representations of $\mathrm{SAut}(F_n)$ is well known in the community.

Proposition 2.2. *Let $\rho : \mathrm{SAut}(F_n) \to \mathrm{GL}_d(K)$ be a linear representation of degree d over a field K.*

- *(i) If $char(K) = 0$, $n \geq 6$ and $d < n - 1$, then ρ is trivial.*
- *(ii) If $char(K) = 2$, $n \geq 9$ and*
  - (a) *n is odd and $d < n - 1$, then ρ is trivial.*
  - (b) *n is even and $d < n - 2$, then ρ is trivial .*
- *(iii) If $char(K) = p$ is odd, $n \geq 7$ and*
  - (a) *$p \nmid n$ and $d < n - 1$, then ρ is trivial.*
  - (b) *$p \mid n$ and $d < n - 2$, then ρ is trivial.*

It can be proven directly by combining Proposition 2.1 (iii) with the following lemma.

Lemma 2.3. *(*[Bur55], [Wag76], [Wag77]*) Let $\mathrm{Alt}(n)$ be the alternating group and $\rho : \mathrm{Alt}(n) \to \mathrm{GL}_d(K)$ be a faithful linear representation of degree d over a field K.*

- *(i) If $char(K) = 0$ and $n \geq 6$, then $d \geq n - 1$.*

(*ii*) *If char*$(K) = 2$*,* $n \geq 9$ *and*
  (a) n *is odd, then* $d \geq n-1$.
  (b) n *is even, then* $d \geq n-2$.
(*iii*) *If char*$(K) = p$ *is odd,* $n \geq 7$ *and*
  (a) $p \nmid n$ *, then* $d \geq n-1$.
  (b) $p \mid n$*, then* $d \geq n-2$.

In order to get the more general result of Theorem A, we need to argue differently. Note that the group $\mathrm{SAut}(F_n)$ is perfect, therefore the image of a linear representation of $\mathrm{SAut}(F_n)$ is a subgroup of $\mathrm{SL}_d(K)$. We divide the proof into two steps. In the first step we show that for $d < n$ any homomorphism $\widetilde{\rho}: \mathrm{SL}_n(\mathbb{Z}_2) \to \mathrm{SL}_d(K)$ is trivial. In the second step we prove by induction on n that for $d < n$ any homomorphism $\rho: \mathrm{SAut}(F_n) \to \mathrm{SL}_d(K)$ factors through $\mathrm{SL}_n(\mathbb{Z}_2)$.

$$\begin{tikzcd}
\mathrm{SAut}(F_n) \arrow[rr, "\rho"] \arrow[dr, "\pi"'] & & \mathrm{SL}_d(K) \\
& \mathrm{SL}_n(\mathbb{Z}_2) \arrow[ur, "\widetilde{\rho}"'] &
\end{tikzcd}$$

A key observation is the following.

Proposition 2.4. *Let* K *be a field with* $\mathrm{char}(K) \neq 2$ *and let* $\phi: \mathbb{Z}_2^m \to \mathrm{SL}_d(K)$ *be a group homomorphism. If* $d \leq m$*, then* ϕ *is not injective.*

Proof. First, we recall an important characterization of diagonalizable linear maps: a linear map is diagonalizable over the field K if and only if its minimal polynomial is a product of distinct linear factors over K, see [Hof71, Thm. 6, p. 204]. The minimal polynomial of an involution is a divisor of $x^2 - 1 = (x+1)(x-1)$ and it is characterized by this if $\mathrm{char}(K) \neq 2$. We observe that all elements in the image of ϕ have order less or equal to two, therefore the elements in the image of ϕ are diagonalizable.

Next, we note that all elements in the image of ϕ commute, therefore they are simultaneously diagonalizable, see [Hor90, p. 51-53].

For A in the image of ϕ, we have $\mathrm{Eig}(A,1) \oplus \mathrm{Eig}(A,-1) = K^d$, where we denote by $\mathrm{Eig}(A,1)$ resp. $\mathrm{Eig}(A,-1)$ the eigenspace for the eigenvalue 1 resp. -1. We note that $\det(A) = 1$. The number of diagonal matrices in $\mathrm{SL}_d(K)$ with entries $+1$ and -1 and even number of entries equal to -1 is given by

$$\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{d}{2i} = 2^{d-1}.$$

Therefore the image of ϕ contains at most 2^{d-1} elements. But we have

$$\mathrm{ord}(\mathbb{Z}_2^m) = 2^m > 2^{d-1} \geq \mathrm{ord}(\mathrm{im}(\phi(\mathbb{Z}_2^m)))$$

and therefore ϕ is not injective. □

Using Proposition 2.4 we obtain the following result.

Proposition 2.5. *Let* $n \geq 3$ *and* $\widetilde{\rho}: \mathrm{SL}_n(\mathbb{Z}_2) \to \mathrm{SL}_d(K)$ *be a linear representation of degree* d *over a field* K *with* $\mathrm{char}(K) \neq 2$*. If* $d < n$*, then* $\widetilde{\rho}$ *is trivial.*

Proof. By simplicity of $\mathrm{SL}_n(\mathbb{Z}_2)$ the kernel of the map $\widetilde{\rho}$ is either trivial or all of $\mathrm{SL}_n(\mathbb{Z}_2)$. The group $\mathrm{SL}_n(\mathbb{Z}_2)$ contains a subgroup U which is isomorphic to $\mathbb{Z}_2^{n-1}$, $U = \langle\{\mathrm{E}_{12}, \ldots, \mathrm{E}_{1n}\}\rangle$, where we denote by E_{1i} the matrix which has ones on the main diagonal and in the entry $(1,i)$ and zeros elsewhere. By Proposition 2.4 the restriction of $\widetilde{\rho}$ to this group is not injective, therefore the kernel of the map $\widetilde{\rho}$ is equal to $\mathrm{SL}_n(\mathbb{Z}_2)$. □

First, we show Theorem A for $n = 3$ and $n = 4$ and then proceed by induction on n.

Lemma 2.6. *Let $\rho : \mathrm{SAut}(F_3) \to \mathrm{SL}_d(K)$ be a linear representation of degree d over a field K with $\mathrm{char}(K) \neq 2$. If $d < 3$, then ρ is trivial.*

Proof. The group SN_3 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and by Proposition 2.4 it follows that the restriction of ρ to this group is not injective. The element $e_1e_2e_3$ is not in SN_3 and therefore by Proposition 2.1 the map ρ factors through $\mathrm{SL}_3(\mathbb{Z}_2)$. Using Proposition 2.5 it follows that ρ is trivial. □

Lemma 2.7. *Let $\rho : \mathrm{SAut}(F_4) \to \mathrm{SL}_d(K)$ be a linear representation of degree d with $\mathrm{char}(K) \neq 2$. If $d < 4$, then ρ is trivial.*

Proof. The subgroup SN_4 is isomorphic to $\mathbb{Z}_2^3$ and by Proposition 2.4 it follows that $\rho_{|\mathrm{SN}_4}$ is not injective. Therefore there exists an element $\alpha \in \mathrm{SN}_4 - \{\mathrm{id}_{F_4}\}$ with $\rho(\alpha) = \mathrm{I}_d$, where I_d is the identity matrix. If α is not equal to $e_1e_2e_3e_4$, then by Proposition 2.1 it follows that ρ factors through $\mathrm{SL}_4(\mathbb{Z}_2)$ and the triviality of ρ follows by Proposition 2.5. If α is equal to $e_1e_2e_3e_4$ then by Proposition 2.1 the map ρ factors through $\mathrm{PSL}_4(\mathbb{Z})$.

$$\begin{tikzcd} \mathrm{SAut}(F_4) \arrow[rr, "\rho"] \arrow[dr, "\pi"'] & & \mathrm{SL}_d(K) \\ & \mathrm{PSL}_4(\mathbb{Z}) \arrow[ur, "\widetilde{\rho}"'] & \end{tikzcd}$$

The group $\mathrm{PSL}_4(\mathbb{Z})$ contains a subgroup U which is isomorphic to $\mathbb{Z}_2^4$, namely

$$U = \langle\{[\mathrm{E}_{-1}\mathrm{E}_{-2}], [\mathrm{E}_{-2}\mathrm{E}_{-3}], [\mathrm{P}_{12}\mathrm{P}_{34}], [\mathrm{P}_{13}\mathrm{P}_{24}]\}\rangle,$$

where we denote by E_{-i} the matrix which has ones on the main diagonal except the entry (i,i) and zeros elsewhere and the entry (i,i) is equal to -1. The matrix P_{ij} is a permutation matrix.

By Proposition 2.4 it follows that there exists an element $[A] \in U - \{[\mathrm{I}_4]\}$ with $\widetilde{\rho}([A]) = \mathrm{I}_d$. We consider a preimage of $[A]$ under π and note that there exists an element $\beta \in \mathrm{SW}_n - \{\mathrm{id}_{F_n}, e_1e_2e_3e_4\}$ with $\rho(\beta) = \widetilde{\rho} \circ \pi(\beta) = \widetilde{\rho}([A]) = \mathrm{I}_d$. By Proposition 2.1 it follows that ρ factors through $\mathrm{SL}_4(\mathbb{Z}_2)$ and by Proposition 2.5 we obtain triviality of ρ. □

Proof of Theorem A.
We proceed by induction on n. By 2.6 and 2.7, we assume that $n > 4$. If $\rho(e_1e_2) = \mathrm{I}_d$, then by Proposition 2.1 the map ρ factors through $\mathrm{SL}_n(\mathbb{Z}_2)$ and by Proposition 2.5 it follows that ρ is trivial. Otherwise $\rho(e_1e_2)$ is a non-trivial involution. We note that the dimension of the eigenspace $\mathrm{Eig}(\rho(e_1e_2), 1)$ is equal to $d - 2\cdot l$ for some $l \in \mathbb{N}_{>0}$. The centralizer of e_1e_2, which we will denote by $C(e_1e_2)$, contains a subgroup U isomorphic to $\mathrm{SAut}(F_{n-2})$. More precisely: let $\{x_1, \ldots, x_n\}$ be a basis of F_n and let $\{x_3, \ldots, x_n\}$ be a basis of F_{n-2}. We define

$$U := \left\{ f \in C(e_1e_2) \mid f(x_1) = x_1, f(x_2) = x_2 \text{ and} f_{|\{x_3,\ldots,x_n\}} \in \mathrm{SAut}(F_{n-2}) \right\}.$$

Then U is isomorphic to $\mathrm{SAut}(F_{n-2})$. By the induction hypothesis, any homomorphism

$$\rho' : \mathrm{SAut}(F_{n-2}) \to \mathrm{SL}_{d'}(K)$$

for $d' < n-2$ is trivial. The restriction of ρ to U acts on $\mathrm{Eig}(\rho(e_1e_2), 1) \cong K^{d-2l}$ and therefore

$$\rho_{|U} : U \to \mathrm{SL}_{d-2\cdot l}(K)$$

is trivial. In particular, the element e_3e_4 is in U and acts trivially on $\mathrm{Eig}(\rho(e_1e_2), 1)$. It follows that $\mathrm{Eig}(\rho(e_1e_2, 1) \subseteq \mathrm{Eig}(\rho(e_3e_4), 1)$. This argument is symmetric and we obtain

$$\mathrm{Eig}(\rho(e_1e_2), 1) = \mathrm{Eig}(\rho(e_3e_4), 1).$$

But then we have two commuting involutions with equal eigenspaces of eigenvalue 1, therefore $\rho(e_1e_2)$ and $\rho(e_3e_4)$ are equal and the element $e_1e_2e_3e_4$ acts trivially. We have $n > 4$ and by Proposition 2.1 the map ρ factors through $\mathrm{SL}_n(\mathbb{Z}_2)$. By Proposition 2.5 it follows that ρ is trivial.

$\square$

Remark 2.8. *The key ingredient in the proof of Theorem A is that there exist at most d-many pairwise commuting involutions in* $\mathrm{SL}_d(K)$ *when* $\mathrm{char}(K) \neq 2$*, see Proposition 2.4. This is no longer true for any infinite field* K *of characteristic* 2*, as we can for example consider the infinite group*

$$\left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{I}_{d-2} \end{pmatrix} \middle| \ a \in K^* \right\} \subseteq \mathrm{SL}_d(K).$$

3. Proof of Theorem B

We begin the proof of Theorem B with the following proposition by Weinberger, whose proof heavily depends on smothness of the actions under considerations. First, we note that the first singular cohomology group of a torus $\mathrm{H}^1(\mathbb{T}^d;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^d$.

Proposition 3.1. *(*[Wei93, 2]*) Let* G *be a non-abelian finite group and* $\psi : G \to \mathrm{Diff}(\mathbb{T}^d)$ *a smooth action. If*

$$\mathrm{H}^1(\psi) : G \to \mathrm{GL}_d(\mathbb{Z}),$$
$$g \mapsto \mathrm{H}^1(\psi(g)) : \mathrm{H}^1(\mathbb{T}^d;\mathbb{Z}) \to \mathrm{H}^1(\mathbb{T}^d;\mathbb{Z})$$

is trivial, then ψ *is not injective.*

Now we have all ingredients to prove

Theorem B. *Let* $n \geq 3$ *and* $\Phi : \mathrm{SAut}(F_n) \to \mathrm{Diff}(\mathbb{T}^d)$ *be a smooth action. If* $d < n$*, then* Φ *is trivial.*

Proof. According to Theorem A, the following action is trivial for $d < n$:

$$\iota \circ \mathrm{H}^1(\Phi) : \mathrm{SAut}(F_n) \to \mathrm{SL}_d(\mathbb{Z}) \hookrightarrow \mathrm{SL}_d(\mathbb{R})$$
$$\alpha \mapsto \mathrm{H}^1(\Phi(\alpha)) \mapsto \mathrm{H}^1(\Phi(\alpha)).$$

In particular, the map $\mathrm{H}^1(\Phi)$ is trivial.

We consider the subgroup A_n in $\mathrm{SAut}(F_n)$ and the restriction of Φ to this group. The map $\mathrm{H}^1(\Phi_{|\mathrm{A}_n})$ is trivial, therefore by Proposition 3.1 the map $\Phi_{|\mathrm{A}_n}$ is not injective and by Proposition 2.1 it follows that Φ is trivial. $\square$

Remark 3.2. *Note that this bound on the dimension of the torus is sharp, as* $\mathrm{SL}_n(\mathbb{Z})$ *admits a linear faithful action on the flat torus* $\mathbb{R}^n/\mathbb{Z}^n$*. Therefore* $\mathrm{SAut}(F_n)$ *admits a smooth non-trivial action on the flat torus as well:* $\mathrm{SAut}(F_n) \twoheadrightarrow \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Diff}(\mathbb{R}^n/\mathbb{Z}^n)$.

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