Part I

Basic Properties of Topological Groups

1 Topological Groups

This chapter contains basic results about the point-set topology of topological groups. Whenever we use the Hausdorff condition, this will be mentioned explicitly. However, we do assume that locally compact spaces and compact spaces are Hausdorff.

We first consider products, subgroups and coset spaces of general topological groups. We also review two constructions of group topologies, starting from subgroups. Then we introduce the identity component and the group of connected components. We then turn to van Dantzig's Theorem: a totally disconnected locally compact group has arbitrarily small open subgroups, and a totally disconnected compact group is profinite. Lastly, we consider Weil's Lemma and its application to compact subsemigroups of topological groups.

Definition 1.1. A *topological group* $(G, \cdot, \mathcal{T})$ consists of a group $(G, \cdot)$ and a topology $\mathcal{T}$ on G for which the multiplication map

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \cdot h = gh \end{aligned}$$

and the inversion map

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are continuous. We then call $\mathcal{T}$ a *group topology* on G. We can combine these two conditions into one condition by considering the map

$$\begin{aligned} \kappa : G \times G &\longrightarrow G \\ (g, h) &\longmapsto g^{-1}h. \end{aligned}$$

If G is a topological group, then κ is continuous. Conversely, if κ is continuous, then the maps $g \longmapsto g^{-1} = \kappa(g, e)$ and $(g, h) \longmapsto gh = \kappa(\kappa(g, e), h)$ are also continuous.

Suppose that G is a topological group. For every $a \in G$, the *right translation map*

$$\rho_a(g) = ga,$$

the *left translation map*

$$\lambda_a(g) = ag$$

and the *conjugation map*

$$\gamma_a(g) = aga^{-1}$$

are homeomorphisms of G onto itself, with inverses $\lambda_{a^{-1}}$, $\rho_{a^{-1}}$ and $\gamma_{a^{-1}}$, respectively. In particular, the homeomorphism group of G acts transitively on G. It follows that every neighborhood W of a group element $g \in G$ can be written as $W = gU = Vg$, where $U = \lambda_{g^{-1}}(W)$ and $V = \rho_{g^{-1}}(W)$ are neighborhoods of the identity. In what follows, we will most of the time write G for a topological group, without mentioning the topology $\mathcal{T}$ explicitly. A neighborhood of the identity element e will be called an *identity neighborhood* for short. If V is an identity neighborhood, then $V \cap V^{-1} \subseteq V$ is a symmetric identity neighborhood.

Now we study the most basic group-theoretic notions for topological groups: homomorphisms, products, subgroups and quotients.

Definition 1.2. We define a *morphism*

$$f : G \longrightarrow K$$

between topological groups G, K to be a continuous group homomorphism. Occasionally we will have to deal with group homomorphisms between topological groups which are *not* assumed to be continuous. Such homomorphism will be called an *abstract homomorphisms*. The underlying group of a topological group will be called its *abstract group*. This terminology is due to Borel and Tits. In the literature, such homomorphism are sometimes called *algebraic homomorphisms*.

The following local criterion for an abstract homomorphism to be a morphism is useful.

Lemma 1.3. *Let G, K be topological groups and let $f : G \longrightarrow K$ be an abstract homomorphism. Then the following are equivalent.*

(i) *The abstract homomorphism f is continuous and hence a morphism of topological groups.*
(ii) *The abstract homomorphism f is continuous at one point $a \in G$, i.e. for every neighborhood W of $f(a)$, there exists a neighborhood V of a such that $f(V) \subseteq W$.*

Proof. It is clear that (i) $\Rightarrow$ (ii), because a continuous map is continuous at every point. Suppose that (ii) holds and that $U \subseteq K$ is open. If $g \in f^{-1}(U)$, then $f(a) = f(ag^{-1}g) \in f(ag^{-1})U$. Hence there exists a neighborhood V of a with $f(V) \subseteq f(ag^{-1})U$. Then $ga^{-1}V$ is a neighborhood of g, with $f(ga^{-1}V) \subseteq U$. Hence $f^{-1}(U)$ is open. $\square$

Example 1.4. The following are examples of topological groups and morphisms.

(1) The additive groups of the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and the p-adic fields $\mathbb{Q}_p$, endowed with their usual field topologies, are examples of topological groups. Likewise, the multiplicative group of units of each of these fields is a topological group. The exponential maps $\exp : \mathbb{R} \longrightarrow \mathbb{R}^\times$ and $\exp : \mathbb{C} \longrightarrow \mathbb{C}^\times$ are morphisms.

(2) The *circle group* $\mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}^\times$ is another example of a topological group. The map $\mathbb{R} \longrightarrow \mathrm{U}(1)$ that maps t to $\exp(2\pi \boldsymbol{i} t) = \cos(2\pi t) + \boldsymbol{i} \sin(2\pi t)$ is a morphism.

(3) Every morphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ is of the form $f(t) = rt$, for a unique real r. This follows from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, and that an additive homomorphism $f : \mathbb{Q} \longrightarrow \mathbb{R}$ is determined uniquely determined by the element $r = f(1)$, since $\mathbb{Q}$ is uniquely divisible.

(4) Every morphism $f : \mathrm{U}(1) \longrightarrow \mathrm{U}(1)$ is of the form $f(z) = z^m$, for a unique integer $m \in \mathbb{Z}$.

(5) As a vector space over $\mathbb{Q}$, the additive group $\mathbb{R}$ has dimension $2^{\aleph_0}$. Hence there are $2^{2^{\aleph_0}}$ abstract homomorphisms $\mathbb{R} \longrightarrow \mathbb{R}$, most of which are not continuous.

(6) Let H denote the additive group of the reals, endowed with the discrete topology. Then H is a locally compact group and the identity map $\mathrm{id} : H \longrightarrow \mathbb{R}$ is a continuous bijective morphism whose inverse is not continuous.

(7) Let F be a field and let $\mathrm{GL}_n(F)$ denote the *general linear group* of invertible $n \times n$-matrices over F. For an $n \times n$-matrix g, let $g^\#$ denote the matrix with entries $(g^\#)_{i,j} = (-1)^{i+j} \det(g'(j,i))$, where $g'(j,i)$ is the $(n-1) \times (n-1)$-matrix obtained by removing column i and row j from the matrix g. Then $gg^\# = g^\# g = \det(g)\mathbb{1}$. Hence if F is a topological field, then $\kappa(g,h) = g^{-1}h = \frac{1}{\det(g)} g^\# h$ depends continuously on g and h, and therefore $\mathrm{GL}_n(F)$ is a topological group. In particular, the matrix groups $\mathrm{GL}_n(\mathbb{Q})$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, and $\mathrm{GL}_n(\mathbb{Q}_p)$ are topological groups.

(8) Every group G, endowed either with the discrete or with the trivial nondiscrete topology, is a topological group. For the discrete topology, G is locally compact and locally contractible.

Products of topological groups behave well, as the next result shows.

Proposition 1.5. *Suppose that $(G_j)_{j \in J}$ is a family of topological groups. Then the product*

$$G = \prod_{j \in J} G_j,$$

endowed with the product topology, is again a topological group. The product is Hausdorff if and only if each factor G_j is Hausdorff, and compact if and only if each factor G_j is compact.

For each j, the projection map $\mathrm{pr}_j : G \longrightarrow G_j$ is an open morphism. If H is a topological group and if there are morphisms $f_j : H \longrightarrow G_j$, for every $j \in J$, then the diagonal map $f(h) = (f_j(h))_{j \in J}$ is the unique morphism $f : H \longrightarrow G$ with the property that $\mathrm{pr}_j \circ f = f_j$ holds for all $j \in J$,

$$\begin{tikzcd} & G \arrow[d, "\mathrm{pr}_j"] \\ H \arrow[ur, "f"] \arrow[r, "f_j"] & G_j. \end{tikzcd}$$

Proof. In order to show that G is a topological group, we have to show that the map $\kappa : G \times G \longrightarrow G$ that maps (g, h) to $g^{-1}h$ is continuous. Let $\kappa_j : G_j \times G_j \longrightarrow G_j$ denote the corresponding maps, which are by assumption continuous. Then we have for each j a continuous map $\mathrm{pr}_j \circ \kappa = \kappa_j \circ (\mathrm{pr}_j \times \mathrm{pr}_j)$,

$$\begin{tikzcd} G \times G \arrow[r, "\kappa"] \arrow[d, "\mathrm{pr}_j \times \mathrm{pr}_j"'] & G \arrow[d, "\mathrm{pr}_j"] \\ G_j \times G_j \arrow[r, "\kappa_j"] & G_j. \end{tikzcd}$$

By the universal property of the product topology on G, this implies that κ is continuous. The remaining claims follow from the general properties of a product of topological spaces. $\square$

We will later look into separation properties for topological groups more closely. At this point we record the following important fact.

Lemma 1.6. *A topological group G is Hausdorff if and only if some singleton $\{a\} \subseteq G$ is closed.*

Proof. Suppose that $\{a\} \subseteq G$ is closed. The preimage of $\{a\}$ under the continuous map $(g, h) \longmapsto g^{-1}ha$ is the diagonal $\{(g, g) \mid g \in G\} \subseteq G \times G$, which is therefore closed in $G \times G$. Thus G is Hausdorff. Conversely, every singleton set in a Hausdorff space is closed. $\square$

Subgroups

Now we study subgroups of topological groups.

Proposition 1.7. *Let H be a subgroup of a topological group G. Then H is a topological group with respect to the subspace topology. Moreover, the closure $\overline{H}$ is also a subgroup of G. If H is normal in G, then $\overline{H}$ is also normal.*

Proof. Let H be a subgroup of G, endowed with the subspace topology. The product topology on $H \times H$ coincides with the subspace topology on $H \times H \subseteq G \times G$. Therefore the map $H \times H \longrightarrow H$ that maps (g, h) to $g^{-1}h$ is continuous. Hence H is a topological group. The continuity of the map $\kappa(g, h) = g^{-1}h$ ensures that

$$\kappa(\overline{H} \times \overline{H}) = \kappa(\overline{H \times H}) \subseteq \overline{\kappa(H \times H)} = \overline{H}.$$

Thus $\overline{H}$ is a subgroup. Suppose in addition that $H \trianglelefteq G$ is normal. For $a, g \in G$ we put $\gamma_a(g) = aga^{-1}$. Since the conjugation map $\gamma_a : G \longrightarrow G$ is continuous, we have

$$\gamma_a(\overline{H}) \subseteq \overline{\gamma_a(H)} = \overline{H}$$

for all $a \in G$. This shows that $\overline{H}$ is normal in G. □

Lemma 1.8. *Let G be a topological group and let $A \subseteq G$ be a closed subset. Then the normalizer of A*

$$\mathrm{Nor}_G(A) = \{g \in G \mid \gamma_g(A) = A\}$$

is a closed subgroup.

Proof. For $a \in A$ put $c_a(g) = gag^{-1}$. Then $c_a : G \longrightarrow G$ is continuous and hence $c_a^{-1}(A) = \{g \in G \mid gag^{-1} \in A\}$ is closed. Therefore

$$S = \bigcap\{c_a^{-1}(A) \mid a \in A\} = \{g \in G \mid \gamma_g(A) \subseteq A\}$$

is a closed subsemigroup in G, and $\mathrm{Nor}_G(A) = S \cap S^{-1}$ is closed as well. □

The Hausdorff property is needed for the next two results. The trivial nondiscrete topology supplies easily examples which show that both results fail for non-Hausdorff topological groups.

Lemma 1.9. *Let G be a Hausdorff topological group, and let $X \subseteq G$ be any subset. Then the centralizer*

$$\mathrm{Cen}_G(X) = \{g \in G \mid [g, x] = e \textit{ for all } x \in X\}$$

is closed. In particular, the center of G is a closed subgroup.

Proof. Given $x \in X$, the map $g \longrightarrow [g, x] = gxg^{-1}x^{-1}$ is continuous. Therefore $\mathrm{Cen}_G(x) = \{g \in G \mid [g, x] = e\}$ is closed because $\{e\} \subseteq G$ is closed. Then $\mathrm{Cen}_G(X) = \bigcap\{\mathrm{Cen}_G(x) \mid x \in X\}$ is closed as well. □

Lemma 1.10. *Let G be a Hausdorff topological group. If $A \subseteq G$ is an abelian subgroup, then the closure $\overline{A}$ is an abelian subgroup.*

Proof. The commutator map $(g,h) \longrightarrow [g,h]$ is constant on $A \times A$ and hence also constant on the closure $\overline{A \times A} = \overline{A} \times \overline{A}$. $\square$

Lemma 1.11. *Let G be a topological group and suppose that $U \subseteq G$ is an open subset. If $X \subseteq G$ is any subset, then UX and XU are open subsets. In particular, the multiplication map $m : G \times G \longrightarrow G$, $(g,h) \longmapsto gh$ and the map $\kappa : G \times G \longrightarrow G$, $(g,h) \longmapsto g^{-1}h$ are open.*

Proof. For each $x \in X$, the sets $Ux = \rho_x(U)$ and $xU = \lambda_x(U)$ are open. Hence $UX = \bigcup\{Ux \mid x \in X\}$ and $XU = \bigcup\{xU \mid x \in X\}$ are open as well. $\square$

Proposition 1.12. *Let G be a topological group and let $H \subseteq G$ be a subgroup. Then we have the following.*

(i) *The subgroup H is open if and only if it contains a nonempty open set.*
(ii) *If H is open, then H is also closed.*
(iii) *The subgroup H is closed if and only if there exists an open set $U \subseteq G$ such that $U \cap H$ is nonempty and closed in U.*

Proof. For (i), suppose that H contains the nonempty open set U. Then $H = UH$ is open by Lemma 1.11. Conversely, if H is open then it contains the nonempty open set H. For (ii), suppose that $H \subseteq G$ is open. Then $G - H = \bigcup\{aH \mid a \in G - H\}$ is also open and therefore H is closed. For (iii) we note that $H \cap U$ is closed in U if H is a closed subgroup. Conversely, suppose that $U \cap H$ is nonempty and closed in the open set U. Then $U \cap H$ is also closed in the smaller set $U \cap \overline{H} \subseteq U$. Upon replacing G by $\overline{H}$, we may thus assume in addition that H is dense in the ambient group G, and we have to show that $H = G$. The set $U - H = U - (U \cap H)$ is open in U and hence open in $G = \overline{H}$. On the other hand, H is dense in G. Therefore $U - H = \varnothing$ and thus $U \subseteq H$. By (i) and (ii), H is closed in G, whence $H = G$. $\square$

Corollary 1.13. *Let G be a topological group and let $V \subseteq G$ be a neighborhood of some element $g \in G$. Then V generates an open subgroup of G.*

Corollary 1.14. *Suppose that G is a Hausdorff topological group and that $H \subseteq G$ is a subgroup. If H is locally compact in the subspace topology, then H is closed. In particular, every discrete subgroup of G is closed.*

Proof. Let $C \subseteq H$ be a compact set which is an identity neighborhood in the topological group H. Then there exists an open identity neighborhood U in G such that $U \cap H \subseteq C$. Since C is compact, C is closed in G and hence $U \cap H = U \cap C$ is closed in U and nonempty. By Proposition 1.12(iii), the subgroup H is closed. $\square$

We recall that a Hausdorff space is called *σ-compact* if it can be written as a countable union of compact subsets. A σ-compact space need not be locally compact; for example the set of rational numbers $\mathbb{Q}$ is σ-compact in its standard euclidean topology.

Lemma 1.15. *Let G be a locally compact group. Then G has a σ-compact open subgroup. In particular, every connected locally compact group is σ-compact.*

Proof. Let $C \subseteq G$ be a compact symmetric identity neighborhood. Then $H = \langle C \rangle = \bigcup\{C^{\cdot k} \mid k \in \mathbb{N}_1\}$ is σ-compact. Since C contains a nonempty open set, H is open in G. $\square$

To illustrate these results, we determine now the closed subgroups of the additive group $\mathbb{R}$.

Lemma 1.16. *Let $A \subseteq \mathbb{R}$ be a discrete subgroup. Then $A = a\mathbb{Z}$ for a unique real number $a \geq 0$.*

Proof. It suffices to consider the case when $A \neq \{0\} = 0\mathbb{Z}$. We put $a = \inf\{t \in A \mid t > 0\}$. Since A is discrete and closed by Corollary 1.14, $a > 0$ and $a \in A$. Thus $a\mathbb{Z} \subseteq A$. For a general element $b \in A$ there exists $k \in \mathbb{Z}$ such that $ka \leq b < (k+1)a$. Then $0 \leq b-ka < a$ and thus $b = ka$. This shows that $A = a\mathbb{Z}$. If a' is a real number with $0 \leq a' < a$ then $a'\mathbb{Z} \neq a\mathbb{Z}$. $\square$

Lemma 1.17. *Let $A \subseteq \mathbb{R}$ be a non-discrete subgroup. Then A is dense in $\mathbb{R}$.*

Proof. Being nondiscrete, A has an accumulation point $a \in A$. But then 0 is also an accumulation point of A. Hence we find for every $n \in \mathbb{N}_1$ an element $a_n \in A$ with $0 < |a_n| \leq \frac{1}{n}$. Given $r \in \mathbb{R}$ and $n \in \mathbb{N}_1$, we can choose $k_n \in \mathbb{Z}$ in such a way that $|a_n k_n - r| \leq \frac{1}{n}$. Hence r is in the closure of A. $\square$

Corollary 1.18. *Every proper closed subgroup of the additive group $\mathbb{R}$ is of the form $a\mathbb{Z}$, for a unique real number $a \geq 0$.*

We noted above that the product of open sets in a topological group is again open. There is, in general, no similar property for the product of closed subsets, as the following example shows.

Example 1.19. The additive group of the reals contains the the closed subgroups $A = \mathbb{Z}$ and $B = a\mathbb{Z}$, where $a \in \mathbb{R}$ is an irrational number. The subgroup $A + B$ is not of the form $c\mathbb{Z}$, for some real number $c \geq 0$, because a is not a rational number. Hence $A + B$ is a countable dense subgroup of $\mathbb{R}$ by Lemma 1.16 and Lemma 1.17, which is not closed.

However, we have the following.

Lemma 1.20. *Let G be a Hausdorff topological group, and let $A, B \subseteq G$ be closed subsets. If either A or B is compact, then $AB \subseteq G$ is closed.*

The proof uses Wallace's Lemma, which is a very convenient tool from point-set topology.

Lemma 1.21 (Wallace). *Let $X_1, \ldots X_k$ be Hausdorff spaces containing compact sets $A_j \subseteq X_j$, for $j = 1, \ldots, k$. If $W \subseteq X_1 \times \cdots \times X_k$ is an open set containing $A_1 \times \cdots \times A_k$, then there exist open sets U_j with $A_j \subseteq U_j \subseteq X_j$, for $j = 1, \ldots, k$, such that*

$$A_1 \times \cdots \times A_k \subseteq U_1 \times \cdots \times U_k \subseteq W.$$

Proof. There is nothing to show for $k = 1$. Suppose that $k = 2$. We put $A = A_1$ and $B = A_2$ and we fix $a \in A$. For every point $b \in B$, we choose an open neighborhood $U_b \times V_b$ of (a, b) such that $U_b \times V_b \subseteq W$. Since $\{a\} \times B$ is compact, finitely many such neighborhoods $U_{b_1} \times V_{b_1}, \ldots, U_{b_m} \times V_{b_m}$ cover $\{a\} \times B$. We put $U_a = U_{b_1} \cap \cdots \cap U_{b_m}$ and $V_a = V_{b_1} \cup \cdots \cup V_{b_m}$. Then $\{a\} \times B \subseteq U_a \times V_a \subseteq W$. Now we let $a \in A$ vary. Since A is compact, finitely many such strips $U_{a_1} \times V_{a_1}, \ldots, U_{a_n} \times V_{a_n}$ cover $A \times B$. We put $U = U_{a_1} \cup \cdots \cup U_{a_n}$ and $V = V_{a_1} \cap \cdots \cap V_{a_n}$. Then $A \times B \subseteq U \times V \subseteq W$ and the claim is proved for $k = 2$.

For $k \geq 3$ we apply the previous argument to $A = A_1$ and $B = A_2 \times \cdots \times A_k$, and we obtain open sets $U \subseteq X_1$ and $V \subseteq X_2 \times \cdots \times X_k$ with $A \times B \subseteq U \times V \subseteq W$. By induction, we find now open sets $U_2, \ldots U_k$ such that $A_2 \times \cdots \times A_k \subseteq U_2 \times \cdots U_k \subseteq V$. Therefore $A_1 \times \cdots \times A_k \subseteq U \times U_2 \times \cdots U_k \subseteq W$. ☐

Corollary 1.22. *Suppose that $A_1, \ldots, A_k$ are pairwise disjoint compact subsets of a Hausdorff space X. Then there exists pairwise disjoint open sets $U_1, \ldots, U_k$ with $A_j \subseteq U_j$, for $j = 1, \ldots, k$.*

In particular, we may fatten compact subsets in locally compact spaces.

Corollary 1.23. *Suppose that K is a compact subset of a locally compact space X. Then there exists an open set $V \subseteq X$ containing K with compact closure $\overline{V}$.*

Proof. Let $Z = X \cup \{\infty\}$ denote the Alexandrov compactification of X. Then K and ∞ have disjoint open neighborhoods V and W in Z, respectively. In particular, $\overline{V} \subseteq Z - W \subseteq X$ is compact. ☐

Proof of Lemma 1.20. Suppose that A is compact and B is closed, and that $g \in G - AB$. We have to show that $G - AB$ contains a neighborhood of g. By assumption, $A^{-1}g \cap B = \varnothing$. If we put $\kappa(g, h) = g^{-1}h$, then $\kappa(A \times \{g\}) \subseteq G - B$. By Wallace's Lemma 1.21 there exists an open neighborhood V of g such that $\kappa(A \times V) \subseteq G - B$, i.e. $A^{-1}V \cap B = \varnothing$. Hence $V \cap AB = \varnothing$ and the claim follows. The case where B is compact and A is closed follows similarly. ☐

Group Topologies from Subgroups

We review two constructions of group topologies, starting from subgroups.

Definition 1.24. Let G be a topological group. We call a subgroup $H \subseteq G$ *inert* if for every $g \in G$ there is an identity neighborhood $U \subseteq G$ such that

$$H \cap U = gHg^{-1} \cap U.$$

Every normal subgroup of G is thus inert, but also every open and every discrete subgroup is inert.

Construction 1.25. Suppose that H is an inert subgroup of a topological group $(G, \cdot, \mathcal{T})$. We put

$$\mathcal{T}_H = \{U \subseteq G \mid gU \cap H \text{ is open in } H \text{ for every } g \in G\}$$

and we claim that this is a group topology on G.

It is clear that $\mathcal{T}_H$ is a topology on G which refines the topology $\mathcal{T}$ on G and that H is open in this topology. Also, the subspace topologies on H with respect to $\mathcal{T}$ and $\mathcal{T}_H$ coincide,

$$\mathcal{T}_H|H = \mathcal{T}|H.$$

It remains to show that $\mathcal{T}_H$ is a group topology. For this we have to show that the map $\kappa(g, h) = g^{-1}h$ is continuous with respect to $\mathcal{T}_H$.

The set $\mathcal{N} = \{U \subseteq H \mid U \text{ is an identity neighborhood in } H\}$ is a neighborhood basis of the identity element for the topology $\mathcal{T}_H$. Given $W \in \mathcal{N}$, we find $V \in \mathcal{N}$ such that $V^{-1}V \subseteq W$, because H is a topological group. Let $g, h \in G$ and put $a = g^{-1}h$. Since H is inert and since V is an identity neighborhood, there is an open identity neighborhood $Y \in \mathcal{T}$ such that $H \cap Y = a^{-1}Ha \cap Y \subseteq V$. Then $H \cap aYa^{-1} \subseteq aVa^{-1}$ and thus $U = H \cap aYa^{-1} \cap Y \subseteq aVa^{-1} \cap V$ is in $\mathcal{N}$. In the topology $\mathcal{T}_H$, the set gU is a neighborhood of g and the set hV is a neighborhood of h, and

$$\kappa(gU \times hV) = U^{-1}g^{-1}hV = a(a^{-1}U^{-1}a)V \subseteq aV^{-1}V \subseteq g^{-1}hW.$$

This shows that κ is continuous at every $(g, h) \in G \times G$, and thus $(G, \cdot, \mathcal{T}_H)$ is a topological group.

The previous construction refines a given group topology, using an inert subgroup. The following construction is related. It extends a given group topology on a normal subgroup of a group.

Construction 1.26. Suppose that N is a normal subgroup of a group G. Suppose also that $\mathcal{S}$ is a group topology on N. We put

$$\mathcal{T} = \{U \subseteq G \mid gU \cap N \in \mathcal{S} \text{ for every } g \in G\}.$$

Then $\mathcal{T}$ is a topology on G, and the corresponding subspace topology on N coincides with $\mathcal{S}$, i.e. $\mathcal{S} = \mathcal{T}|H$. Moreover, $H \subseteq G$ is open in this topology. We claim that $\mathcal{T}$ is a group topology on G, provided that for every $g \in G$ the conjugation map $\gamma_g(a) = gag^{-1}$ is continuous on N.

The proof of this claim is similar to the previous example. We let $\mathcal{N}$ denote the neighborhood basis of e in N with respect to the topology $\mathcal{S}$. Given $W \in \mathcal{N}$ and $g, h \in G$, we choose $V \in \mathcal{N}$ such that $V^{-1}V \subseteq W$. We put $U = V \cap g^{-1}hVh^{-1}g$. Then U is in $\mathcal{N}$ because $\gamma_{g^{-1}h}|_N$ is a homeomorphism of N. Moreover, gU is a neighborhood of g and hV is a neighborhood of h. As above we conclude that

$$\kappa(gU \times hV) = (gU)^{-1}hV \subseteq g^{-1}hW$$

and this shows that κ is continuous at (g, h). Hence $\mathcal{T}$ is a group topology on G.

The second construction can be applied in particular if G is a group with nontrivial center Z. Every group topology on Z can be extended to a group topology on G, such that Z is an open subgroup.

Quotients

Suppose that H is a subgroup of a topological group G. We endow the set G/H of left cosets with the quotient topology with respect to the natural map

$$p : G \longrightarrow G/H, \qquad g \longmapsto gH.$$

Thus a subset of G/H is open if and only if its p-preimage is open. The next result is elementary, but important.

Proposition 1.27. *Let G be a topological group and let H be a subgroup. Then the quotient map*

$$p : G \longrightarrow G/H$$

is open. The quotient G/H is Hausdorff if and only if H is closed in G, and it is discrete if and only if H is open in G.

Proof. Suppose that $U \subseteq G$ is an open set. Then $p^{-1}(p(U)) = UH$ is open by Lemma 1.11, hence $p(U)$ is open by the definition of the quotient topology.

If G/H is Hausdorff, then $\{H\} \subseteq G/H$ is closed, hence $H = p^{-1}(\{H\}) \subseteq G$ is closed as well. Conversely, suppose that $H \subseteq G$ is closed. The map $p \times p : G \times G \longrightarrow G/H \times G/H$ is open, because p is open and because a cartesian product of two open maps is again open. The open set $W = \{(x, y) \in G \times G \mid x^{-1}y \in G - H\}$ maps under $p \times p$ onto the complement of the diagonal in $G/H \times G/H$. Hence the diagonal $\{(gH, gH) \mid g \in G\}$ is closed in $G/H \times G/H$, and therefore G/H is Hausdorff.

If H is open, then every coset gH is open and hence G/H is discrete. Conversely, if G/H is discrete, then H, being the preimage of the open singleton $\{H\} \subseteq G/H$, is open. ☐

If G is a Hausdorff topological group and if $H \subseteq G$ is a compact subgroup, then the map $p : G \longrightarrow G/H$ is also closed by Lemma 1.20. A more general result is proved in Corollary **??**.

Corollary 1.28. *Suppose that G is a compact group and that $H \subseteq G$ is a closed subgroup. Then H is open if and only if H has finite index in G.*

Proof. A discrete topological space is compact if and only if it is finite. ☐

A more general result is proved below in Lemma **??**, using Baire's Category Theorem. For locally compact groups, we have the following basic result about subgroups and quotients.

Proposition 1.29. *Let G be a locally compact group. Then a subgroup $H \subseteq G$ is closed if and only if it is locally compact. If $H \subseteq G$ is a closed subgroup, then G/H is locally compact.*

Proof. A closed subspace of a locally compact space is again locally compact. Conversely, a locally compact subgroup of a Hausdorff topological group is closed by Corollary 1.14. For the last claim, suppose that $H \subseteq G$ is a closed subgroup, and that $g \in G$ is any element. We have to show that gH has a compact neighborhood in G/H. Let $V \subseteq G$ be an open identity neighborhood with compact closure. Then $p(Vg)$ is an open neighborhood of g because $p : G \longrightarrow G/H$ is open, and thus $p(\overline{V}g)$ is a compact neighborhood of gH. ☐

Proposition 1.30. *Let G be a topological group. If $N \trianglelefteq G$ is a normal subgroup, then the group G/N is a topological group with respect to the quotient topology on G/N. The quotient map $p : G \longrightarrow G/N$ is an open morphism. The group G/N is Hausdorff if and only if N is closed. In particular, $G/\overline{N}$ is a Hausdorff topological group.*

Proof. We put $\bar{\kappa}(gN, hN) = g^{-1}hN$ and $p(g) = gN$. Then the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\kappa} & G \\ {\scriptstyle p\times p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ G/N \times G/N & \xrightarrow{\bar{\kappa}} & G/N \end{array}$$

commutes, and $p\circ\kappa$ is continuous. Since p is open, $p\times p$ is also open and hence a quotient map. It follows from the universal property of quotient maps that $\bar\kappa$ is continuous, and therefore G/N is a topological group. The remaining claims follow from Propositions 1.7 and 1.27. □

Corollary 1.31. *If G is a locally compact group and if $N\trianglelefteq G$ is a closed normal subgroup, then G/N is a locally compact group.*

The next result is the Homomorphism Theorem for topological groups.

Lemma 1.32. *Let $f:G\longrightarrow K$ be a morphism of topological groups, and suppose that the normal subgroup $N\trianglelefteq G$ is contained in $\ker(f)$. Then f factors through the open morphism $p:G\longrightarrow G/N$ by a unique morphism $\bar f$,*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & K. \\ G/N \arrow[ur, "\bar f"'] & \end{tikzcd}$$

The map f is open if and only if $\bar f$ is open.

Proof. The group homomorphism $\bar f$ exists uniquely by the Homomorphism Theorem for groups. Since p is a quotient map, $\bar f$ is continuous and thus a morphism. If f is open and if $W\subseteq G/N$ is an open set, then $f(p^{-1}(W))=\bar f(W)$ is open as well. If $\bar f$ is open, then f is the composite of two open maps and therefore open. □

Corollary 1.33. *Suppose that $f:G\longrightarrow K$ is a morphism of topological groups and that K is Hausdorff. Then f factors uniquely through the open morphism $p:G\longrightarrow G/\overline{\{e\}}$,*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & K. \\ G/\overline{\{e\}} \arrow[ur, "\bar f"'] & \end{tikzcd}$$

Example 1.34. Suppose that $K\subsetneq\mathbb{R}$ is a nontrivial closed subgroup. Then $K=b\mathbb{Z}$ for some $b>0$ by Corollary 1.18. The compact interval $[0,b]$ maps onto $\mathbb{R}/K$ and therefore the quotient $\mathbb{R}/K$ is compact. We recall that the circle group is the compact abelian group

$$\mathrm{U}(1)=\{z\in\mathbb{C}\mid |z|=1\}\subseteq\mathbb{C}^\times.$$

If we put $f(t)=\exp(2\pi\boldsymbol{i}t/b)$, then $f:\mathbb{R}\longrightarrow\mathrm{U}(1)$ is a morphism with kernel K. Since $\mathbb{R}/K$ is compact, the induced morphism

$$\bar f:\mathbb{R}/K\longrightarrow\mathrm{U}(1)$$

that maps $t+b\mathbb{Z}$ to $\exp(2\pi\boldsymbol{i}t/b)$ is an isomorphism of topological groups.

Semidirect Products

Suppose that N is a topological group. We let $\mathrm{Aut}_{\mathbf{tg}}(N)$ denote the group of all automorphisms of N as a topological group. Suppose that K is another topological group and that $\rho : K \longrightarrow \mathrm{Aut}_{\mathbf{tg}}(N)$ is an abstract group homomorphism. If the map $K \times N \longrightarrow N$ that maps (k, n) to $\rho(k)(n)$ is continuous, then the semidirect product

$$G = N \rtimes_\rho K$$

is a topological group with respect to the product topology on the underlying set $N \times K$, and the group multiplication

$$(n_1, k_1)(n_2, k_2) = (n_1\rho(k_1)(n_2), k_1k_2).$$

Furthermore, the canonical morphism $G = N \rtimes_\rho K \longrightarrow K$ that maps (n, k) to k is open. The following result shows that this characterizes internal semidirect products.

Proposition 1.35. *Let G be a topological group and let $K, N \subseteq G$ be subgroups. Assume that K normalizes N, that $G = NK$, and that $N \cap K = \{e\}$. Then the map*

$$f : N \rtimes K \longrightarrow G, \quad (n, k) \longmapsto nk$$

is a bijective morphism of topological groups. The following are equivalent.
(i) *The morphism f is an isomorphism of topological groups.*
(ii) *The morphism f is open.*
(iii) *The canonical morphism $\bar{f} : K \longmapsto G/N$ that maps k to kN is open.*

If G is Hausdorff, then (i) implies that N and K are closed subgroups of G.

Proof. The map f is a bijective group homomorphism since G is, as an abstract group, the semidirect product of N and K. It is continuous because the multiplication in G is continuous. If f is an isomorphism of topological groups, then f is open. If f is open, then the morphism $N \rtimes K \xrightarrow{f} G \xrightarrow{q} G/N$ is open and the commutative diagram

$$\begin{tikzcd} N \rtimes K \arrow[r, "f"] \arrow[d, "p"'] & G \arrow[d, "q"] \\ K \arrow[r, "\bar{f}"] & G/N \end{tikzcd}$$

shows that the morphism $\bar{f}$ is open as well. If the bijective morphism $\bar{f}$ is open, then its inverse $j : G/N \longrightarrow K$ is continuous. The inverse of f is the continuous map $g = nk \longmapsto (n, k) = (gj(gN)^{-1}, j(gN))$. $\square$

In order to decompose a topological group topologically as a semidirect product, one has therefore to check that the bijective morphism $\bar{f}$ is open. For this, open mapping theorem are often helpful.

Split short exact sequences of topological groups behave well, as we show now.

Proposition 1.36. *Suppose that*

$$1 \longrightarrow N \hookrightarrow G \xrightarrow{f} K \longrightarrow 1$$

is a short exact sequence of morphisms of topological groups, with $N = \ker(f)$. If there is a morphism $s : K \longrightarrow G$ with $f \circ s = \mathrm{id}_K$, then G is as a topological group isomorphic to the semidirect product $N \rtimes_\rho K$, where $\rho(k)(n) = s(k)ns(k)^{-1}$.

Proof. We define a morphism $N \rtimes_\rho K \longrightarrow G$ via $(n, k) \longmapsto ns(k)$. Its inverse is the morphism $g \longmapsto (gs(f(g))^{-1}, f(g))$. □

The following example shows that an abstract product decomposition of a topological group need not be a topological decomposition. We modify Example 1.19 as follows.

Example 1.37. Let $a \in \mathbb{R}$ be an irrational number and consider the free abelian group $G = \mathbb{Z} + a\mathbb{Z} \subseteq \mathbb{R}$, endowed with the subspace topology in $\mathbb{R}$. We put $A = \mathbb{Z}$ and $B = a\mathbb{Z}$. Then A and B are closed subgroups of the group G which is, as an abstract group, the direct product of A and B. The canonical map $f : A \times B \longrightarrow G$ that maps (x, ay) to $x + ay$ is therefore bijective and continuous. It is not an isomorphism of topological groups because G is not a discrete group, see Lemma 1.17.

Connected Components

We recall that a topological space X is called *connected* if $\varnothing$ and X are the only subsets of X which are both open and closed. A subset $A \subseteq X$ is called connected if A is connected in the subspace topology. The closure of a connected subset is again connected.

Definition 1.38. Let x be a point in a topological space X. The *connected component* of x is the union of all connected subsets of X containing x. This union is closed and connected. We call a topological space X *totally disconnected* if the only connected nonempty subsets of X are the singletons. The terminology varies and some authors call such spaces hereditarily disconnected.

The connected component of the identity element of a topological group G will be denoted by G°, and we call G° the *identity component* of G. Since G acts via left translation transitively on G, the group G is totally disconnected if and only if $G^\circ = \{e\}$. Every subgroup of a totally disconnected group is again totally disconnected. We note also that a totally disconnected group is Hausdorff.

An example of a totally disconnected group is the additive group of the rational integers $\mathbb{Q}$. More interesting is the additive group of the p-adic field $\mathbb{Q}_p$. If F is a finite group, endowed with the discrete topology, then $G = F^{\mathbb{N}}$ is a compact totally disconnected group with respect to the topology of pointwise convergence.

Proposition 1.39. *Let G be a topological group. Then the identity component G° is a closed normal subgroup, and G/G° is a totally disconnected Hausdorff topological group.*

We call G/G° the *group of components* of G.

Proof. We put $\kappa(g,h) = g^{-1}h$ and we note that a continuous image of a connected set is connected. Since $G^\circ \times G^\circ$ is connected and contains the identity element, $\kappa(G^\circ \times G^\circ) \subseteq G^\circ$. This shows that G° is a subgroup. By the remark above, G° is closed. For every $a \in G$, the set $\gamma_a(G^\circ) = aG^\circ a^{-1}$ is connected and contains the identity, whence $aG^\circ a^{-1} \subseteq G^\circ$. This shows that G° is a closed normal subgroup.

It remains to show that G/G° is totally disconnected. We consider the canonical morphism $p : G \longrightarrow G/G^\circ$ and we put $H = (G/G^\circ)^\circ$ and $N = p^{-1}(H)$. Then N is a closed normal subgroup of G containing G°. We claim that $N = G^\circ$. If we have proved this claim, then $H = \{G^\circ\}$ and thus G/G° is totally disconnected. The restriction-corestriction $p|_N^H : N \longrightarrow H$ is an open map, hence H carries the quotient topology with respect to $p|_N^H$. Suppose that $V \subseteq N$ is closed and open in N and contains the identity. Since G° is connected and contains e, we have $vG^\circ \subseteq V$ for all $v \in V$. Hence $V = p^{-1}(p(V))$, and therefore $p(V)$ is closed and open in H. But H is connected, whence $H = p(V)$ and thus $V = N$. It follows that N is connected, whence $N = G^\circ$. ☐

Corollary 1.40. *Let $f : G \longrightarrow K$ be a morphism of topological groups. If K is totally disconnected, then f factors through the open morphism $p : G \longrightarrow G/G^\circ$,*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & K. \\ G/G^\circ \arrow[ur, "\bar f"'] & \end{tikzcd}$$

Proof. Since $f(G^\circ) \subseteq K$ is connected, G° is contained in the kernel of f. ☐

Van Dantzig's Theorem

We show the existence of arbitrarily small open subgroups in locally compact totally disconnected groups.

Lemma 1.41. *Let G be a locally compact group and suppose that V is a compact open identity neighborhood. Then V contains an open subgroup $H \subseteq G$.*

Proof. By Wallace's Lemma 1.21, applied to the compact set $V \times \{e\} \subseteq V \times V$, there exists an open symmetric identity neighborhood $U \subseteq V$ such that $VU \subseteq V$. In particular, $UU \subseteq V$, and thus $UUU \subseteq V$. By induction we conclude that for every $k \geq 1$ the k-fold product $U^{\cdot k}$ is contained in V. Hence the open subgroup $H = \langle U \rangle = \bigcup_{k \in \mathbb{N}} U^{\cdot k}$ is contained in V. ☐

In order to put this lemma to work, we need a result about totally disconnected locally compact spaces. A nonempty Hausdorff space is called *zero-dimensional* if every point has arbitrarily small closed open neighborhoods. A zero-dimensional space is therefore totally disconnected. The converse holds for locally compact spaces, as we prove below in Proposition 1.43. We note that every nonempty subspace of a zero-dimensional space is again zero-dimensional. Also, every zero-dimensional space is a Tychonoff space.

Lemma 1.42. *Suppose that X is a compact space and that $x \in X$. Then the set*

$$Q(x) = \bigcap \{D \subseteq X \mid D \text{ contains } x \text{ and } D \text{ is closed and open}\}$$

is connected.

In a general topological space, the set $Q(x)$ as defined above is called the *quasi-component* of x.

Proof. Clearly $Q(x)$ is closed and contains x. Suppose that $Q(x) = A \cup B$, with $x \in A$ and A, B closed and disjoint. We have to show that $B = \varnothing$. By Corollary 1.22, there exist disjoint open sets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$. We put $C = X - (U \cup V)$ and we note that C and $Q(x)$ are disjoint. For every $c \in C$ we can therefore choose an open and closed set W_c containing x, but not containing c. The open sets $X - W_c$ cover C. Since C is compact, there exists $c_1, \ldots, c_m \in C$ such that $C \subseteq (X - W_{c_1}) \cup \cdots \cup (X - W_{c_m})$. Hence C is disjoint from the set $W = W_{c_1} \cap \cdots \cap W_{c_m}$. Also, W is closed and open, and therefore $Q(x) \subseteq W$. Since W is disjoint from C, we have $W \subseteq U \cup V$. Now $Y = U \cup (X - W)$ is open and $Z = V \cap W$ is open and contains B. Since $X = Y \cup Z$ and $Y \cap Z = \varnothing$, the set Y is closed and open and thus $Q(x) \subseteq Y$. It follows that $B = \varnothing$. ☐

Proposition 1.43. *A nonempty locally compact space is totally disconnected if and only if it is zero-dimensional.*

Proof. Every zero-dimensional space is totally disconnected. Suppose that X is locally compact and totally disconnected. Let V be a neighborhood of a point $x \in X$. We claim that V contains a compact open neighborhood U of x. Passing to a smaller neighborhood if necessary, we may assume in addition that V is open and that $\overline{V}$ is compact. We put $A = \overline{V} - V$ and we note that we are done if $A = \varnothing$, with $U = V$. If $A \neq \varnothing$, we make use of Lemma 1.42. Since X is totally disconnected and $Q(x)$ is connected, $Q(x) = \{x\}$. Hence for each $a \in A$, there exists a compact neighborhood $U_a \subseteq \overline{V}$ of x which does not contain a, and which is open in $\overline{V}$. Then $\bigcap\{U_a \mid a \in A\} \cap A = \varnothing$. Hence there exist finitely many points $a_1, \ldots, a_m$ such that $U = U_{a_1} \cap \cdots \cap U_{a_m}$ is disjoint from A. Then U is closed and open in $\overline{V}$, and $U \subseteq V$. But then U is also open in X. ☐

Now we can prove van Dantzig's Theorem.

Theorem 1.44 (van Dantzig). *Let G be a locally compact group. Then the following are equivalent.*
(i) *G is totally disconnected.*
(ii) *G is zero-dimensional.*
(iii) *Every identity neighborhood in G contains an open subgroup.*

Proof. By Proposition 1.43, (i) $\Leftrightarrow$ (ii). We show that (ii) $\Rightarrow$ (iii). Suppose that $U \subseteq G$ is an identity neighborhood. Since G is zero-dimensional, U contains a compact open identity neighborhood V. By Lemma 1.41, there exists an open subgroup $H \subseteq V$ and (iii) follows. Suppose that (iii) holds and that $g \in G - \{e\}$. There exists an open and closed subgroup $H \subseteq G - \{g\}$, hence there is no connected set $C \subseteq G$ containing e and g, and (i) follows. ☐

Corollary 1.45. *In every locally compact group G, the identity component G° is the intersection of all open subgroups of G.*

Proof. In general, every open subgroup $H \subseteq G$ contains G°, because H is also closed. We consider the open morphism $p : G \longrightarrow G/G^\circ$ and we note that G/G° is totally disconnected by Proposition 1.39. If $g \in G - G^\circ$, then there exists an open subgroup $K \subseteq G/G^\circ - p(g)$ by Theorem 1.44. Then $H = p^{-1}(K)$ is an open subgroup of G which does not contain g. ☐

For compact groups there is a stronger form of van Dantzig's Theorem.

Theorem 1.46 (van Dantzig). *Let G be a compact group. Then the following are equivalent.*
(i) *G is totally disconnected.*
(ii) *G is zero-dimensional.*
(iii) *Every identity neighborhood contains an open normal subgroup.*

Proof. We show that (ii) $\Rightarrow$ (iii). Suppose that (ii) holds and that U is an identity neighborhood. By Theorem 1.44, there exist an open subgroup $H \subseteq G$ which is contained in U. Since $G = \bigcup G/H$ is compact, G/H is finite. Let N denote the kernel of the action of G on G/H. Then N has finite index in G, and $N = \bigcap\{aHa^{-1} \mid a \in G\}$ is closed. Since G/N is finite, G/N is discrete and therefore N is open, see Proposition 1.27. Claim (iii) follows now, since $N \subseteq H \subseteq U$. The remaining implications follow from Theorem 1.44. □

A compact group satisfying the three equivalent conditions in Theorem 1.46 is commonly called a *profinite group*. The next result is a weak version of the Peter–Weyl Theorem.

Lemma 1.47. *Let G be a profinite group and suppose that $g \in G$ is not the identity element. Then there exist an integer $n \geq 1$ and a morphism $\rho : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ with $\rho(g) \neq \mathbb{1}$.*

Proof. By Theorem 1.46 there exists an open normal subgroup $N \subseteq G - \{g\}$. Then $F = G/N$ is a finite group and $gN \neq N$. Let $f : F \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be an injective homomorphism, for some $n \geq 1$. Such a homomorphism exists, for example via the natural left action of F on the finite dimensional vector space $\mathbb{C}^F$. Then the composite

$$\rho = f \circ p : G \longrightarrow G/N \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

is a morphism with $\rho(g) \neq \mathbb{1}$. □

Suppose that $(F_i)_{i \in I}$ is a family of finite groups. If we endow each group F_i with the discrete topology, then the F_i are compact and the product

$$G = \prod_{i \in I} F_i$$

is a compact group. From the definition of the product topology we see that G is a profinite group. Conversely, every profinite group is a closed subgroup of such a product.

Proposition 1.48. *Let G be a profinite group. Then there exists a family of finite groups $(F_i)_{i \in I}$ and a closed injective morphism $f : G \longrightarrow \prod_{i \in I} F_i$. Hence G is isomorphic as a topological group to a closed subgroup of a product of finite groups.*

Proof. Let I denote the set of all open normal subgroups of G. We put $F_N = G/N$ and $f_N(g) = gN$, for every $N \in I$. Then each F_N is a finite group and the f_N fit together to a diagonal morphism $f : G \longrightarrow \prod_{N \in I} F_N$. Since the open normal subgroups form a neighborhood base of the identity in G by Theorem 1.46, the morphism f is injective. Since G is compact, f is closed. □

Weil's Lemma

Suppose G is a Hausdorff topological group and that $A \subseteq \mathbb{R}$ is a closed subgroup. Thus $A = a\mathbb{Z}$ for some real number $a \geq 0$ or $A = \mathbb{R}$ by Corollary 1.18. If $f : A \longrightarrow G$ is a morphism with nontrivial kernel K, then f factors through the canonical morphism $p : A \longrightarrow A/K$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & G \\ A/K. \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Then $A/K \subseteq \mathbb{R}/K$ is compact, as we showed in Example 1.34. Therefore the induced morphism $A/K \longrightarrow f(A)$ is an isomorphism of topological groups. If we put

$$A_{\geq r} = \{a \in A \mid a \geq r\}$$

for $r \in A$, then $K + A_{\geq r} = A$ and thus $f(A) = f(A_{\geq r})$. Weil's Lemma extends these facts.

Theorem 1.49 (Weil's Lemma). *Let $A \subseteq \mathbb{R}$ be a closed subgroup and suppose that $f : A \longrightarrow G$ is a morphism from A to a locally compact group G. Then either $f(A)$ is a closed noncompact subgroup and the corestriction $A \longrightarrow f(A)$ is an isomorphism of topological groups, or $\overline{f(A)}$ is compact and $\overline{f(A)} = \overline{f(A_{\geq r})}$ holds for every $r \in A$.*

Proof. If f is not injective, then the remarks preceding this theorem show that $f(A)$ is compact and then the claim is true. It remains to consider the case where f is injective. If $A = \{0\}$ there is nothing to show. Hence we may assume that $A \neq \{0\}$. Replacing G by the locally compact abelian group $\overline{f(A)}$, we may as well assume that $f(A)$ is dense in G and that G is abelian, see Lemma 1.10. We note that $U \cap f(A) \neq \varnothing$ holds whenever $U \subseteq G$ has nonempty interior and we distinguish two cases.

Case 1. There is a nonempty open set $U \subseteq G$ such that $f^{-1}(U)$ has an upper bound.

We choose $u \in f(A) \cap U$. Then $W = (u^{-1}U) \cap (U^{-1}u)$ is an open identity neighborhood and $V = f^{-1}(W)$ is bounded and open. Hence $f(\overline{V}) \supseteq f(A) \cap W$ is a compact identity neighborhood of $f(A)$. Then $f(A) \subseteq G$ is locally compact and therefore closed by Corollary 1.14. Therefore $f(A) = G$ and f is bijective. The restriction-corestriction $f : \overline{V} \longrightarrow f(\overline{V})$ is a homeomorphism because $\overline{V}$ is compact, and $W \subseteq f(\overline{V})$ is an identity neighborhood. Hence the inverse of f is continuous at the identity, and therefore the inverse of f is a morphism by Lemma 1.3.

Case 2. $\varnothing \neq f^{-1}(U) \cap A_{\geq r}$ holds for every $r \in A$ and every nonempty open set $U \subseteq G$.

Then $\overline{f(A_{\geq r})} = G$. We claim that G is compact. Let $V \subseteq G$ be a compact identity neighborhood. For every $g \in G$ and $r > 0$ we have $gV^{-1} \cap f(A_{\geq r}) \neq \varnothing$, whence $G =$

$f(A_{\geq r})V$. Therefore there is a finite set $X \subseteq A_{\geq r}$ such that $V \subseteq f(X)V$. We put $m = \max(X)$ and we claim that

$$G = f(A \cap [0, m])V^{-1}. \tag{1}$$

Given $g \in G$, we put $a = \min(A_{\geq 0} \cap f^{-1}(gV))$. Then $f(a) \in gV$ and thus $f(a) \in gf(x)V$ for some $x \in X$. Hence $f(a - x) \in gV$. Since $x \geq r > 0$, we have $a - x < a$ and therefore $a - x < 0$, whence $0 \leq a < m$. Therefore $g \in f(a)V^{-1} \subseteq f(A \cap [0, m])V^{-1}$ and (1) is true. Since $f(A \cap [0, m])$ is compact and V^{-1} is compact, G is compact. □

The following result about compact subsemigroups in topological groups is an immediate consequence.

Proposition 1.50. *Suppose that G is a Hausdorff topological group and that $\varnothing \neq S \subseteq G$ is a compact subsemigroup. Then S is a subgroup.*

Proof. If $s \in S$, then the cyclic group $\langle s \rangle$ generated by s is contained in the compact set $S^{-1} \cup \{e\} \cup S$. Hence $\langle s \rangle$ has compact closure $H = \overline{\langle s \rangle}$ in G. We consider the morphism $f : \mathbb{Z} \longrightarrow H$ that maps k to $f(k) = s^k$ and we note that $C = \overline{\{s^k \mid k \geq 1\}} \subseteq S \cap H$ is compact. By Weil's Lemma 1.49 we have $\langle s \rangle \subseteq H = C$. Hence S is a subgroup. □