# Holomorphic polygons 

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Holomorphic projective planes have been classified by Breitsprecher; he showed that up to holomorphic isomorphism, there is only one such plane, namely the projective plane over the complex numbers [B1], [B2, 1.5.7]. The idea of his proof is roughly as follows: every point row is a Riemann sphere, hence the group of all projectivities is isomorphic to $\mathbf{P G L}_{2} \mathbb{C}$. It follows by von Staudt's Theorem that the plane is Pappian and thus isomorphic to the projective plane over the commutative field $\mathbb{C}$.

Projective planes are generalized triangles. In this paper we prove that up to holomorphic isomorphism, there are precisely five holomorphic polygons (three up to duality), namely the Moufang polygons associated to the simple complex Lie groups $\mathbf{P G L}_{3} \mathbb{C}, \mathbf{P S p}_{4} \mathbb{C} \cong \mathbf{P S O}_{5} \mathbb{C}$, and $\mathbf{G}_{2}^{\mathbb{C}}$ (2.11).

Now for generalized polygons, no analogue of von Staudt's Theorem is presently known (although it seems conceivable that a corresponding result holds). Thus our proof follows a different and much more analytic line than Breitsprecher's proof. We show directly (using the Riemann-Roch Theorem) that the derived incidence structure $\mathfrak{A}_{p}$ of a holomorphic polygon $\mathfrak{P}$ is isomorphic to the projective plane over $\mathbb{C}$. Using results of Schroth and Schroth-Van Maldeghem, this leads to a classification for $n=4,6$. The Riemann-Roch Theorem yields also a new (and completely analytic) proof of Breitsprechers result.

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## 1 Generalized polygons

Recall that an incidence structure $\mathfrak{P}=(\mathscr{P}, \mathscr{C}, \mathscr{F})$ is a triple consisting of a set $\mathscr{P}$ of points, a set $\mathscr{C}$ of lines, and a set $\mathscr{F} \subseteq \mathscr{P} \times \mathscr{B}$ of flags, describing the incidence relation. A (polygonal) $k$-chain $\left(x_{0}, x_{1}, \ldots, x_{k}\right)$ is a sequence of
elements $x_{i} \in \mathscr{P} \cup \mathscr{B}$ with the property that $x_{i}$ is incident with $x_{i-1}$ for $i=$ $1, \ldots, k$.We say that the $k$-chain stammers if $x_{i}=x_{i-2}$ for some $i$ with $1 \leq i \leq k$.
1.1 Definition An incidence structure $\mathfrak{P}=(\mathscr{P}, \mathscr{B}, \mathscr{F})$ is called a generalized $n$-gon if it satisfies the following three axioms:
(i) Every element $x \in \mathscr{P} \cup \mathscr{L}$ is incident with at least three other elements.
(ii) Any two elements $x, y \in \mathscr{P} \cup \mathscr{L}$ can be joined by a polygonal chain of length $\leq n$. We denote by $d(x, y)$ the length of a minimal chain joining $x$ and $y$.
(iii) If $d(x, y)<n$, then there is exactly one minimal chain joining $x$ and $y$.

Generalized digons are not interesting, therefore we assume always that $n>2$.

If $x, y$ are elements with $d(x, y)=n-1$, then there is a unique element $z$ incident with $y$ such that $d(x, z)=n-2$ (the projection of $x$ to $y$ ). We put $f_{n-1}(x, y)=z$.

The generalized triangles are precisely the projective planes; for two points $p, q \in \mathscr{P}$, the element $f_{2}(p, q)$ is simply the line $p \vee q$ joining $p$ and $q$.

For a point $p \in \mathscr{P}$ we let $\mathscr{L}_{p}$ denote the pencil of lines passing through $p$. Similarly, for a line $\ell \in \mathscr{B}$, we let $L$ denote the point row corresponding to $\ell$, consisting of all points lying on $\ell$.

Finally we put

$$
\begin{aligned}
p^{\perp} & =\{q \in \mathscr{P} \mid q \text { lies on some line passing through } p\} \\
& =\{q \in \mathscr{P} \mid d(p, q) \leq 2\} .
\end{aligned}
$$

For our purposes it is convenient to make the following definition.
1.2 Definition Let $\mathfrak{P}=(\mathscr{P}, \mathscr{B}, \mathscr{F})$ be a generalized $n$-gon, and let $p \in \mathscr{P}$ be a point. We define an incidence structure $\mathfrak{A}_{p}$ as follows (cp. [Sch], [S-vM]): the points of $\mathfrak{A}_{p}$ are the points in $p^{\perp}$, the lines of $\mathfrak{A}_{p}$ are the point rows passing through $p$, as well as the sets

$$
p^{x}=\left\{q \in p^{\perp} \mid d(q, x)=n-2\right\}
$$

where $x \in \mathscr{P} \cup \mathscr{C}$ is an element with $d(x, p)=n$. The incidence relation of $\mathfrak{A}_{p}$ is given by the membership relation $\epsilon$.

For $n=3$, i.e. in the case of projective planes, the incidence structure $\mathfrak{A}_{p}$ is of course isomorphic to $\mathfrak{P}$. It is immediate from the definition that $\mathfrak{A}_{p}$ has the following property ( ${ }^{*}$ ):
$\left.{ }^{( }\right)$any two points in $p^{\perp}$ can be joined by a (possibly not unique) line of $\mathfrak{A}_{p}$.

## 2 Holomorphic polygons

2.1 Definition Let $\mathfrak{P}=(\mathscr{P}, \mathscr{B}, \mathscr{F})$ be a generalized $n$-gon. Suppose that $\mathscr{P}$ and $\mathscr{L}$ are manifolds of positive dimension, and that the map $f_{n-1}$ is continuous
on its domain. Then $\mathscr{P}$ and $\mathscr{B}$ are compact and connected, and the domain of $f_{n-1}$ is open [ $\left.\mathrm{G}-\mathrm{vM}, 2.1(\mathrm{~b})\right]$, [ $\mathrm{Kr}, \mathrm{Ch} .2$ ]. We call $\mathfrak{P}$ a holomorphic polygon if the manifolds $\mathscr{P}$ and $\mathscr{C}$ carry complex structures, and if the map $f_{n-1}$ is holomorphic on its domain.
2.2 Examples The complex projective plane is an example of a holomorphic projective plane. Here is another one: consider the symplectic form $\omega=\left({ }_{-1}{ }^{1}\right)$ on $\mathbb{C}^{4}$. Put $\mathscr{P}=\mathbb{C} P^{3}$ and let $\mathscr{B}=\left\{L \in G_{2}\left(\mathbb{C}^{4}\right)|w|_{L \times L}=0\right\}$ denote the set of all totally isotropic two-dimensional subspaces of $\mathbb{C}^{4}$ with respect to $\omega$. Then $(\mathscr{P}, \mathscr{L}, \subseteq)$ is a holomorphic quadrangle, the complex symplectic quadrangle. For $P \in \mathscr{P}$ and $L \in \mathscr{L}$ with $P \not \subset L$ we have $f_{3}(P, L)=P^{\perp} \cap L$, and $f_{3}(L, P)=$ $f_{3}(P, L) \oplus P$, where $\perp$ is taken with respect to $\omega$. Clearly, the group $\mathbf{P S} \mathbf{p}_{4} \mathbb{C}$ acts as a group of automorphisms on this incidence structure.
2.3 Proposition Suppose that $\mathfrak{P}=(\mathscr{P}, \mathscr{B}, \mathscr{F})$ is a generalized polygon, that $\mathscr{P}$ and $\mathscr{L}$ are complex manifolds, that $\mathscr{F} \subseteq \mathscr{P} \times \mathscr{L}$ is a complex submanifold, and that the projections $\mathrm{pr}_{1}: \mathscr{F} \rightarrow \mathscr{P}$ and $\mathrm{pr}_{2}: \mathscr{F} \rightarrow \mathscr{C}$ are submersions. Then $\mathfrak{P}$ is a holomorphic $n$-gon. In particular, the Moufang polygons associated to the simple complex Lie groups $\mathbf{P G L}_{3} \mathbb{C}, \mathbf{P S p}_{4} \mathbb{C} \cong \mathbf{P S O}{ }_{5} \mathbb{C}$, and $\mathbf{G}_{2} \mathbb{C}$ are holomorphic polygons.

Proof. First, $\mathfrak{P}$ is a compact connected polygon by [G-vM, 2.1(a)], [Kr, 2.5.4].
It follows readily from the assumptions and transversality that the set of all ( $n-1$ )-chains is a complex manifold. The set of all non-stammering ( $n-1$ )chains is an open subset and hence also a complex manifold. The map that sends a non-stammering $(n-1)$-chain $\left(x_{0}, x_{1}, \ldots, x_{n-1}\right)$ to ( $x_{0}, x_{n-1}$ ) is a holomorphic homeomorphism onto the complex manifold consisting of all pairs of elements $(x, y)$ with distance $d(x, y)=n-1$. Hence the inverse of this map is also holomorphic, see [G-H, p.19] e.g., and thus $x_{n-2}$ depends holomorphically on $\left(x_{0}, x_{n-1}\right)$.

### 2.4 Lemma The sphere $\mathbb{S}^{4 k}$ does not admit a complex structure for $k \geq 1$.

Proof. It follows from the Hirzebruch Signature Theorem [M-S, 19.4] that the total Pontrjagin class $\mathbf{p}=1+\mathbf{p}_{k}$ of $\mathbb{S}^{4 k}$ (with respect to any differentiable structure) is trivial. On the other hand, the total Chern class of the tangent bundle of a complex sphere is necessarily of the form $\mathbf{c}=1+2 x$, where $x$ is the orientation class of the tangent bundle, since the Euler number of an even-dimensional sphere is 2 , see [M-S, $14.1, \mathrm{p} .158,11.12$ ]. If the dimension of a complex sphere is $4 k$, for some $k>0$, then we have the relation

$$
1=1+(-1)^{k} \mathbf{p}_{k}=\left(1+\mathbf{c}_{2 k}\right)^{2}=(1+2 x)^{2}=1+4 x
$$

by [M-S, 15.5], a contradiction.
2.5 Proposition Let $\mathfrak{P}=(\mathscr{P}, \mathscr{C}, \mathscr{F})$ be a holomorphic n-gon. Every point row $L \subseteq \mathscr{P}$ is a complex submanifold and biholomorphically equivalent to the Riemann sphere $\mathbb{C} P^{1}$. Thus $n \in\{3,4,6\}$ by Knarr's result [Kn2], and $\mathfrak{P}$ has topological parameters $(2,2), c p .[K r, 3.3 .6]$. (See also Breitsprecher [B1] for the case of holomorphic projective planes.)

Proof. Let $x$ be a vertex opposite to the line $\ell$ corresponding to the point row $L$. The set $U=\{p \in \mathscr{P} \mid d(p, x)=n-1\}$ is open and connected (since it is the union of two big cells, cp . $[\mathrm{Kr}, 2.2,2.4]$ ), and it contains the point row $L$. The map

$$
p \mapsto f_{n-1}\left(f_{n-1}(p, x), \ell\right)
$$

is a holomorphic retraction of $U$ onto $L$, hence $L$ is a complex submanifold of $U$, cp. [B-J, 5.13], [H, p. 20], [B-K, 4.2] (the proof given there is also vaild in the complex case, since it uses mainly the implicit function theorem).

By [Kn2, 2.1], [Kr, 4.1.2] the point row $L$ is a topological sphere, and since the dimension $m$ of the point rows and the dimension $m^{\prime}$ of the pencils of lines is even, we conclude that $m=m^{\prime} \in\{2,4,8\}$, see [Kn2], [G-K-K, 1.7], [Kr, 3.3.6]. By 2.4 , we have $m=m^{\prime}=2$.

Finally, a complex 2-sphere has genus $g=0$, and thus it is biholomorphically equivalent to $\mathbb{C} P^{1}$ by [G-H, p. 222].
2.6 Proposition Let $\mathfrak{P}$ be a holomorphic polygon, and let $\ell \in \mathscr{L}$ be a line. The group of all projectivities $\Pi(\ell)$ acting on the point row $L$ is isomorphic to $\mathbf{P G L}_{2} \mathbb{C}$ acting in the usual way on the complex projective line $\mathbb{C} P^{1}$. Thus, $\Pi(\ell)$ is sharply 3-transitive on the point row L. The same holds for the group of all projectivities $\Pi I(p)$ acting on $\mathscr{L}_{p}$, where $p$ is a point.

Proof. The group of all projectivities $\Pi(\ell)$ is doubly transitive on $L$ by [Kn1, 1.2], [G-K-K, 1.4], or [Kr, 1.8.2]. On the other hand, every projectivity is a biholomorphic map of $L \cong \mathbb{C} P^{1}$. Thus $\Pi(\ell) \subseteq \operatorname{Aut}\left(\mathbb{C} P^{1}\right)=\mathbf{P G L}{ }_{2} \mathbb{C}$ (see [G-H, p. 64]) is a doubly transitive subgroup, and hence $\Pi(\ell)=\mathbf{P S L}_{2} \mathbb{C}=\mathbf{P G L}_{2} \mathbb{C}$ by [St, Satz 1].
2.7 Lemma Consider the usual action of $\mathbf{P G L}_{2} \mathbb{C}$ on the topological sphere $\mathbb{S}^{2}$. There are precisely two complex structures on $\mathbb{S}^{2}$ such that every element of $\mathbf{P G L}_{2} \mathbb{C}$ acts as a biholomorphic map on $\mathbb{S}^{2}$.

Proof. We endow $\mathrm{PGL}_{2} \mathbb{C}$ with the compact-open topology; thus it becomes in a unique way a Lie transformation group. It is well-known that the group of all continuous (and hence real-analytic) automorphisms of $\mathbf{P G L} L_{2} \mathbb{C}$ is $\langle\sigma\rangle \cdot \mathbf{P G L}_{2} \mathbb{C}$, where $\sigma$ denotes complex conjugation. Thus there are precisely two complex analytic Lie group structures on the topological group $\mathbf{P G L}_{2} \mathbb{C}$. On the other hand, each complex structure on $\mathbb{S}^{2}$ compatible with the action of $\mathbf{P G L}_{2} \mathbb{C}$ determines a complex structure on this group.

The following may be compared to the results in $[B-K]$.
2.8 Theorem Let $\mathfrak{P}$ be a holomorphic polygon. The complex structure on the point space $\mathscr{P}$ and the line space $\mathscr{B}$ is uniquely determined by the complex
structure on a single row point $L \subseteq \mathscr{P}$. In particular, the underlying topological polygon of a holomorphic polygon admits precisely two complex structures which make it into a holomorphic polygon, and the identity map is an anti-holomorphic isomorphism between these two holomorphic polygons.

Proof. On $L$, there are precisely two complex structures compatible with the action of the transformation group $\mathbf{P G L}_{2} \mathbb{C}$ by 2.7 .

Given any other point row $L^{\prime}$, there exists a (necessarily biholomorphic) projectivity between $L$ and $L^{\prime}$. Thus the complex structure on $L$ determines the complex structure on every other point row. Finally, there exists a nonconstant holomorphic map of $L$ into some pencil of lines $\mathscr{E}_{p}$, see [G-K-K, 1.1], [B-K, 1.7], [ $\mathrm{Kr}, 1.8 .1$ ], hence the complex structure on $L$ determines the complex structure on each pencil of lines. Thus a continuous automorphism of $\mathscr{P}$ is either holomorphic or anti-holomorphic on each point row and on each pencil of lines. It is an immediate consequence of the coordinatization of the polygon, see [B-K, 1.6], [ $\mathrm{Kr}, 1.7 .2$ ], that an automorphism which is (anti-) holomorphic on each point row and on each pencil of lines is (anti-) holomorphic on $\mathscr{P}$ and $\mathscr{L}$.
2.9 Corollary Let $\psi: \mathfrak{P} \rightarrow \mathfrak{P}^{\prime}$ be a continuous isomorphism between two holomorphic n-gons, $c p$. [B-K]. Then $\psi$ is either holomorphic or anti-holomorphic.

Our next aim is the following result:
2.10 Theorem Let $\mathfrak{P}=(\mathscr{P}, \mathscr{C}, \mathscr{F})$ be a holomorphic polygon. Then the following statement holds up to duality, i.e. up to exchanging $\mathscr{P}$ and $\mathscr{C}$ :

For every point $p \in \mathscr{P}$, there exists a homeomorphism of $p^{\perp}$ onto the complex projective plane $\mathbb{C} P^{2}$ that maps the lines of $\mathfrak{A}_{p}$ bijectively onto the point rows of $\mathbb{C} P^{2}$.
2.11 Corollary Let $\mathfrak{P}$ be a holomorphic generalized n-gon. Then $\mathfrak{P}$ is biholomorphically isomorphic to one of the following holomorphic Moufang polygons associated to the groups $\mathbf{P G L}_{3} \mathbb{C}, \mathbf{P S p}_{4} \mathbb{C} \cong \mathbf{P S O}_{5} \mathbb{C}$, and $\mathbf{G}_{2}^{\mathbb{C}}$, respectively:
$\mathbf{3}_{\mathbb{C}}$ the projective plane over $\mathbb{C}$.
$\mathbf{4}_{\mathbb{C}}$ the symplectic quadrangle over $\mathbb{C}$ or its dual, the complex orthogonal quadrangle in $\mathbb{C} P^{4}$.
$\mathbf{6}_{\mathbb{C}}$ the complex split Cayley hexagon or its dual.

Proof of the Corollary. $(\mathbf{n}=\mathbf{3})$ : By 2.10 there exists an abstract isomorphism which is a homeomorphism on the point space. By [B-K, 3.5], this isomorphism is a homeomorphism on the line space as well. Hence it is either holomorphic or anti-holomorphic by 2.9 . Composing the homomorphism with an antiholomorphic automorphism of the complex projective plane, if necessary, we get a holomorphic isomorphism.
( $\mathbf{n}=4$ ): By 2.10 and Schroth's characterization of the symplectic quadrangle over $\mathbb{C}$ [Sch], [S-vM, 4.5], we get a homeomorphic isomorphism onto the complex symplectic quadrangle or its dual. The other steps are the same as in the case ( $n=3$ ).
( $\mathbf{n}=\mathbf{6}$ ): Again, by 2.10 and by Schroth's and Van Maldeghem's characterization [S-vM, 1.1] of the split Cayley hexagon over $\mathbb{C}$, we get a homeomorphic isomorphism onto the complex split Cayley hexagon or its dual. The other steps are as in the case $(n=3)$.

We divide the proof of 2.10 into several lemmata.
In the sequel it is convenient to have the following explicit construction of the holomorphic Hopf line bundle $\eta_{\mathbb{C}}$ over $\mathbb{C} P^{1}$. We use homogeneous coordinates in $\mathbb{C} P^{2}$.

### 2.12 Lemma Consider the projective line

$$
\mathbb{C} P^{1}=\left\{\left[z_{1}, z_{2}, z_{3}\right] \in \mathbb{C} P^{2} \mid z_{3}=0\right\} \subseteq P^{2}
$$

Put $o=[0,0,1]$. The map $\left[z_{1}, z_{2}, z_{3}\right] \mapsto\left[z_{1}, z_{2}, 0\right]$ defines a holomorphic line bundle $\mathbb{C} P^{2}-\{o\} \rightarrow \mathbb{C} P^{1}$, the Hopf line bundle $\eta_{\mathbb{C}}$. The map $z \mapsto\left[z_{1}, z_{2}, z\right]$ induces a natural vector space structure on the fiber $E_{\left[z_{1}, z_{2}, 0\right]}$ of $\eta_{\mathbb{C}}$ over $\left[z_{1}, z_{2}, 0\right]$.

Every point row $H \subseteq \mathbb{C} P^{2}$ that does not meet $o$ is of the form $H=$ $\left\{\left[z_{1}, z_{2}, z_{3}\right] \in \mathbb{C} P^{2} \mid a_{1} z_{1}+a_{2} z_{2}+a_{3} z_{3}=0\right\}$ for some complex numbers $a_{1}, a_{2} \in \mathbb{C}$. The map $\mathbb{C} P^{1} \rightarrow H,\left[z_{1}, z_{2}, 0\right] \mapsto\left[z_{1}, z_{2},-a_{1} z_{1}-a_{2} z_{2}\right]$ provides a holomorphic section of this bundle. Thus we may define a surjective homomorphism $\phi$ from the complex vector space $\mathbf{H}^{0}\left(\mathbb{C} P^{1} ; \eta_{\mathbb{C}}\right)$ of all holomorphic sections of $\eta_{\mathbb{C}}^{2}$ as follows: pick two distinct fibers $E_{a}, E_{b}$ of $\eta_{\mathbb{C}}$ and consider the map

$$
\begin{array}{rlrl}
\phi: \quad \mathbf{H}^{0}\left(\mathbb{C} P^{1} ; \eta_{\mathbb{C}}\right) & \rightarrow & E_{a} \times E_{b} & \cong \mathbb{C}^{2} \\
s & \mapsto & \mapsto(a), s(b)) .
\end{array}
$$

From the Riemann-Roch Theorem we get the following result:
2.13 Lemma Let $\eta_{\mathbb{C}}$ denote the holomorphic Hopf line bundle over $\mathbb{C} P^{1}$, and let $\bar{\eta}_{\mathbb{C}}$ denote the conjugate bundle, cp. [M-S, p.167]. The complex vector space $\mathbf{H}^{0}\left(\mathbb{C} P^{1} ; \eta_{\mathbb{C}}\right)$ of all holomorphic sections of $\eta_{\mathbb{C}}$ is two-dimensional, and the complex vector space $\mathbf{H}^{0}\left(\mathbb{C} P^{1} ; \bar{\eta}_{\mathbb{C}}\right)$ of all holomorphic sections of $\bar{\eta}_{\mathbb{C}}$ is zerodimensional.

Proof. We use the notation of [G-H].
Recall that a divisor on a smooth algebraic curve $L$ over $\mathbb{C}$ is just a finite formal linear combination $D=\sum n_{i} p_{i}$ of points of $L$. The degree of a divisor is $\operatorname{deg}(D)=\sum n_{i}$; a divisor is called effective, if all coefficients $n_{i}$ are non-negative, cp. [G-H, p. 130].

To each divisor $D$ one can associate a holomorphic line bundle $[D]$ over $L$; this yields a group homomorphism from the additive group $\operatorname{Div}(L)$ of all divisors on $L$ to the set of all equivalence classes of holomorphic line bundles on $L$ (with the tensor product as group operation), see [G-H], p. 132]. Given a divisor $D$ let $h^{0}(D)$ denote the dimension of the complex vector space $\mathbf{H}^{0}(L ; \mathscr{O}(D))$ of
all holomorphic sections of the line bundle [ $D$ ], see [G-H, p. 137]. If $D$ is not effective (in particular, if $\operatorname{deg}(D)<0$ ), then $h^{0}(D)=0[\mathrm{G}-\mathrm{H}, \mathrm{p} .136]$.

Let $D$ be the divisor associated to $\eta_{\mathbb{C}}$, and let $K$ denote the canonical divisor of $\mathbb{C} P^{1}$, i.e. the divisor associated to the cotangent bundle of $\mathbb{C} P^{1}, \mathrm{cp}$. [G-H, p. 146]. The degree of $D$ is $\operatorname{deg}(D)=1$, and $\operatorname{deg}(K)=-2$ by $[\mathrm{G}-\mathrm{H}, \mathrm{p}$. 144]. Thus $\operatorname{deg}(K-D)=-3$, and hence $h^{0}(K-D)=0$, because the divisor $K-D$ is not effective. The genus of $\mathbb{C} P^{1}$ is $g=0$. From the Riemann-Roch Theorem

$$
h^{0}(D)-h^{0}(K-D)=\operatorname{deg}(D)-g+1
$$

[G-H, p. 245] we conclude that $h^{0}(D)=\operatorname{dim}_{\mathbb{C}} \mathbf{H}^{0}\left(\mathbb{C} P^{1} ; \eta_{\mathbb{C}}\right)=2$.
Now $-D$ is the divisor associated to $\bar{\eta}_{\mathbb{C}}$; it has degree $\operatorname{deg}(-D)=-1$, hence we have $h^{0}(-D)=\operatorname{dim}_{\mathbb{C}} \mathbf{H}^{0}\left(\mathbb{C} P^{1} ; \bar{\eta}_{\mathbb{C}}\right)=0$.

Combining the two lemmata above, we get the following corollary:
2.14 Corollary The map $\phi$ defined in 2.12 is a vector space isomorphism.

Now we return to the holomorphic polygon $\mathfrak{P}=(\mathscr{P}, \mathscr{L}, \mathscr{F})$.
2.15 Lemma Let $p \in \mathscr{P}$ be a point, and choose an element $x_{0} \in \mathscr{P} \cup \mathscr{C}$ with $d\left(x_{0}, p\right)=n$. The map

$$
\begin{aligned}
p^{\perp}-\{p\} & \rightarrow \mathscr{L}_{p} \\
q & \mapsto p \vee q
\end{aligned}
$$

defines a holomorphic line bundle $\eta$ over $\mathscr{E}_{p} \cong \mathbb{C} P^{1}$ with zero-section $p^{x_{0}}$ (we use the notation of 1.2).

For every element $x \in \mathscr{P} \cup \mathscr{C}$ with $d(x, p)=n$, there exists a holomorphic section of $\eta$ that has $p^{x}$ as its image.

Proof. Choose a line $\ell \in \mathscr{C}_{p}$ and put $y=f_{n-1}\left(\ell, x_{0}\right)$. Next, put $\infty_{K}=f_{n-1}(p, y)$, and let ( $z_{0}=\infty_{K}, z_{1}, \ldots, z_{n-2}=p$ ) denote the unique ( $n-2$ )-chain joining $\infty_{K}$ and $p$. Finally, let $K$ denote the set of all elements incident with $y$, and let $L$ denote the point row corresponding to $\ell$. For each $q \in p^{\perp}-L$ we have $p \vee q=f_{n-1}\left(z_{1}, q\right)$, and thus we get a trivialization of this bundle over $\mathscr{L}_{p}-\{\ell\}$ by

$$
\begin{aligned}
& p^{\perp}-L \rightarrow \\
& q \mapsto \\
& q\left(\mathscr{C}_{p}-\{\ell\}\right) \times\left(K-\left\{\infty_{K}\right\}\right) \\
&\left(f_{n-1}\left(z_{1}, q\right), f_{n-1}(q, y)\right) .
\end{aligned}
$$

Next, choose a biholomorphic map $\left(K-\left\{\infty_{K}\right\}, x_{0}\right) \stackrel{\cong}{\rightrightarrows}(\mathbb{C}, 0)$. Note that this map is unique up to some scalar factor; hence we get a well-defined complex vector space structure on $K-\left\{\infty_{K}\right\}$ compatible with the linear action of $\Pi(y)_{\infty_{K}, x_{0}} \cong$ $\mathbb{C}^{*}$ on $K-\left\{\infty_{K}\right\}$. It is readily verified that the structure group of this bundle with respect to the charts of this type is $\Pi(y)_{\infty_{K}, x_{0}} \cong \mathbb{C}^{*}$, hence we get indeed a holomorphic line bundle.

Finally, if $x$ is an element of $\mathscr{P} \cup \mathscr{L}$ with $d(x, p)=n$, we may define a holomorphic section of $\eta$ by putting $\ell \rightarrow f_{n-1}(x, \ell)$.
2.16 Lemma Up to duality we may assume that for each $p \in \mathscr{P}$ the cohomology ring of $p^{\perp}$ is given by $\mathbf{H}^{\bullet}\left(p^{\perp}\right)=\mathbb{Z}[x] /\left(x^{3}\right)$. In this case the holomorphic line bundle $\eta$ defined in 2.15 is holomorphically equivalent to the complex Hopf line bundle.

Proof. Up to duality, the cohomology ring of $\mathscr{P}$ is given as $\mathbb{Z}[x] /\left(x^{3}\right), \mathbb{Z}[x] /\left(x^{4}\right)$, and $\mathbb{Z}[x, y] /\left(x^{3}-2 y, y^{2}\right)$ for $n=3,4,6$, respectively, where $x$ is an element of degree 2 and $y$ has degree 6 , see [G-K-K, App.]. The inclusion $p^{\perp} \subseteq \mathscr{P}$ induces an isomorphism for the cohomology groups in dimension $\leq 2 m$, and kills the cohomology groups of degree $>2 m$, see $[\mathrm{Kr}, \mathrm{Ch} .4]$, hence the claim about the cohomology ring follows.

The cohomology ring of the 2 -sphere is $\mathbf{H}^{\bullet}\left(\mathbb{S}^{2}\right)=\mathbb{Z}[x] /\left(x^{2}\right)$. If $\xi$ is a complex line bundle over $\mathbb{S}^{2}$ with total Chern class $\mathbf{c}=1+a x$, for some $a \in \mathbb{Z}$, then the Euler class of the underlying oriented real vector bundle is given by $\mathbf{e}=a x$, cp. [M-S, p. 158]. Let $U_{\xi} \in \mathbf{H}^{2}\left(E, E_{0}\right)$ denote the orientation class of $\xi$, and let $\theta: \mathbf{H}^{\bullet}\left(\mathbb{S}^{2 k}\right) \rightarrow \mathbf{H}^{\bullet}\left(E, E_{0}\right)$ denote the Thom isomorphism [M-S, p. 110]. Now $\theta(1)=U_{\xi}=u$ generates $\mathbf{H}^{2}\left(E, E_{0}\right)$, and $\theta(x)=v$ generates $\mathbf{H}^{4}\left(E, E_{0}\right)$, since 1 and $x$ generate $\mathbf{H}^{0}\left(\mathbb{S}^{2}\right)$ and $\mathbf{H}^{2}\left(\mathbb{S}^{2}\right)$, respectively. On the other hand, $U_{\xi}^{2}=\theta(\mathbf{e})=$ $a \theta(x)$, see [M-S, p. 99]. Therefore the relation $u^{2}=a v$ holds in $\mathbf{H}^{\bullet}\left(E, E_{0}\right)$. But the reduced cohomology of the Thom space $\xi$ is isomorphic to that of $\left(E, E_{0}\right)$; thus the cohomology ring of the Thom space of $\xi$ (which is in our case just $p^{\perp}$, the one-point compactification of $E)$ is given by $\mathbb{Z}[u, v] /\left(v^{2}, u v, u^{2}-a v\right)$.

Hence in our case the total Chern class of $\eta$ is given by $\mathbf{c}=1 \pm x$. The holomorphic line bundles over $\mathbb{C} P^{1}$ are classified up to holomorphic equivalence by their first Chern class [G-H, p. 145]. Thus the holomorphic line bundle $\eta$ is holomorphically equivalent to the Hopf line bundle $\eta_{\mathbb{C}}$ or to its dual $\bar{\eta}_{\mathbb{C}}$. It follows from 2.15 and the property $\left(^{*}\right)$ in 1.2 that $\eta$ has nontrivial sections, hence it is indeed equivalent to the Hopf line bundle.

Proof of Theorem 2.10. Let $\mathscr{S}$ denote the collection of all holomorphic sections of $\eta \cong \eta_{\mathbb{C}}$ which we obtain by 2.15 . From property $\left(^{*}\right)$ in 1.2 , we have $\phi(\mathscr{S})=\mathbb{C}^{2}$. Thus $\mathscr{S}=\mathbf{H}^{0}\left(\mathbb{C} P^{1} ; \eta\right)$ is precisely the set of all holomorphic sections of the Hopf bundle $\eta$, and therefore the isomorphism $\eta \cong \eta_{\mathbb{C}}$ induces an homeomorphism between $p^{\perp}$ and $\mathbb{C} P^{2}$ that maps the lines of $\mathfrak{A}_{p}$ bijectively onto the point rows of $\mathbb{C} P^{2}$.

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