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Algebraic Polygons

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In this paper we prove the following: Over each algebraically closed field K of characteristic 0 there exist precisely three algebraic polygons (up to duality), namely the projective plane, the symplectic quadrangle, and the split Cayley hexagon over K (Theorem 3.3). As a corollary we prove that every algebraic Tits system over K is Moufang and obtain the following classification:

THEOREM. *Let (G, B, N, S) be an irreducible effective spherical Tits system of rank ≥ 2. If G is a connected algebraic group over an algebraically closed field of characteristic 0, and if B is closed in G, then G is simple and B is a standard Borel subgroup of G.* © 1996 *Academic Press, Inc.*

INTRODUCTION

Generalized polygons were introduced by Tits in [22] in order to give a geometric interpretation for certain simple algebraic groups. More generally, spherical buildings provide a beautiful and uniform geometric interpretation for all simple algebraic groups, even the exceptional ones. Generalized polygons are exactly the spherical buildings of rank 2. The irreducible spherical buildings of rank ≥ 3 have been classified by Tits;

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they are essentially the buildings arising in a standard way from the simple algebraic groups or the classical groups [23]. This generalizes the well-known fact that a projective space of dimension ≥ 3 is desarguesian (projective k-space is a building of type $\mathbf{A}_k$).

In contrast to this, nonclassical (that is non-Moufang) generalized polygons abound. Hence one has to impose some additional conditions on the polygon in order to obtain more specific results. Probably one of the earliest results in that direction is the famous Feit–Higman Theorem: *finite* generalized n-gons exist only for $n = 3, 4, 6, 8$ [7]. Later, Weiss and Tits proved that *Moufang* n-gons exist only for $n = 3, 4, 6, 8$ [28, 26]. The Moufang 3, 6, and 8-gons have been classified explicitly. The classification of the Moufang quadrangles was announced in [24] although a complete proof has not been published yet (but see [27], and [16, Ap. 1] for an overview of the classification).

Yet another approach is to impose topological conditions on generalized polygons. Knarr proved that *compact* generalized n-gons of finite and positive topological dimension exist only for $n = 3, 4, 6$, see [11, 12]. As in the finite case, there is no hope to classify all compact polygons. The picture changes drastically for *holomorphic* polygons: the only holomorphic polygons are the three Moufang polygons mentioned above over the complex numbers [13]. Complex manifolds are close to complex algebraic varieties; thus it seems reasonable to conjecture that the only algebraic polygons over an algebraically closed field are the three Moufang polygons over that field.

In this paper we prove the following: over each algebraically closed field K of characteristic 0 there exist precisely three algebraic polygons (up to duality), namely the projective plane, the symplectic quadrangle, and the split Cayley hexagon over K (Theorem 3.3). As a corollary we prove that every algebraic Tits system over K is Moufang; this corresponds to the result of Burns and Spatzier that every BN-pair consisting of Lie groups is Moufang [5] (they call this "topologically Moufang," which is a bit misleading, since it is merely the Tits condition).

Algebraic projective planes have been investigated and classified by Strambach, see [21], as well as [8]; however, our approach is slightly more general in characteristic 0, because we do not make any assumptions like completeness or one-dimensionality of the point rows.

The classification of the holomorphic polygons depends very much on the well-developed tools from topological geometry, which are not available over fields other than $\mathbb{C}$. Therefore we follow a different line: we consider first the special case of algebraic polygons over $\mathbb{C}$ (which is a rather straightforward consequence of the holomorphic case). The general case then follows by some standard model theoretic reasoning. For example, the existence of an algebraic n-gon over an algebraically closed field K of characteristic 0 implies the existence of an algebraic n-gon over $\mathbb{C}$;

hence $n = 3, 4, 6$. Similarly, the existence of two nonisomorphic algebraic n-gons over K would imply the existence of two nonisomorphic algebraic n-gons over $\mathbb{C}$.

We conjecture that the same result holds over algebraically closed fields of positive characteristic; however, the proof would certainly require a very different kind of reasoning. We do not touch upon any rationality questions. Presumably the result does not hold over fields which are not algebraically closed.

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1. GENERALIZED POLYGONS

1.1. DEFINITION. Recall that an incidence structure is a triple $(\mathscr{P}, \mathscr{L}, \mathscr{F})$ consisting of a set $\mathscr{P}$ of *points*, a set $\mathscr{L}$ of *lines*, and a set $\mathscr{F} \subseteq \mathscr{P} \times \mathscr{L}$ of *flags*. We assume always that $\mathscr{P}$ and $\mathscr{L}$ are nonempty disjoint sets. If (p, l) is a flag, then we say that the line l passes through p, and that the point p lies on l, or that p and l are incident. The set of all lines passing through p is called the *line pencil* $\mathscr{L}_p$; the set of all points lying on l is called the *point row* L. The dual incidence structure is given by $(\mathscr{P}, \mathscr{L}, \mathscr{F})^{\text{dual}} = (\mathscr{L}, \mathscr{P}, \mathscr{F}^{-1})$, where $\mathscr{F}^{-1} = \{(l, p) \mid (p, l) \in \mathscr{F}\}$.

A *k-chain* joining $x_0, x_k \in \mathscr{P} \cup \mathscr{L}$ is a sequence $(x_0, x_1, \ldots, x_k)$ of points and lines with the property that x_i is incident with x_{i+1} for $0 \le i < k$. The k-chain is called *minimal* if there is no i-chain joining x_0 and x_k for $i < k$; in that case we say that x_0 and x_k have *distance* $d(x_0, x_k) = k$.

If $x_0 = x_k$, and if $x_i \neq x_j$ for all $0 \le i < j < k$, then we call the set $\{x_0, \ldots, x_{k-1}\}$ an *ordinary k/2-gon*.

1.2. DEFINITION. An incidence structure $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$ is called a *generalized n-gon* if it satisfies the following three axioms:

(i) Every element is incident with at least three other elements.

(ii) Any two elements $x, y \in \mathscr{P} \cup \mathscr{L}$ can be joined by some k-chain, for some $k \le n$.

(iii) If $x_0, x_1, \ldots, x_k$ is a minimal k-chain, and if $k < n$, then this chain is uniquely determined by x_0 and x_k.

These geometries are sometimes also called "thick" generalized polygons.

If $x, y \in \mathscr{P} \cup \mathscr{L}$ are elements with $d(x, y) = k < n$, then axiom (iii) says that there is a unique element z incident with y, with $d(x, z) = k - 1$. We put $f_k(x, y) = z$.

Note that $(\mathscr{P}, \mathscr{L}, \mathscr{F})$ is a generalized n-gon if and only if $(\mathscr{P}, \mathscr{L}, \mathscr{F})^{\text{dual}}$ is a generalized n-gon; therefore, we will sometimes state and prove a result only for the set of points.

The generalized digons are trivial incidence structures $(\mathscr{P}, \mathscr{L}, \mathscr{P} \times \mathscr{L})$, or in other words complete bipartite graphs; hence we will assume from now on that $n \geq 3$.

The generalized triangles are precisely the projective planes.

Generalized n-gons exist for every $n \geq 2$; in fact one can start with any collection of points and lines satisfying axiom (iii) and complete this incidence structure via some free construction to a generalized n-gon, see [25].

1.3. EXAMPLES. Let K be a commutative field (in (3) we assume for simplicity that the characteristic of K is $\neq 2$).

(1) If we take for $\mathscr{P}$ the set of all one-dimensional subspaces of K^3 and for $\mathscr{L}$ the set of all two-dimensional subspaces of K^3, then $(\mathscr{P}, \mathscr{L}, \subseteq)$ is a generalized triangle, the projective plane over K. This incidence structure is self-dual, and the simple group $\mathbf{PSL}_3(K)$ acts as a group of automorphisms on it.

(2) Consider the symplectic polarity on K^4 given by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $\mathscr{P}$ denote the set of all one-dimensional subspaces of K^4, and let $\mathscr{L}$ denote the set of all two-dimensional totally isotropic subspaces of K^4. Then $(\mathscr{P}, \mathscr{L}, \subseteq)$ is a generalized quadrangle, the symplectic quadrangle over K, and the simple group $\mathbf{PSp}_4(K)$ acts as a group of automorphisms on it.

(3) Let $K^8 = O$ denote the split Cayley algebra over K. It consists of all matrices of the form $\left(\begin{smallmatrix} a & u \\ v & b \end{smallmatrix}\right)$, where $a, b \in K$ and $u, v \in K^3$. The multiplication is defined as

$$\begin{pmatrix} a & u \\ v & b \end{pmatrix}\begin{pmatrix} a' & u' \\ v' & b' \end{pmatrix} = \begin{pmatrix} aa' - u \cdot v' & au' + b'u + v \times v' \\ a'v + bv' + u \times u' & bb' - v \cdot u' \end{pmatrix},$$

where $\cdot$ and $\times$ denote the standard dot and cross product on K^3. Now let $\mathscr{P}$ denote the set of all one-dimensional subspaces X of O with $XX = 0$ and let $\mathscr{L}$ denote the set of all two-dimensional subspaces Y of O with $YY = 0$. Then $(\mathscr{P}, \mathscr{L}, \subseteq)$ is a generalized hexagon, the split Cayley hexagon. The simple exceptional group $\mathbf{G}_2(K)$ acts as a group of automorphisms on it, see [17].

We have listed the Moufang polygons (1), (2), (3) rather explicitly, because they will eventually turn out to be the only algebraic polygons over an algebraically closed field of characteristic 0.

1.4. DEFINITION. Let $x, y \in \mathscr{P} \cup \mathscr{L}$ be elements of a generalized n-gon $\mathfrak{P}$ with distance $d(x, y) = n$. Let X, Y denote the set of all elements incident with x and y, respectively. Then there is a *perspectivity* $[x, y]$: $Y \to X$, given by $z \mapsto f_{n-1}(z, x)$. Note that $[x, y]$ is a bijection with inverse $[y, x]$. A concatenation of perspectivities is called a *projectivity*. The set of all projectivities of $\mathfrak{P}$ forms a groupoid Π. The group $\Pi(x)$ of all projectivities from x to x is a doubly transitive permutation group on X, see [10]; moreover, if x_1, x_2 are elements of the same type (i.e., both x_1 and x_2 are points, or both x_1 and x_2 are lines), then there always exists a projectivity between x_1 and x_2. If n is odd, then there exist also projectivities between points and lines.

1.5. DEFINITION. Let (p, l) be a flag of the generalized n-gon $(\mathscr{P}, \mathscr{L}, \mathscr{F})$. Let $\mathscr{P}_{2k}(l, p)$ denote the set of all points $q \in \mathscr{P}$ with $d(p, q) = 2k$ and $d(l, q) = 2k + 1$, for $k \geq 0$. Similarly, let $\mathscr{P}_{2k+1}(p, l)$ denote the set of all points $q \in \mathscr{P}$ with $d(l, q) = 2k + 1$ and $d(p, q) = 2k + 2$, for $k \geq 0$. Thus we have a partition $\mathscr{P} = \mathscr{P}_0(l, p) \cup \mathscr{P}_1(p, l) \cup \mathscr{P}_2(l, p) \cup \cdots$ of $\mathscr{P}$ into n *Schubert cells*. Similarly, the line space can be partitioned into n Schubert cells.

2. ALGEBRAIC POLYGONS

Throughout this section, we let K denote an algebraically closed commutative field.

In modern terminology a K-variety is in an integral separated K-scheme of finite type, i.e., a sheaf of rings with special properties, see Hartshorne [9] and Mumford [15]. However, in our case the older definition which states that a variety X is patched together from a finite collection of affine varieties living in some K^m is more convenient (see Section 3). The two concepts are equivalent (see, e.g., [15, Chap. II.3, Theorem 2 and II.6, Corollary I]). The "concrete" variety X corresponds precisely to the set of all closed points of the associated scheme.

2.1. DEFINITION. Let $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$ be a generalized n-gon, for some $n \geq 3$. Assume that $\mathscr{P}$ and $\mathscr{L}$ are K-varieties (and hence infinite). Assume moreover that the domain $\{(x, y) \mid x, y \in \mathscr{P} \cup \mathscr{L},\ d(x, y) = n - 1\}$ of the map f_{n-1} is locally closed (i.e., the intersection of an open and a closed set) in $(\mathscr{P} \cup \mathscr{L}) \times (\mathscr{P} \cup \mathscr{L})$ (note that this is the scheme theoretic product, hence it does *not* carry the product topology). We call $(\mathscr{P}, \mathscr{L}, \mathscr{F})$ an *algebraic polygon* (over K) if the map f_{n-1} is a K-morphism on each irreducible component of its domain. We say that two algebraic polygons $\mathfrak{P}, \mathfrak{P}'$ are *isomorphic up to duality* if there exists an incidence-preserving K-isomorphism between $\mathfrak{P}$ and $\mathfrak{P}'$ or between $\mathfrak{P}^{\text{dual}}$ and $\mathfrak{P}'$.

Note that the domain of f_{n-1} has at least two irreducible components, depending on whether the first argument of f_{n-1} is a point or a line. Since we are dealing with noetherian spaces, the domain of f_{n-1} decomposes uniquely into a finite collection of irreducible components.

Note also that an algebraic polygon, endowed with the Zariski topology, is *not* a topological polygon. Indeed, the point set of a topological polygon is always a Hausdorff space, whereas an algebraic variety of positive dimension is not a Hausdorff space. However, if K happens to be a topological (locally compact) field, then $\mathfrak{P}$ can be made into a topological (locally compact) polygon, see [15, Chap. I.10].

2.2. LEMMA. *Let $(\mathscr{P}, \mathscr{L}, \mathscr{F})$ be an algebraic n-gon, let u be a line if n is odd, or a point if n is even, and let v be incident with u. Then the "big" Schubert cell $\mathscr{P}_{n-1}(u, v)$ is open in $\mathscr{P}$.*

Proof. The proof in [12] for topological polygons goes through without any changes.

Let $p \in \mathscr{P}_{n-1}(u, v)$, and let $X_0, \ldots, X_{n-1}$ denote the Schubert cells with respect to (u, v). For every Schubert cell X_i different from $\mathscr{P}_{n-1}(u, v)$ we can find an element v_i such that $d(v_i, p) = n - 1 = d(v_i, r)$ holds for all $r \in X_i$, and such that the map $r \mapsto f_{n-1}(r, v_i)$ has constant value different from $f_{n-1}(p, v_i)$ on X_i.

Indeed, suppose that $n - i$ is even. Inductively we may choose elements $(x_0 = v, x_1 = u, x_2, \ldots, x_{n-i-1})$ with the property that $d(x_j, p) \geq n - 1$ and such that the relation $d(x_j, r) = i + j$ holds for all $r \in X_i$. Thus we have $f_{n-1}(r, x_{n-i-1}) = x_{n-i-2} \neq f_{n-1}(p, x_{n-i-1})$ for all $r \in X_i$, and we may put $v_i = x_{n-i-1}$. The case that $n - i$ is odd is similar.

Hence there is an open neighborhood U_i of p that does not meet X_i. The open neighborhood $U_0 \cap \cdots \cap U_{n-2}$ of p is therefore contained in $\mathscr{P}_{n-1}(u, v)$. ∎

2.3. LEMMA. *Let $(\mathscr{P}, \mathscr{L}, \mathscr{F})$ be an algebraic polygon. Then every point row and every line pencil is a nonsingular, closed, and doubly homogeneous subvariety.*

Proof. Let l be a line, and let v be an element with $d(v, l) = n$. Choose two elements u_1, u_2 incident with v. Now the union of the two big cells $U = \mathscr{P}_{n-1}(v, u_1) \cup \mathscr{P}_{n-1}(v, u_2) = \{q \in \mathscr{P} \mid d(q, v) = n - 1\}$ is an open neighborhood of the point row L corresponding to l, and the map $q \mapsto f_{n-1}(f_{n-1}(q, v), l)$ is a retraction of U onto L. Therefore L is closed in U. We may cover $\mathscr{P}$ by neighborhoods of L of this kind, since given any point $q \in \mathscr{P}$, there exists an element u with $d(u, p) = n - 1$ and $d(u, l) = n$, see [23, 3.30]; therefore the point row L is closed in $\mathscr{P}$.

Now L is doubly homogeneous and hence either irreducible or discrete. But if the point rows are discrete, then the line pencils are discrete as well,

since there are nonconstant maps from line pencils to point rows which may be expressed by means of the map f_{n-1}, see [2]. This would imply that the big Schubert cells $\mathscr{P}_{n-1}(u,v)$ are discrete (and even finite), contradicting the fact that $\mathscr{P}$ is a variety; therefore L is a closed subvariety of $\mathscr{P}$, and being homogeneous it cannot have singular points. ∎

2.4. LEMMA. *Let $(\mathscr{P},\mathscr{L},\mathscr{F})$ be an algebraic polygon. Then the varieties $\mathscr{P}$ and $\mathscr{L}$ are nonsingular.*

Proof. Let X be a point row with one point removed, and let Y be a line pencil with one line removed. There exists a bijection of each big Schubert cell $\mathscr{P}_{n-1}(u,v)$ onto $X \times Y \times X \ldots$ ($n-1$ factors) which can in both directions be expressed in terms of the map f_{n-1}, see [2, 1.6] or [11, 2.5]. Thus each big cell is a nonsingular open subvariety, and since $\mathscr{P}$ may be covered by n big cells, it is nonsingular as well.

2.5. PROPOSITION. *Let $\mathfrak{P}$ be an algebraic polygon over the field of complex numbers $\mathbb{C}$. Then (up to duality) $\mathfrak{P}$ is isomorphic to the complex projective plane, the complex symplectic quadrangle, or the complex split Cayley hexagon.*

Proof. Since $\mathscr{P}$ and $\mathscr{L}$ are nonsingular complex algebraic varieties, we can make them into complex manifolds, thus obtaining a holomorphic polygon, see [20, Chap. VII; 15, I.10; 9, App. B]. Now the claim follows from [13]. ∎

A group G that acts transitively on the ordinary ordered n-gons of a generalized n-gon has a BN-pair, see [4, 16, 23]. Conversely, given a Tits system (G,B,N,S), the group G acts strongly transitively on the building consisting of the lattice of all parabolic subgroups of G, see [16, Chap. 5]. We call a Tits system spherical (irreducible, of rank k) if its Coxeter system is finite (irreducible, of rank k), and we call it effective if the action of G on the building is effective i.e., if B contains no normal subgroup of G.

2.6. PROPOSITION. *Let (G,B,N,S) be an irreducible spherical Tits system of rank 2. Assume that G is an algebraic group over an algebraically closed field K of characteristic 0, and that the Borel subgroup B is closed in G. Then the corresponding generalized polygon is an algebraic polygon. In particular, the generalized polygons corresponding to the simple algebraic groups $\mathbf{PSL}_3(K)$, $\mathbf{PSp}_4(K) \cong \mathbf{PSO}_5(K)$, and $\mathbf{G}_2(K)$, i.e, the projective plane, the symplectic quadrangle, and the split Cayley hexagon over K, are algebraic polygons.*

Proof. The assumptions about G and K allow us to identiy a G-orbit $G \cdot y$ on some variety with G/G_y.

Let $S = \{s_1, s_2\}$ be the set of involutive generators of the Coxeter group $W = N/(N \cap B)$ and let $P_i = B \cup Bs_iB$ be the standard parabolic subgroups of G, for $i = 1, 2$. Now we may define $\mathscr{P} = G/P_1$ and $\mathscr{L} = G/P_2$, and $\mathscr{F} = \{(gP_1, gP_2) \mid g \in G\}$. Thus $\mathscr{P}$ and $\mathscr{L}$ are nonsingular algebraic varieties, and the group G acts transitively on the ordered ordinary n-gons of the generalized n-gon $(\mathscr{P}, \mathscr{L}, \mathscr{F})$ (note that N is just the stabilizer of an ordinary n-gon in $(\mathscr{P}, \mathscr{L}, \mathscr{F})$).

It remains to show that the map f_{n-1} is a K-morphism.

Let aP_i be an element with distance $n - 1$ from the point $P_1 \in \mathscr{P}$ (thus $i = 1$ if n is odd, and $i = 2$ if n is even). The G-orbit $X = \{(gP_1, gaP_i) \mid g \in G\} \subseteq G/P_1 \times G/P_i$ consists of all pairs of elements of distance $n - 1$, where the first element is a point.

Now put $bP_j = f_{n-1}(P_1, aP_i)$. We want to show that the map f_{n-1}: $(gP_1, gaP_i) \mapsto gbP_j$ is a K-morphism. Since $P_1 \cap aP_ia^{-1}$ fixes the $(n - 1)$-chain from P_1 to aP_i, we have $P_1 \cap aP_ia^{-1} \subseteq bP_jb^{-1}$. Now we may decompose f_{n-1} as $(gP_1, gaP_i) \mapsto g(P_1 \cap aP_ia^{-1}) \mapsto gbP_jb^{-1} \mapsto gbP_j$. The first map is a K-isomorphism, since we have $X \cong G/(P_1 \cap aP_ia^{-1})$. The second map is the K-morphism $G/(P_1 \cap aP_ia^{-1}) \to G/bP_jb^{-1}$, and the last map is just right translation by b, which is again a K-isomorphism.

3. SOME MODEL THEORY

Since Proposition 2.5 relies on [13] where a corresponding result is proved over $\mathbb{C}$ using [11] and the machinery of holomorphic function theory, there is no direct way to transfer this result to other algebraically closed fields.

There have been several attempts to formalize the so-called *Lefschetz's principle*, which in its strongest form states that any statement of algebraic geometry which is true over the complex numbers is true over any algebraically closed field of characteristic 0, see [14, Appendix]. This problem lends itself to model theory in a very natural way since if the statements that one is concerned about can be expressed in an appropriate language, model theory provides tools for transferring them from one algebraically closed field to another. Unfortunately it quickly turns out that finitary first order languages, i.e., languages that allow only finite conjunctions and disjunctions, finite sequences of quantifiers, and no quantification over subsets, are not able to capture the general concepts of algebraic geometry. On the other hand infinitary and/or higher order languages don't have the same nice model theoretic properties. The (infinitary) language proposed by Barwise and Eklof [1], for example, is able to express concepts like that of a variety or a K-morphism, but at the

same time it can distinguish between finite and infinite transcendence degrees, so that statement in this language which are true over $\mathbb{C}$ transfer only to algebraically closed fields of infinite transcendence degree. Since we are interested in a result that concerns all algebraically closed fields of characteristic 0, we have to find a finitary language instead.

The important observation in [1] is the fact that the concepts of algebraic geometry are finitely determined, e.g., polynomials are determined by finitely many coefficients; varieties and K-morphisms are each determined by finitely many polynomials, etc.

We use this fact to express particular instances of algebraic geometry in a finitary language designed explicitly to capture that specific instance. From the description below it seems reasonable to expect that (at least most) statements of algebraic geometry true over $\mathbb{C}$ could be dealt with in a similar (if not uniform) manner.

In order to transfer Proposition 2.5 from $\mathbb{C}$ to arbitrary algebraically closed fields we need the following model theoretic facts (see, e.g., [6]).

3.1. *Facts and Definitions*. (i) Let T be a theory (i.e,. a set of sentences) in some finitary first order language $\mathbf{L}$. A model of T is an $\mathbf{L}$-structure in which all the sentences of T hold. The theory T is consistent if it has a model in every cardinality $\kappa \geq |T|$. In particular any countable, consistent theory has a model in every infinite cardinality (see [6, 3.1.5, 3.1.6]).

(ii) Conversely, if we start with an $\mathbf{L}$-structure M, the theory $Th(M)$ is the collection of all L-sentences true in M. By definition, such a theory is consistent.

(iii) The theory $T_{\mathrm{acf},0}$ of algebraically closed fields of characteristic 0 is $\aleph_1$-categorical; i.e., there is a unique model in every uncountable cardinality (up to isomorphism). This is a generalization of the well-known fact that $\mathbb{C}$ is up to isomorphism the unique algebraically closed field of characteristic 0 in cardinality $2^{\aleph_0}$ and follows directly from Steinitz' Exchange Lemma.

3.2. PROPOSITION. *If* $(\mathscr{P}, \mathscr{L}, \mathscr{F})$ *is an algebraic n-gon over an algebraically closed field of characteristic* 0, *then* $n \in \{3, 4, 6\}$.

Proof. Assume that there is an algebraically closed field K over which there is an algebraic n-gon $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$ for n other than 3, 4, or 6.

We start with the language of fields $\mathbf{L} = \{+, \cdot\}$. Following the definition of variety in [15, Chap. I], we will add new function and relation symbols to $\mathbf{L}$ so that the theory of K in this extended language specifies that $(\mathscr{P}, \mathscr{L}, \mathscr{F})$ is an algebraic n-gon.

Let $\mathscr{P}$ and $\mathscr{L}$ be pieced together from affine varieties $U_1, \ldots, U_{k_1}$ and $V_1, \ldots, V_{k_2}$, respectively. Recall that an affine variety is the zero-set of a prime ideal in some polynomial ring $K[X_1, \ldots, X_m]$. We can assume that all these live in K^m, hence each of these affine varieties is given by a finite collection of polynomials in $K[X_1, \ldots, X_m]$. Similarly, there is a finite collection of polynomials defining the K-isomorphic transition maps between open subsets of the U_i's (V_j's resp.). An open subset of an affine variety is the complement of a zero-set of finitely many polynomials. A subset of a product variety is open if and only if its intersection with each affine patch $U \times V$ is open in $U \times V$, hence the fact that the diagonal in $\mathscr{P} \times \mathscr{P}$ (and $\mathscr{L} \times \mathscr{L}$) is closed may be expressed by finitely many polynomials. And finally the K-morphism f_{n-1} is locally given by polynomials on the affine charts.

All this information is given by finitely many polynomials over K for each of which we add a function symbol to the language. The theory of K in the language extended by these function symbols says that each of these functions is a polynomial in m variables of some fixed degree.

Next we add names for the affine varieties $U_1, \ldots, U_{k_1}, V_1, \ldots, V_{k_2}$, i.e,. m-ary relation symbols (which we should denote by $\hat{U}_i$ to distinguish between the name and its interpretation.) The sentences stating that each of these is the zero set of the corresponding polynomials are part of the theory as are sentences expressing the irreducibility: Remember that an affine variety is irreducible if and only if every two nonempty open subsets have nonempty intersection.

For each q and each U_i (resp. V_j), there are sentences saying that any two nonempty open subsets of U_i given by q polynomials in m variables of degree q have nonempty intersection in U_i. These clearly are sentences in the extended language true in K.

In the same way we can add to the language function symbols $\phi_{i,j}$, $1 \leq i, j \leq k_1$ and $\psi_{i,j}$, $1 \leq i, j \leq k_2$ for the transition maps which the theory will specify as K-isomorphisms given by the appropriate polynomials on affine open subsets of the U_i (resp. V_j) (where again the theory says that the open subsets are defined by the appropriate polynomials.)

We also add m-ary predicates $\mathscr{P}$ and $\mathscr{L}$ (to be interpreted in the obvious way as the union of the corresponding affine sets) and a $2m$-ary predicate $\mathscr{F}$ for the incidence relation. The theory contains sentences saying that $\mathscr{F}$ defines an incidence relation satisfying Definition 1.2 above and respecting the coherency conditions imposed by the transition maps, namely if $\mathscr{F}(\bar{x}, \bar{y})$ then also, e.g., $\mathscr{F}(\phi_{i,j}(\bar{x}), \phi_{i',j'}(\bar{y}))$ for all transition maps $\phi_{i,j}, \phi_{i',j'}$ and $\bar{x}, \bar{y}$ in their respective domains. The fact that the diagonals in $\mathscr{P} \times \mathscr{P}$ and $\mathscr{L} \times \mathscr{L}$ are closed will also be part of the theory since this is again expressible in the language.

Obviously the map f_{n-1} as defined above is definable in this language so its definition is part of the theory as are the coherency conditions imposed by the transition maps, namely, e.g., $f_{n-1}(\bar{x}) = \bar{y}$ implies $f_{n-1}(\phi_{i,j}(\bar{x})) = \phi_{i',j'}(\bar{y})$, etc.

Let T denote the theory of K in this extended language. Obviously $T_{\mathrm{acf},0} \subseteq T$. It is now clear that in any other model of this theory the interpretation of the language symbols $(\mathscr{P}, \mathscr{L}, \mathscr{F})$ will be an algebraic n-gon. By Fact 3.1 there must be a model of power $2^{\aleph_0}$. But any model of this theory in particular has to be an algebraically closed field of characteristic 0, so the only possible model in that cardinality is $\mathbb{C}$ contradicting Proposition 2.5. ∎

Next we turn to showing that the algebraic n-gons for $n = 3, 4, 6$ are unique up to algebraic isomorphism and duality:

3.3. THEOREM. *If $\mathfrak{P}$ is an algebraic polygon over some algebraically closed field K of characteristic* 0, *then* (*up to duality*) *$\mathfrak{P}$ is isomorphic to the projective plane, the symplectic quadrangle, or the split Cayley hexagon* (*over K*).

Proof. By Proposition 2.6, the projective plane, the symplectic quadrangle, and the split Cayley hexagon over K are algebraic n-gons for $n = 3, 4$, and 6, respectively, and by Proposition 3.2 there are no algebraic n-gons for n other than $3, 4, 6$. So it is left to show that for $n \in \{3, 4, 6\}$ there is a unique algebraic n-gon over K (up to isomorphism), which then, clearly, is isomorphic to the specific n-gon from this list.

Suppose otherwise and let K be an algebraically closed field of characteristic 0 over which for some $n \in \{3, 4, 6\}$ there are two nonisomorphic algebraic n-gons.

As in Proposition 3.2 we can extend the language of fields to a language $\mathbf{L}'$ such that the theory of K in the language $\mathbf{L}'$ says that there are two algebraic n-gons. The theory also specifies that there is no algebraic isomorphism betwen these two polygons as for every $q < \omega$ there is a sentence in $\mathbf{L}'$ saying that whenever there is a K-isomorphism between these pairs of algebraic varieties, possibly switching the roles of points and lines, such that all the "data" (i.e., all the pieces of the K-morphisms and the open subsets) are given by at most q polynomials of degree $\leq q$, there is an instance where the incidence relation (or its dual) is not preserved.

As above, by assumption this theory is consistent, hence $\mathbb{C}$ would have to be a model, contradicting Proposition 2.5.

3.4. *Remark*. For two nonisomorphic fields K, K', the associated algebraic polygons are not isomorphic. This can be seen as follows: the derived

incidence structure of such a polygon is isomorphic to the projective plane over the corresponding field, see [18, 19], hence the field can be recovered from the geometry.

3.5. COROLLARY. *Let K be an algebraically closed field of characteristic* 0, *and let* (G, B, N, S) *be an irreducible effective spherical Tits system of rank* 2. *Assume that G is a connected algebraic group over K, and that the parabolic subgroups are closed. Then* (G, B, N, S) *is Moufang, i.e.,* $G \in \{\mathbf{PSL}_3(K), \mathbf{PSp}_4(K), \mathbf{G}_2(K)\}$, *and B is a standard Borel subgroup, i.e., a maximal solvable subgroup of G, see, e.g.,* [3, *Chap. IV*].

Combining this with Tits' classification of the irreducible spherical buildings of rank ≥ 3, we get the following result:

3.6. COROLLARY. *Let* (G, B, N, S) *be an irreducible effective spherical Tits system of rank* ≥ 2. *If G is a connected algebraic group over an algebraically closed field of characteristic* 0, *and if B is closed in G, then G is simple and B is a standard Borel subgroup of G.*

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