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The Sylow structure of scalar automorphism groups

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A B S T R A C T

We shall review basically known facts about periodic locally compact abelian groups. For any periodic locally compact abelian group A, its automorphism group contains (as a subgroup) those automorphisms that leave invariant every closed subgroup of A; to be denoted by $\mathrm{SAut}(A)$. This subgroup is profinite in the g-Arens topology and hence allows a decomposition into its p-primary subgroups for the primes p for which topological p-elements in this automorphism subgroup exist. The interplay between the p-primary decomposition of $\mathrm{SAut}(A)$ and A can be encoded in a bipartite graph, the *mastergraph* of A. Properties and applications of this concept are discussed.

Introduction

This text deals with widely known observations about periodic locally compact abelian groups. A topological group is called *periodic* if it is locally compact and totally disconnected and if it is the union of compact subgroups. The ring $\mathbb{Z}$ of integers acts on every abelian group A via scalar multiplication. The ring $\mathbb{Z}$ has a universal compactification to a compact totally disconnected topological ring $\widetilde{\mathbb{Z}} \supseteq \mathbb{Z}$, and, if A is

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a periodic locally compact abelian group, then the scalar multiplication of A by $\mathbb{Z}$ extends to a continuous scalar multiplication

$$(z, a) \mapsto z{\cdot}a : \widetilde{\mathbb{Z}} \times A \to A.$$

(As a consequence of [7, Lemma 4.1.1], every profinite abelian group turns out to be a $\widetilde{\mathbb{Z}}$-module and so does A being the union of its compact and hence profinite open subgroups.) The automorphism group $\mathrm{Aut}(A)$ is of considerable interest to group theoreticians. Its center contains all automorphisms of the form $a \mapsto r{\cdot}a$ for any (multiplicatively) invertible element $r \in \widetilde{\mathbb{Z}}$. Such elements are called *units* and they form a compact multiplicative subgroup $\widetilde{\mathbb{Z}}^\times$ of $\widetilde{\mathbb{Z}}$. The profinite abelian group $\widetilde{\mathbb{Z}}^\times$ has a remarkably rich structure. So, for each prime number p the compact ring $\mathbb{Z}_p$ is a subring of $\widetilde{\mathbb{Z}}$, and so its group of units $\mathbb{Z}_p^\times$ is a subgroup of $\widetilde{\mathbb{Z}}^\times$. It contains a compact open multiplicative subgroup which is isomorphic to the additive group $\mathbb{Z}_p$, but, if $p > 2$ then it also contains a finite cyclic group of order $p - 1$ of roots of unity which therefore contains elements of order $q^{\nu_p(q-1)}$ if $q|(p-1)$, where $\nu_p(q-1)$ is the largest natural number n such that $p = q^n m + 1$ for a natural number m. The simple task of finding the p-Sylow subgroups of $\widetilde{\mathbb{Z}}^\times$ appears to be a mind boggling problem at first sight.

We solve this problem by describing a countably infinite bipartite labeled graph that is easily depicted and imagined as drawn in the real plane. It supplies a very good organization of the set of all procyclic (and cyclic) subgroups of $\widetilde{\mathbb{Z}}$ that are compact p-groups (i.e., are pro-p groups) and this allows us to find the maximal p-subgroups. Indeed, the essential cyclic and procyclic subgroups are lucidly indexed by the labeled edges of the graph, which we call *the mastergraph*. For a group G let $\mathrm{tor}(G)$ denote the set of its torsion elements. A locally compact p-group has only elements contained in compact p-subgroups. With the help of the tools that it provides it is, for instance, possible to argue that the multiplicative group $\widetilde{\mathbb{Z}}^\times$ is isomorphic to the additively written group $\widetilde{\mathbb{Z}} \times \overline{\mathrm{tor}(\widetilde{\mathbb{Z}}^\times)}$ and that the group $\overline{\mathrm{tor}(\widetilde{\mathbb{Z}}^\times)}$ contains a dense subgroup algebraically isomorphic to the large torsion-free group $(\widetilde{\mathbb{Z}}, +)^{(\mathbb{N})}$. (See Corollary 21.)

Given a periodic locally compact abelian group A we let $\mathrm{SEnd}(A) \subseteq \mathrm{End}(A)$ denote the subring of all endomorphisms implemented by scalar multiplication. Then the natural homomorphism $\zeta: \widetilde{\mathbb{Z}} \to \mathrm{SEnd}(A)$ defined by $\zeta(r)(a) = r{\cdot}a$ will be shown to be a quotient morphism of profinite rings, and we call the ring $\mathcal{R}(A) := \widetilde{\mathbb{Z}}/\ker(\zeta)$ the *ring of scalars of* A. Then ζ factors through $\mathcal{R}(A)$ with an isomorphism $\mathcal{R}(A) \to \mathrm{SEnd}(A)$ of rings. The group of units of $\mathrm{SEnd}(A)$ is denoted $\mathrm{SAut}(A)$, and we have $\mathcal{R}(A)^\times \cong \mathrm{SAut}(A)$. We shall clarify the structure of $\mathcal{R}(A)^\times$ completely in the way it depends on the exponents of the A_p. (See Theorem 39.)

Let G be a locally compact group with a closed normal subgroup A. Let $\mathrm{Int}(A)$ denote the group of all inner automorphisms. There is a natural representation $G \to \mathrm{Int}(A)$ sending g to the inner automorphism $a \mapsto gag^{-1}$ whose kernel is the centralizer of A in G.

Proposition 1. *For a locally compact group G with a periodic abelian closed normal subgroup A the following statements are equivalent:*

(i) $\mathrm{Int}(A) \subseteq \mathrm{SAut}(A)$*, i.e., every inner automorphism induced on A is a scalar automorphism.*
(ii) *Every closed subgroup of A is normal in G.*
(iii) *There is a morphism $\rho: G \to \mathcal{R}(A)^\times$ such that*

$$(\forall g \in G,\, a \in A)\ gag^{-1} = \rho(g){\cdot}a$$

For a proof of this see Proposition 34.

We emphasize here again that in Theorem 39 we shall give an explicit structure theory of $\mathcal{R}(A)^\times \cong \mathrm{SAut}(A)$.

1. The Sylow structure of the compactified ring of integers

By the “ring of compactified integers” we mean the profinite completion of the ring $\mathbb{Z}$ and we denote it by $\widetilde{\mathbb{Z}}$. Technically, if $B = \alpha(G)$ is the Bohr compactification of a topological group G, then B/B_0 (with the identity component B_0 of B) is the *zero dimensional compactification* or the *profinite completion* of G. Since $\mathbb{Z}$ is a ring it turns out that $\widetilde{\mathbb{Z}} = B/B_0$ in this case carries a ring structure. The profinite ring $\widetilde{\mathbb{Z}}$ is at the focus of the present discussion.

The set of all prime numbers is denoted by π. For each profinite abelian group A the p-primary component or p-Sylow subgroup A_p is the largest p-subgroup of A, and one has the Sylow decomposition $A = \prod_{p\in\pi} A_p$. Note, however that the ring $\mathbb{Z}$ of integers in its discrete topology is not profinite, allowing the standard notation $\mathbb{Z}_p$ for the ring of p-adic integers to be an exception to this convention. Accordingly, we shall formulate the equation $(\widetilde{\mathbb{Z}})_p = \mathbb{Z}_p$. The compact ring $\widetilde{\mathbb{Z}}$ then satisfies

$$\widetilde{\mathbb{Z}} = \prod_{p\in\pi} (\widetilde{\mathbb{Z}})_p \cong \prod_{p\in\pi} \mathbb{Z}_p. \tag{1}$$

This is the Sylow decomposition (or primary decomposition) of $\widetilde{\mathbb{Z}}$.

2. The group of p-adic units $\mathbb{Z}_p^\times$

For a unital commutative ring R we denote by $R^\times$ the multiplicative group of its units, i.e., invertible elements. We clarify this concept for $R = \mathbb{Z}_p$ by a reminder of some elementary structural information of $\mathbb{Z}_p$. Recall that under suitable circumstances in a topological ring R the sequence $1 + x + \frac{1}{2}{\cdot}x + \frac{1}{3!}x^3 \cdots$ converges for x from a suitable domain D and defines a function

$$\exp\colon D \to 1 + D, \quad 1 + D \subseteq R^\times.$$

If $p \in \pi$ is a prime and $m \in \mathbb{N}$, then

$$\nu_p(m) = \max\{n \in \mathbb{N}_0 : p^n | m\} \tag{2}$$

is that unique nonnegative integer n for which $m = p^n m'$ and $(m', p) = 1$.

For the following information on the ring $\mathbb{Z}_p$ of p-adic integers see e.g. [5].

Lemma 2. (i) *For each prime $p \neq 2$, the function*

$$\exp\colon p{\cdot}\mathbb{Z} \to (1 + p\mathbb{Z}_p), \quad 1 + p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$$

is an isomorphism of profinite groups and $1 + p\mathbb{Z}_p$ is an open subgroup of $\mathbb{Z}_p^\times$. In particular,

$$z \mapsto \exp pz : (\mathbb{Z}_p, +) \to (1 + p\mathbb{Z}_p, \times) \tag{3}$$

is an isomorphism of profinite groups.

(ii) *The factor ring $\mathbb{Z}_p/p\mathbb{Z}_p$ is the field* GF(p) *of p elements, and so $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ is a cyclic group of $p-1$ elements. The ring $\mathbb{Z}_p^\times$ contains a cyclic group C_p of $p-1$ elements (called roots of unity) such that*

$$(x, c) \mapsto xc : (1 + p\mathbb{Z}_p) \times C_p \to \mathbb{Z}_p^\times \text{ is an isomorphism,}$$

and

$$(\mathbb{Z}_p^\times, \times) \cong \left(\mathbb{Z}_p \oplus \bigoplus_{q\in\pi} \mathbb{Z}\left(q^{\nu_q(p-1)}\right), +\right). \tag{4}$$

In particular, for $q \in \pi$ the q-Sylow subgroup of $\mathbb{Z}_p^\times$ is procyclic and

$$(\mathbb{Z}_p^\times)_q \cong \begin{cases} \mathbb{Z}\left(q^{\nu_q(p-1)}\right) & \text{if } q < p, \\ \mathbb{Z}_p & \text{if } q = p, \\ \{0\} & \text{if } p < q. \end{cases}$$

From a formalistic point of view it is regrettable that the case $p = 2$ is not exactly subordinate to the scheme. However, here it is:

Lemma 3. (i) *The function*

$$\exp\colon 4\mathbb{Z}_2 \to (1 + 4\mathbb{Z}_2), \quad 1 + 4\mathbb{Z}_2 \subseteq \mathbb{Z}_2^\times$$

is an isomorphism of profinite groups and $1 + 4\mathbb{Z}_2$ is an open subgroup of $\mathbb{Z}_2^\times$. In particular,

$$z \mapsto \exp 4z : (\mathbb{Z}_2, +) \to (1 + 4\mathbb{Z}_2, \times) \tag{5}$$

is an isomorphism of profinite groups.

(ii) *The factor ring $\mathbb{Z}_2/2\mathbb{Z}_2$ is the field* GF(2) *of 2 elements, and the group of units of $(\mathbb{Z}_2/4\mathbb{Z}_2)^\times$ is a group of 2 elements. The group $\mathbb{Z}_2^\times$ contains a cyclic group C_2 of 2 elements (called roots of unity) such that*

$$(x, c) \mapsto xc : (1 + 4\mathbb{Z}_2) \times C_2 \to \mathbb{Z}_2^\times \text{ is an isomorphism,}$$

and

$$(\mathbb{Z}_2^\times, \times) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}(2), +). \tag{6}$$

In particular, $\mathbb{Z}_2^\times$ is a nonprocyclic 2-group.

The product representation (1) $\widetilde{\mathbb{Z}}_p = \prod_{p\in\pi} \mathbb{Z}_p$ immediately yields

$$\widetilde{\mathbb{Z}}^\times = \prod_{p\in\pi} \mathbb{Z}_p^\times. \tag{7}$$

Since for $p \neq 2$ the profinite group $\mathbb{Z}_p^\times$ is not a p-group, the product representation of the profinite group $\widetilde{\mathbb{Z}}^\times$ in (7) is not its Sylow decomposition. Our first and foremost goal is now to determine the Sylow decomposition of $\widetilde{\mathbb{Z}}^\times$ and to describe it in an intuitive and useful form.

3. Some helpful facts on groups and numbers

The information contained in (1) through (6) suggests rather clearly that products $G = \prod_{j\in J} \mathbb{Z}(p^{n_j})$ (for families $(n_j)_{j\in J}$ of natural numbers and for a fixed prime number p will play a role in the structure of $\widetilde{\mathbb{Z}}^\times$). Lemma 3.9 of [2] provides the following standard information:

Lemma 4. *The group $\prod_{j\in J} \mathbb{Z}(p^{n_j})$ is a torsion group if and only if $(n_j)_{j\in J}$ is a bounded family.*

Accordingly we first collect some general facts on groups G for general families $(n_j)_{j\in J}$ and keep in mind as a special example the family $\mathbb{N} = (1, 2, 3, \ldots)$ and, accordingly, the group

$$P = \mathbb{Z}(p) \times \mathbb{Z}(p^2) \times \mathbb{Z}(p^3) \times \cdots \tag{8}$$

Definition 5. Let $p \in \pi$. Given $G = \prod_{j\in J} \mathbb{Z}(p^{n_j})$ for a family $(n_j)_{j\in J}$, for each $m \in \mathbb{N}$ we define

$$n_{jm} = \begin{cases} n_j & \text{if } n_j \le m, \\ 0 & \text{otherwise.} \end{cases}$$

Now we set $G_m = \prod_{j\in J} \mathbb{Z}(p^{n_{jm}})$. For any finite subset $F \subseteq J$ we let

$$n_{jF} = \begin{cases} n_j & \text{if } j \in F, \\ 0 & \text{otherwise} \end{cases}$$

and set

$$G_F = \prod_{j\in J} \mathbb{Z}(p^{n_{jF}}) \cong \bigoplus_{j\in F} \mathbb{Z}(p^{n_j}).$$

We see that $m \le n$ implies $G_m \le G_n$, and for any finite subset $F \subseteq J$ there is an m such that $G_F \le G_m$. Since $\bigcup_{F\subseteq J,\ F \text{ finite}} G_F$ is dense in $\prod_{j\in J} \mathbb{Z}(p^{n_j})$, we have

Remark 6. For any family $(n_j)_{j\in J}$ of natural numbers, the profinite p-group $G = \prod_{j\in J} \mathbb{Z}(p^{n_j})$ has the dense torsion subgroup $\bigcup_{m\in\mathbb{N}} G_m$ of the ascending sequence G_m, $m = 1, 2, \ldots$ of compact torsion subgroups.

Let us consider the character group $A := \widehat{G}$ of G. Then $A \cong \bigoplus_{j\in J} \mathbb{Z}(p^{n_j})$.

From Propositions 8.2 and 8.3 in [3] we cite

Lemma 7. *If Γ is any compact or any discrete group, then*

$$\overline{\operatorname{tor}\Gamma} = \mathit{Div}(\widehat{\Gamma})^{\perp}, \tag{9}$$

the annihilator of the group of all divisible elements of the character group of Γ.

We consider the special group

$$\Sigma_p = \mathbb{Z}(p) \oplus \mathbb{Z}(p^2) \oplus \mathbb{Z}(p^3) \oplus \cdots, \tag{10}$$

the character group of the group P in (8) above. In [2] Σ_p emerges as the torsion subgroup of the remarkable locally compact p-group ∇_p (see [2], Theorem 3.16) and it shows some surprising features itself.

Firstly we cite Lemma 3.17 of [2] known to Prüfer:

Lemma 8. *Let e_n be the generator of $\mathbb{Z}(p^n)$ in Σ_p and let $\phi\colon \Sigma_p \to \Sigma_p$ be the endomorphism defined by $\phi(e_n) = e_n - p\cdot e_{n+1}$. Then ϕ is injective and its cokernel $\Sigma_p/\phi(\Sigma_p)$ is, up to isomorphism, the Prüfer group $\mathbb{Z}(p^\infty)$. That is, the following sequence is exact:*

$$0 \to \Sigma_p \xrightarrow{\phi} \Sigma_p \to \mathbb{Z}(p^\infty) \to 0.$$

We can iterate ϕ and set $S_n = \phi^n(\Sigma_p)$, $n = 0, 1, 2, \ldots$. Then $\Sigma_p = S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$. Since ϕ is injective, all S_n are isomorphic to Σ_p.

Proposition 9. *The countable torsion group Σ_p is filtered by a sequence $S_0 = \Sigma_p \supseteq S_1 \supseteq S_2 \supseteq \cdots$ of isomorphic subgroups such that*

(i) $S_{n-1}/S_n \cong \mathbb{Z}(p^\infty)$ *for* $n \in \mathbb{N}$, *and*
(ii) $\bigcap_{n\in\mathbb{N}} S_n = \{0\}$.

Proof. We have to prove (i) and (ii). For each $n \in \mathbb{N}$, set $K_n = S_{n-1}/S_n$; in particular $K_0 = \mathbb{Z}(p^\infty)$. The injective endomorphism $\phi: S_0 \to S_0$ leaves S_n invariant and induces an injective endomorphism $\phi_n: S_n \to S_n$ with cokernel K_n. We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & S_0 & \xrightarrow{\phi} & S_0 & \to & K_0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & S_1 & \xrightarrow{\phi_2} & S_1 & \to & K_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \to & S_n & \xrightarrow{\phi_n} & S_n & \to & K_n & \to & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

in which all rows are exact and the vertical morphisms $S_{n-1} \to S_n$ $n \in \mathbb{N}$ are the isomorphisms induced by $\phi|S_n$ Since the downarrows $S_{n-1} \to S_n$ are isomorphisms, and $K_1 = \mathbb{Z}(p^\infty)$, it follows, inductively, that $K_n \cong \mathbb{Z}(p^\infty)$ for all $n \in \mathbb{N}$.

(ii) By the definition of ϕ in Lemma 8 we have $\phi(e_n) = e_n - p{\cdot}e_{n+1}$. We define $\ell: \Sigma_p \to \mathbb{N}$ as follows: let $x = \sum_{n\in\mathbb{N}} x_m$ with $x_m \in \mathbb{Z}(p^m)$. Then

$$\ell(x) = \begin{cases} 0 & \text{if } x = 0, \\ \max\{m \in \mathbb{N} \mid 0 \neq x_m \in \mathbb{Z}(p^m)\} & \\ - \min\{m \in \mathbb{N} \mid 0 \neq x_m \in \mathbb{Z}(p^m)\} + 1 & \text{otherwise.} \end{cases}$$

In the definition of ϕ in Lemma 8 we set $\phi(e_n) = e_n - p{\cdot}e_{n+1}$. Thus let $y = \sum_{n\in\mathbb{N}} y_n$ be $\phi(x)$ and assume $x \neq 0$. Then

$$\min\{m \in \mathbb{N} \mid 0 \neq y_m \in \mathbb{Z}(p^m)\} = \min\{m \in \mathbb{N} \mid 0 \neq x_m \in \mathbb{Z}(p^m)\}$$

and

$$\max\{m \in \mathbb{N} \mid 0 \neq y_m \in \mathbb{Z}(p^m)\} = \max\{m \in \mathbb{N} \mid 0 \neq x_m \in \mathbb{Z}(p^m)\} + 1.$$

Thus

$$\ell\big(\phi^n(x)\big) = \ell(x) + n. \tag{11}$$

Now assume that $y \in \bigcap_{m\in\mathbb{N}} S_m$. Suppose that $y \neq 0$ and set $n = \ell(y) \in \mathbb{N}$. Then $y \in \bigcap_{m\in\mathbb{N}} S_m \subseteq S_n$, and so there is an $x \neq 0$ such that $\phi^n(x) = y$. Thus (11) shows that $\ell(y) = \ell\big(\phi^n(x)\big) = \ell(x) + n = \ell(x) + \ell(y)$, that is, $\ell(x) = 0$ and hence $x = 0$ which is impossible. $\square$

This proposition dualizes comfortably according to the Annihilator Mechanism of locally compact abelian groups (see [3], 7.17 ff., notably Corollary 7.22, all of which fully applies to locally compact abelian groups). So let P of (8) be the dual of Σ_p and let $H_n \leq P$ be the annihilator $(S_n)^\perp$ of $S_n \leq \Sigma_p$. Since the S_n are descending, the H_n are ascending, and since $\bigcap_{n\in\mathbb{N}} S_n = \{0\}$ we know that

$$P = \overline{\bigcup_{n\in\mathbb{N}} H_n}. \tag{12}$$

For all $n \in \mathbb{N}$ we deduce via duality from $S_{n-1}/S_n \cong \mathbb{Z}(p^\infty)$ that $H_n/H_{n-1} \cong \mathbb{Z}_p$ for $n \in \mathbb{N}$. However, at this point we can utilize the fact that in the category of compact p-groups, the group $\mathbb{Z}_p$ is projective (since its dual $\mathbb{Z}(p^\infty)$ is divisible hence injective in the category of discrete p-groups; see also [3], Theorem 8.78.) Therefore, for each $n \in \mathbb{N}$, the compact group H_n contains a compact subgroup $K_n \cong \mathbb{Z}_p$ such that

$$(\forall n \in \mathbb{N})\, H_n = H_{n-1}K_n \cong H_{n-1} \times K_n. \tag{13}$$

By induction we conclude at once that

$$(\forall n \in \mathbb{N})\, H_n = C_1 \cdots C_n = C_1 \times \cdots \times C_n \cong \mathbb{Z}_p^n \tag{14}$$

and there are algebraic isomorphisms

$$\bigcup_{n\in N} H_n = \left\langle \bigcup_{n\in\mathbb{N}} C_n \right\rangle \cong \bigoplus_{n\in\mathbb{N}} C_n \cong \mathbb{Z}_p^{(\mathbb{N})} \tag{15}$$

Let us collect this information:

Corollary 10. *The group $P = \mathbb{Z}(p)\times\mathbb{Z}(p^2)\times\mathbb{Z}(p^3)\times\cdots$ contains a dense $\mathbb{Z}_p$-submodule which is algebraically isomorphic to the $\mathbb{Z}_p$-module $\mathbb{Z}_p^{(\mathbb{N})}$.*

Corollary 11. *For any family $(n_j)_{j\in J}$ of natural numbers, the profinite p-group $G = \prod_{j\in J}\mathbb{Z}(p^{n_j})$ is either a torsion group or else it contains a $\mathbb{Z}_p$-submodule isomorphic to $\mathbb{Z}_p^{(\mathbb{N})}$ whose closure is isomorphic to $P = \mathbb{Z}(p) \times \mathbb{Z}(p^2) \times \mathbb{Z}(p^3) \times \cdots$*

Proof. Either the family $(n_j)_{j\in J}$ is bounded, in which case G is a torsion group, or else it is unbounded. In that case there is an increasing unbounded subsequence $(n_{j(m)})_{m\in\mathbb{N}}$. Set $k_m = n_{j(m)}$. Since the k_n are increasing, we have $n \leq k_n$. The cyclic group $\mathbb{Z}(p^{k_m}) = \mathbb{Z}(p^{n_{j(m)}})$ contains a subgroup $B_m \cong \mathbb{Z}(p^m)$. Then group $B_1 \times B_2 \times B_3 \times \cdots$ is clearly isomorphic to a subgroup B of G which is isomorphic to $\mathbb{Z}(p) \times \mathbb{Z}(p^2) \times \mathbb{Z}(p^3) \times \cdots$. Then it follows from Corollary 10 that B contains a dense $\mathbb{Z}_p$-submodule algebraically isomorphic to $\mathbb{Z}_p^{(\mathbb{N})}$, as asserted. □

We shall need the following pieces of information. The first one is number theoretical. As in (2), for a prime p and a natural number r, let $\nu_p(r)$ be the exponent of the largest p-power dividing r.

Lemma 12. *Let $p \in \pi$ be an arbitrary prime number and n an arbitrary natural number. Then there is a prime number q such that $n \leq \nu_p(q-1)$. Accordingly, $p^n|(q-1)$. In particular $p|(q-1)$.*

Proof. Fix $p \in \pi$ and an arbitrary natural number n. The numbers $a = p^n$ and $b = 1$ are relatively prime. Hence the arithmetic progression $(am + b)_{m\in\mathbb{N}}$ contains infinitely many primes q by the Dedekind Prime Number Theorem. Let q be one of them. Then $q-1 = p^n m$, that is $\nu_p(q-1) \geq n$. In particular, $p|(q-1)$. □

Lemmas 2 and 3 imply via (7) that $\widetilde{\mathbb{Z}}^\times$ contains for each fixed prime p a product

$$E := \prod_{q\in\pi} \mathbb{Z}(p^{\nu_p(q-1)}),$$

where we note that $\nu_p(q-1) = 0$ if p fails to divide $q-1$. Therefore the following conclusion of the preceding Lemma 12 is relevant:

Proposition 13. *Let $p \in \pi$ be an arbitrary prime number. Then the group E contains a subgroup isomorphic to*

$$P = \mathbb{Z}(p) \times \mathbb{Z}(p^2) \times \mathbb{Z}(p^3) \times \cdots$$

which in turn contains a dense subgroup and $\mathbb{Z}_p$-module $D \cong \mathbb{Z}_p^{(\mathbb{N})}$.

Proof. By Lemma 12 for each n there is a $q \in \pi$ such that $n \leq \nu_p(q-1)$. Hence the group $\mathbb{Z}(p^{\nu_p(q-1)})$ contains a subgroup $B_n \cong \mathbb{Z}(p^n)$. Thus E contains an isomorphic copy of

$$B = \prod_{n\in\mathbb{N}} B_n \cong \mathbb{Z}(p) \times \mathbb{Z}(p^2) \times \mathbb{Z}(p^3) \times \cdots$$

The remainder then follows from Corollary 10. □

4. The mastergraph

We introduce a bipartite edge-labeled graph $\mathcal{G}$ as follows:

Definition 14. A *bipartite graph* consists of two disjoint sets U and V and a binary relation $\mathcal{E} \subseteq (U \cup V)^2$ such that $(u, v) \in \mathcal{E}$ implies $u \in U$ and $v \in V$. The elements of $U \cup V$ are called *vertices* and the elements of $\mathcal{E}$ are called *edges*. Any triple $(U, V, \mathcal{E})$ of this type is called a *bipartite graph*.

An *edge labeled* graph is a quadruple $(U, V, \mathcal{E}, \lambda)$ such that $(U, V, \mathcal{E})$ is a bipartite graph and λ is a function $\lambda: \mathcal{E} \to L$ for some set L of labels.

Labels could be numbers, or symbols like ∞.

Now we define a particular edge labeled graph $\mathcal{G}$. Recall the definition of $\nu_p(m)$ from (2) above.

Definition 15. The following bipartite edge labeled graph

$$\mathcal{G} = (U, V, \mathcal{E}, \lambda), \quad \mathcal{E} \subseteq U \times V,$$

will be called the *prime mastergraph* or *mastergraph* for short:

(i) $U = \pi \times \{1\} \subseteq \pi \times \{0, 1\}$,
(ii) $V = \pi \times \{0\} \subseteq \pi \times \{0, 1\}$,
(iii) $\mathcal{E} = \{\big((p, 1), (q, 0)\big) : p = q \text{ or } p|(q-1)\}$,
(iv) $\lambda: \mathcal{E} \to \mathbb{N} \cup \{\infty\},\ \lambda\big(((p, 1), (q, 0))\big) = \begin{cases} \infty, & \text{if } p = q, \\ \nu_p(q-1), & \text{if } p < q. \end{cases}$

We shall call the vertices in U the *upper* and those in V the *lower* vertices. The edges $\big((p, 1), (p, 0)\big)$, $p \in \pi$ are said to be *vertical*, all others are called *sloping*. We say that $e = ((p, 1), (q, 0))$ is *the edge from p to q* (see Fig. 1 and Fig. 2).

The labels of the sloping edges are all equal to 1 with the exception 2—5 where it is 2 (see Fig. 2).

The “geometric” terminology is chosen because $\mathcal{G}$ has an intuitive representation in the plane $\mathbb{R}^2$ preserving the order:

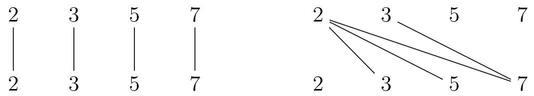

Fig. 1. Vertical and sloping edges.

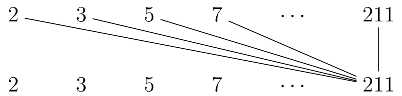

Fig. 2. The five vertices in U connected to the lower vertex numbered "211" in V.

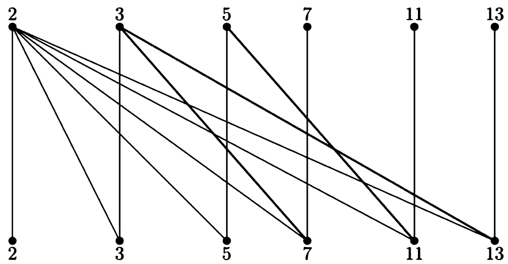

Fig. 3. The initial part of the master-graph.

Proposition 16. *Let $\omega\colon \pi \to \mathbb{N}$ be the bijection inverse to the usual enumeration $n \mapsto p_n$ of primes according to their natural ordering according to their size. Let* id *be the identity map of the set $\{0, 1\}$. There is a faithful representation of the configuration of $\mathcal{G}$ into the plane $\mathbb{R}^2$ preserving the componentwise order which is induced by the injection*

$$\pi \times \{0,1\} \xrightarrow{\omega\times\mathrm{id}} \mathbb{N} \times \{0,1\} \xrightarrow{\mathrm{incl}} \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

and taking U to $\mathbb{N} \times \{1\}$ and V to $\mathbb{N} \times \{0\}$.

In Fig. 3 the label of the edge from $(2, 1)$ to $(13, 0)$ is 2.

Definition 17. Let p and q be any primes. Then

$$\mathcal{E}_p = \{e : e = ((p, 1), (p', 0)) \in \mathcal{E} \text{ such that } p = p' \text{ or } p|(p' - 1)\},$$

the set of all edges emanating downwards from the vertex $(p, 1) \in U$ will be called the *cone peaking at* p. Further the set

$$\mathcal{F}_q = \{e : e = ((q', 1), (q, 0)) \in \mathcal{E} \text{ such that } q'|(q-1)\},$$

the set of edges ending below in the vertex $(q, 0) \in V$, is called the *funnel pointing to* q.

Both the cones and the funnels provide a partition of the set of edges. It is instantly clear that each funnel is finite, and so the funnels, are not as important as the cones. The structure of a cone is more interesting than that of a funnel as the following translation of Lemma 12 into the language of the mastergraph $\mathcal{G}$ shows.

Proposition 18. *Let p be any prime. Accordingly, in the graph $\mathcal{G}$, the cone $\mathcal{E}_p$ is peaking at the upper vertex $(p, 1)$, and for each natural number n, it contains an edge $e = ((p, 1), (q, 0))$ labeled $\nu_p(q-1) \geq n$. In particular, $\mathcal{E}_p$ contains infinitely many edges.*

5. The Sylow decomposition of $\widetilde{\mathbb{Z}}$ indexed by $\mathcal{G}$

We recall that $\mathcal{E}$ is the set of all edges of the mastergraph $\mathcal{G} = (U, V, \mathcal{E}, \lambda)$. We start the indexing by attaching to each edge $e = ((p, 1), (q, 0)) \in \mathcal{E}$ a profinite group $\mathbb{S}_e$ being, up to a natural isomorphism, a subgroup of $\widetilde{\mathbb{Z}}$:

Definition 19. For each edge $e \in \mathcal{E}$ from p to q we set

$$\mathbb{S}_e = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}(2), & \text{if } p = q = 2, \\ \mathbb{Z}_p, & \text{if } 2 < p = q, \\ \mathbb{Z}(p^{\nu_p(q-1)}), & \text{if } p < q. \end{cases} \tag{16}$$

We noted in (7) that

$$\widetilde{\mathbb{Z}}^\times = \prod_{q \in \pi} \mathbb{Z}_q^\times$$

and in Lemmas 2 and 3 a procyclic p-group occurs precisely as a subgroup of some $\mathbb{S}_e$ for an edge e with upper vertex p. Therefore, the p-Sylow subgroup of $\widetilde{\mathbb{Z}}^\times$ is represented by the cone $\mathcal{E}$ peaking in p. One has

$$(\mathbb{Z}_q)^\times = \prod_p (\mathbb{Z}_q^\times)_p = \prod_{e \in \mathcal{F}_q} \mathbb{S}_e, \tag{17}$$

as well as

$$(\widetilde{\mathbb{Z}}^\times)_p \cong \left(\prod_{q \in \pi} \mathbb{Z}_q^\times \right)_p = \prod_{e \in \mathcal{E}_p} \mathbb{S}_e, \tag{18}$$

$$p \mapsto \mathcal{E}_p \tag{19}$$

is a bijection from the set of primes to the set $\mathcal{C}$ of cones such that $\mathcal{C} = \bigcup_{p \in \pi} \mathcal{E}_p$ in the mastergraph.

Taking these matters and Proposition 13 into account, we can summarize:

Theorem 20. (i) *The group $\widetilde{\mathbb{Z}}^\times$ of units of the universal procyclic compactification $\widetilde{\mathbb{Z}}$ of the ring of integers $\mathbb{Z}$ is the product*

$$\widetilde{\mathbb{Z}}^\times \cong \prod_q (\mathbb{Z}^\times)_q = \prod_q \prod_{e\in\mathcal{F}_q} \mathbb{S}_e = \prod_{e\in\mathcal{E}} \mathbb{S}_e \tag{20}$$

extended over the set $\mathcal{E}$ of all edges of the mastergraph, where $\mathbb{S}_e$ is the profinite group given in (16) *above.*

(ii) *Its p-Sylow subgroup is the subproduct extended over the cone peaking in p:*

$$(\widetilde{\mathbb{Z}}^\times)_p = \prod_{e\in\mathcal{E}_p} \mathbb{S}_e \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}(2) \oplus \prod_{q>2} \mathbb{Z}(2^{\nu_2(q-1)}), & \text{if } p = 2, \\ \mathbb{Z}_p \oplus \prod_{q>p} \mathbb{Z}(p^{\nu_p(q-1)}), & \text{otherwise.} \end{cases} \tag{21}$$

(iii) *For each $p \in \pi$ fixed,*

$$(\widetilde{\mathbb{Z}}^\times)_p \cong \mathbb{Z}_p \oplus T_p, \qquad \text{where } T_p = \overline{\operatorname{tor}(\widetilde{\mathbb{Z}}^\times)_p}, \tag{22}$$

and where T_p contains a $\mathbb{Z}_p$-submodule algebraically isomorphic to $\mathbb{Z}_p^{(\mathbb{N})}$ whose closure is isomorphic to $\prod_{n\in\mathbb{N}} \mathbb{Z}(p^n)$.

Let $T = \overline{\operatorname{tor}(\widetilde{\mathbb{Z}}^\times)}$. For each prime p, define

$$\mathbb{ZP}_p = \prod_{n\in\mathbb{N}} \mathbb{Z}(p^n), \qquad \mathbb{ZP} = \prod_{p\in\pi} \mathbb{ZP}_p = \left(\prod_{(p,n)\in\pi\times\mathbb{N}} \mathbb{Z}(p^n). \right) \tag{23}$$

Corollary 21. (i) $\mathbb{ZP}$ *contains a dense copy of the torsion-free $\widetilde{\mathbb{Z}}$-module $M := \widetilde{\mathbb{Z}}^{(\mathbb{N})}$.*

(ii) *The closure T of the torsion subgroup of $\widetilde{\mathbb{Z}}^\times$ contains a copy of M.*

Proof. (i) The group $\mathbb{ZP}_p$ contains a dense copy of $\mathbb{Z}_p^{(\mathbb{N})}$ (see Theorem 20 (iii) above). Hence $\mathbb{ZP} = \prod_{p\in\pi} \mathbb{ZP}_p$ contains a dense copy of $\prod_{p\in\pi} \mathbb{Z}_p^{(\mathbb{N})}$ which contains a copy of $\widetilde{\mathbb{Z}}^{(\mathbb{N})} \cong \prod_{p\in\pi} \mathbb{Z}_p$ and this copy is still dense in $\mathbb{ZP}$.

(ii) From Theorem 20 (iii) implies that for each prime, T_p contains a copy of $\mathbb{ZP}_p$. Hence T contains a copy of $\mathbb{ZP}$. □

6. The Sylow decomposition of $\mathbb{Z}(n)^\times$ indexed by $\mathcal{G}$

We record $n = \prod_{p|n} p^{\nu(n)}$ (finite product: almost all $\nu_p(n) \neq 0$ only if $p|n$) and accordingly $\mathbb{Z}(n) = \prod_{p|n} \mathbb{Z}(p^{\nu_p(n)})$. Hence $\mathbb{Z}(n)^\times = \prod_{p|n} \mathbb{Z}(p^{\nu_p(n)})^\times$, and it suffices to recall the case that $n = p^m$. This we assume for the remainder of this section, and we fix a prime $\mathbf{p}$.

While the structure of $\mathbb{Z}(\mathbf{p}^m)^\times$ is usually dealt with in elementary number theory (see e.g. [1, Chapter 4]) we show how its structure can be determined also by interpreting $\mathbb{Z}(\mathbf{p}^m)$ as a $\mathbf{p}$-adic Lie group and thus use the exponential function from Section 2.

Here we have $\mathbb{Z}(\mathbf{p}^m) = \mathbb{Z}_\mathbf{p}/\mathbf{p}^m{\cdot}\mathbb{Z}_\mathbf{p}$. Let $\mu\colon \mathbb{Z}_\mathbf{p} \to \mathbb{Z}_\mathbf{p}$ denote the scalar endomorphism given by $\mu(x) = \mathbf{p}^m x$. Then

$$0 \to \mathbb{Z}_\mathbf{p} \xrightarrow{\mu} \mathbb{Z}_\mathbf{p} \to \mathbb{Z}(\mathbf{p}^m) \to 0$$

is exact and μ induces a quotient morphism $\mu^\times\colon \mathbb{Z}_\mathbf{p}^\times \to \mathbb{Z}(\mathbf{p}^m)^\times$. We recall that the morphism $\mathbb{Z}_\mathbf{p} \to \mathbb{Z}_\mathbf{p}/\mathbf{p}\mathbb{Z}_\mathbf{p} \cong \mathrm{GF}(\mathbf{p})$ maps $C_\mathbf{p}$ of Lemmas 2 and 3 faithfully because $\mathbf{p}^m\mathbb{Z}_\mathbf{p} \subseteq \mathbf{p}\mathbb{Z}_\mathbf{p}$ unless $\mathbf{p} = 2$ and $m \leq 2$, in which case $\mathbf{p}^m = 2$ or $= 4$, in which case we have $\mathbb{Z}(2)^\times = \{1\}$, respectively, $\mathbb{Z}(4)^\times = \{\pm 1\}$. If $\mathbf{p} > 2$ then we know that

$$\exp\colon (\mathbf{p}\mathbb{Z}_{\mathbf{p}}, +) \to (1 + \mathbf{p}\mathbb{Z}_{\mathbf{p}}, \times) \text{ is an isomorphism,}$$

whence by applying μ

$$\exp\colon \left(\frac{\mathbf{p}\mathbb{Z}_{\mathbf{p}}}{\mathbf{p}^m\mathbb{Z}_{\mathbf{p}}}, +\right) \to (\mu(1 + \mathbf{p}\mathbb{Z}_{\mathbf{p}}), \times) \text{ is an isomorphism.}$$

Since $\frac{\mathbf{p}\mathbb{Z}_{\mathbf{p}}}{\mathbf{p}^m\mathbb{Z}_{\mathbf{p}}} \cong \mathbb{Z}(\mathbf{p}^{m-1})$ in view of Lemma 2 we have

$$\mathbb{Z}(\mathbf{p}^m)^\times \cong \mathbb{Z}(\mathbf{p}^{m-1}) \oplus \mathbb{Z}(\mathbf{p} - 1).$$

Analogously, for $\mathbf{p} = 2$ and $m > 2$, from Lemma 3 we obtain

$$\mathbb{Z}(2^m)^\times \cong \mathbb{Z}(2^{m-2}) \oplus \mathbb{Z}(2) \tag{24}$$

Summarizing, we have

Lemma 22. *The group of units of $\mathbb{Z}(\mathbf{p}^m)$ is*

$$\mathbb{Z}(\mathbf{p}^m)^\times \cong \begin{cases} \{0\}, & \text{if } \mathbf{p}^m = 2, \\ \mathbb{Z}(2), & \text{if } \mathbf{p}^m = 4, \\ \mathbb{Z}\left(2^{m-2}\right) \oplus \mathbb{Z}(2), & \text{if } \mathbf{p} = 2, m > 2, \\ \mathbb{Z}\left(\mathbf{p}^{m-1}\right) \oplus \mathbb{Z}(\mathbf{p} - 1), & \text{if } \mathbf{p} > 2. \end{cases} \tag{25}$$

We may use $\mathcal{G}$ as index set for describing the p-Sylow decomposition of $A = \mathbb{Z}(\mathbf{p}^m)^\times$ as follows:

We index subgroups $\mathbb{S}_e \leq A$ by attaching again to each edge $e = ((p, 1), (q, 0)) \in \mathcal{E}$ a profinite group $\mathbb{S}_e$ being, up to a natural isomorphism, a subgroup of $\widetilde{\mathbb{Z}}^\times$:

Definition 23. For each edge $e \in \mathcal{E}$ from p to q we set

$$\mathbb{S}_e = \begin{cases} \{0\}, & \text{if } \mathbf{p}^m = 2 \text{ or } q > \mathbf{p}^m, \\ \mathbb{Z}(2), & \text{if } \mathbf{p}^m = 4 \text{ and } p = q = 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}(2), & \text{if } p = q = \mathbf{p} = 2, \\ \mathbb{Z}\left(p^{m-2}\right), & \text{if } 2 < \mathbf{p} \text{ and } q \leq \mathbf{p}, \\ \mathbb{Z}\left(p^{\nu_p(q-1)}\right), & \text{if } p < q \leq \mathbf{p}. \end{cases} \tag{26}$$

With this indexing we can formulate

Theorem 24. *For a fixed prime* $\mathbf{p}$ *and a fixed natural number* m*,*

(i) *the group $\mathbb{Z}(\mathbf{p}^m)^\times$ of units of the universal cyclic group $\mathbb{Z}(\mathbf{p}^m)$ is*

$$\mathbb{Z}(\mathbf{p}^m)^\times = \prod_{e \in \mathcal{E}} \mathbb{S}_e \tag{27}$$

extended over the set $\mathcal{E}$ of all edges of the mastergraph, where $\mathbb{S}_e$ is the profinite group given in (26) above.

(ii) *Its p-Sylow subgroup is the subproduct extended over the cone peaking in* $\mathbf{p}$*:*

$$\left(\mathbb{Z}(\mathbf{p}^m)^\times\right)_p = \prod_{e \in \mathcal{E}_p} \mathbb{S}_e \cong \begin{cases} \mathbb{Z}(4) \oplus \mathbb{Z}(2) \oplus \bigoplus_{\mathbf{p} \geq q > 2} \mathbb{Z}\left(2^{\nu_2(q-1)}\right), & \text{if } p = 2, \\ \mathbb{Z}(p^{m-2}) \oplus \quad \bigoplus_{\mathbf{p} \geq q > p} \mathbb{Z}\left(p^{\nu_p(q-1)}\right), & \text{otherwise.} \end{cases} \tag{28}$$

7. The mastergraph of a periodic abelian group

Recall that for a locally compact group G an element g is called *compact* if it is contained in a compact subgroup. The set of compact elements is called $\mathrm{comp}(G)$. If G is abelian, then $\mathrm{comp}(G)$ is a fully characteristic subgroup. For details see [3], Chapter 7 and [2]. The identity component of a topological group is written G_0.

Definition 25. A locally compact group G is said to be *periodic*, if it satisfies the following conditions:

(i) $G = \mathrm{comp}(G)$,
(ii) $G_0 = \{0\}$.

In other words, G is the union of its compact subgroups and is totally disconnected. In fact, if G is abelian, then G is the directed union of its compact open subgroups, and if C and K are two of them, then C and K are *commensurable*, that is both $C/(C \cap K)$ and $K/(C \cap K)$ are finite.

If $(G_j)_{j\in J}$ is a family of topological groups and $C_j \leq G_j$ is a compact open subgroup for each j, then the set of all $(g_j)_{j\in J} \in T = \prod_{j\in J} G_j$ such that $\{j \in J |\ g_j \notin C_j\}$ is finite forms a subgroup $L \leq T$ of the product containing $C = \prod_{j\in J} C_j$. Then L is a locally compact topological group for the unique topology for which C is open in G. This group L is called the *local product* of the family $(G_j, C_j)_{j\in J}$ and is written

$$L = \prod_{j\in J}^{\mathrm{loc}} (G_j, C_j).$$

We shall write abelian groups additively in general, unless the context demands otherwise, e.g. in the case of the group of units of a ring, such as $\mathbb{Z}_p$.

With this notation it is easy to reproduce Braconnier's theorem on the Sylow decomposition of a periodic locally compact abelian group A into its p-Sylow subgroups A_p, $p \in \pi$:

Theorem 26. (J. Braconnier) *Let A be a periodic locally compact abelian group and C any compact open subgroup of A. Then A is isomorphic to the local product*

$$\prod_{p}^{\mathrm{loc}} (A_p, C_p). \tag{29}$$

If A is a periodic locally compact abelian group, then every endomorphism α leaves the Sylow subgroup A_p invariant. We write $\alpha_p = \alpha|G_p : A_p \to A_p$. If C is a compact open subgroup, let $\mathrm{End}(G, C)$ denote the subring of the endomorphism ring $\mathrm{End}(G)$ of all endomorphisms leaving C invariant.

In view of Theorem 26 we may identify A with its canonical local product decomposition of the pair (A, C).

Every locally compact abelian p-group A is a $\mathbb{Z}_p$-module for a multiplication $(r_p, g_p) \mapsto r_p{\cdot}g_p$. If we identify $\widetilde{\mathbb{Z}}$ and $\prod_{p\in\pi} \mathbb{Z}_p$ by (1) and a periodic locally compact abelian group A with $\prod_{p\in\pi}^{\mathrm{loc}} (A_p, C_p)$ for any compact open subgroup C, we see at once that we have a continuous module multiplication, a map from $\widetilde{\mathbb{Z}} \times A$ to A given by

$$(r, g) = ((r_p)_p, (g_p)_p) \mapsto (r_p{\cdot}g_p)_p = r{\cdot}g. \tag{30}$$

In a similar vein we observe

Proposition 27. *For a periodic locally compact abelian group A, the componentwise application κ defined by*

$$\alpha \mapsto (\alpha_p)_p : \mathrm{End}(A, C) \to \prod_p \mathrm{End}(A_p, C_p)$$

is an isomorphism of groups, and $\alpha((g_p)_p) = (\alpha_p(g_p))_p$.

Proof. After identifying (A, C) and $\prod_p^{\mathrm{loc}}(A_p, C_p)$ according to Theorem 26, it is straightforward to verify that κ is an injective morphism of groups. Moreover, if

$$(\phi_p)_p \in \prod_p \mathrm{End}(A_p, C_p),$$

then the morphism

$$\phi\colon \prod_p A_p \to \prod_p A_p \text{ defined by } \phi((g_p)_p) = (\phi_p(g_p))_p$$

leaves $C = \prod_p C_p$ fixed as a whole and does the same with $\prod_p^{\mathrm{loc}}(A_p, C_p)$ and so $\kappa(\phi) = (\phi_p)_p$. Thus κ is surjective as well. □

We noted in (30) that every $r \in \widetilde{\mathbb{Z}}$ yields an endomorphism $a \mapsto r{\cdot}a$ of the periodic locally compact abelian group A, giving us a morphism of rings $\zeta\colon \widetilde{\mathbb{Z}} \to \mathrm{End}(A)$. In particular, since scalar multiplication $\widetilde{\mathbb{Z}} \times A \to A$ is continuous, $\ker(\zeta)$ is a closed ideal of $\widetilde{\mathbb{Z}}$.

Definition 28. For a locally compact abelian group A we denote the factor ring $\widetilde{\mathbb{Z}}/\ker(\zeta)$ by $\mathcal{R}(A)$ and call it *the ring of scalars* of A. There is an obvious scalar multiplication $\mathcal{R}(A) \times A \to A$.

The ring morphism ζ factors through an isomorphism of rings

$$\mathcal{R}(A) \xrightarrow{\cong} \mathrm{End}(A). \tag{31}$$

We note that

$$\mathcal{R}(A) \cong \prod_p \mathcal{R}(A)_p, \tag{32}$$

and scalar multiplication operates componentwise on $A \cong \prod_p(A_p, C_p)$.

We shall be mostly interested in the scalar multiplication by units $r \in \widetilde{\mathbb{Z}}$. In this context it is clear that (32) induces an isomorphism

$$\mathcal{R}(A)^\times \cong \prod_p (\mathcal{R}(A)_p)^\times, \tag{33}$$

where $(R(A)_p)^\times$ is isomorphic to a quotient group of $(\mathbb{Z}_p)^\times$.

One verifies easily the following piece of information:

Example 29. Let A be a locally compact abelian p-group. Then

$$\mathcal{R}(A) = \begin{cases} \mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}(p^m), & \text{if } A \text{ has finite exponent } p^m, \\ \mathbb{Z}_p, & \text{otherwise.} \end{cases} \tag{34}$$

Lemma 4 shows that among the *compact* abelian groups A the torsion groups are exactly the ones having finite exponent.

8. Scalar multiplication on a periodic locally compact abelian group

The following lemma is straightforward:

Lemma 30. *For a continuous endomorphism α of a locally compact group G the following conditions are equivalent:*

(i) $\alpha(H) \subseteq H$ *for all closed subgroups* H *of* G.
(ii) $\alpha(\overline{\langle g\rangle}) \subseteq \overline{\langle g\rangle}$ *for all* $g \in G$.
(iii) $\alpha(g) \in \overline{\langle g\rangle}$ *for all* $g \in G$.

Definition 31. An endomorphism α of a locally compact group G is called *scalar* if it satisfies the equivalent conditions of Lemma 30.

In [5] it is shown that on a compact abelian p-group G, for any automorphism α which is scalar in the sense of Definition 31 there is an $r \in \mathbb{Z}_p^\times$ such that $\alpha(g) = r{\cdot}g$ for all $g \in G$. The proof through Lemma 2.21 and Proposition 2.22 in [5] works for endomorphisms as well and thus yields

Lemma 32. *Let A be a compact abelian p-group. Then for any scalar endomorphism α there is an $r \in \mathbb{Z}_p$ such that $\alpha(a) = r{\cdot}a$ for all $a \in A$. Accordingly, α is an automorphism iff $r \in \mathbb{Z}_p^\times$.*

In [2], Lemma 4.6 it is shown for a locally compact abelian p-group A for any automorphism α for which any restriction to a compact-open subgroup is scalar, there is an $r \in \mathbb{Z}_p$ such that $\alpha(a) = r{\cdot}a$ for all $a \in A$. Again this proof works for endomorphisms as well as for automorphisms. Therefore we have

Lemma 33. *Let A be a locally compact abelian p-group. Then for any scalar endomorphism α there is an $r \in \mathbb{Z}_p$ such that $\alpha(a) = r{\cdot}a$ for all $a \in A$. Accordingly, α is an automorphism iff $r \in \mathbb{Z}_p^\times$.*

Finally, if $A = \prod_p^{\mathrm{loc}}(A_p, C_p)$ is any periodic locally compact abelian group we observe that every closed subgroup H is of the form $\prod_p^{\mathrm{loc}}(H_p, C_p \cap H_p)$, and so an endomorphism α of A is scalar iff every restriction α_p to A_p is scalar. If this is the case, then for every p there is an $r_p \in \mathbb{Z}_p$ such that $\alpha_p(a_p) = r_p{\cdot}a_p$ for all $a_p \in A_p$. So if $r = (r_p)_p$ in $\widetilde{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, for the scalar endomorphism α we have an $r \in \widetilde{\mathbb{Z}}$ such that $\alpha(a) = r{\cdot}a$ for $a \in A$. Thus we have the following classification of scalar endomorphisms, justifying the nomenclature:

Proposition 34. *Let A be a periodic locally compact abelian group and $\alpha\colon A \to A$ an endomorphism of locally compact abelian groups such that $\alpha(H) \subseteq H$ for all closed subgroups of A. Then there is an $r \in \widetilde{\mathbb{Z}}$ such that $\alpha(a) = r{\cdot}a$ for all $a \in A$.*

Definition 35. The group of scalar automorphisms of a locally compact group G is denoted by $\mathrm{SAut}(G)$.

If A is abelian and is written additively, then the subgroup

$$\{\mathrm{id}_A, -\mathrm{id}_A\} \subseteq \mathrm{SAut}(A)$$

is said to consist of *trivial scalar automorphisms*. All other scalar automorphisms are called *nontrivial*.

Notice that we shall not only call the identity automorphism, but also the inversion automorphism "$-\mathrm{id}_G$" *trivial*.

For periodic locally compact abelian groups A we have seen in Proposition 34 that all scalar automorphisms are indeed scalar multiplications in the traditional sense (see [2], Proposition 4.15):

We topologize $\mathrm{SAut}(G)$ with the Braconnier topology (see [4, (26.3)]).

Proposition 36. *Let the locally compact abelian group G be periodic. Then we have the following conclusions:*

(i) *The natural map $\zeta\colon \widetilde{\mathbb{Z}}^{\times} \to \mathrm{SAut}(G)$ (such that $\zeta(r)(g) = r{\cdot}g$) is surjective. In particular, $\mathrm{SAut}(G)$ is a profinite group and a homomorphic image of $\widetilde{\mathbb{Z}}^{\times}$.*

(ii) *The subsequent two statements are equivalent:*
  (a) $\mathrm{SAut}(G) = \{\mathrm{id}_G, -\mathrm{id}_G\}$.
  (b) *The exponent of G is 2, 3, or 4.*

Notably: The exponent of G is 2 if and only if $-\mathrm{id}_G = \mathrm{id}_G$.

Indeed, periodicity and the existence of nontrivial scalar multiplications are related as follows (see [2], Theorem 4.16):

Theorem 37. *For a locally compact abelian group G, we consider the following statements:*

(i) *G has nontrivial scalar automorphisms.*
(ii) *G is periodic.*

Then (i) implies (ii), and if G does not have exponent 2, 3, or 4, then both statements are equivalent.

The Sylow decomposition of $\mathrm{SAut}(G)$ is described in the following theorem (see [2], Theorem 4.17):

Theorem 38 *(Mukhin, Theorem 2 in [6]). Let G be a locally compact abelian group written additively.*

(a) *$\mathrm{SAut}(G)$ is a homomorphic image of $\widetilde{\mathbb{Z}}^{\times}$.*
(b) *If G is not periodic, then $\mathrm{SAut}(G) = \{\mathrm{id}, -\mathrm{id}\}$.*
(c) *If G is periodic, then $\mathrm{SAut}(G) = \prod_p \mathrm{SAut}(G_p)$, where $\mathrm{SAut}(G_p)$ may be identified with the group of units of the ring $\mathcal{R}(G_p)$ of scalars of G_p, namely, $\mathcal{R}(G_p)^{\times}$ is isomorphic to*

$$\begin{cases} \mathbb{Z}_p \times \mathbb{Z}(p-1), & \text{if } p > 2 \text{ and the exponent of } G_p \text{ is infinite},\\ \mathbb{Z}(p^{m-1}) \times \mathbb{Z}(p-1), & \text{if } p > 2 \text{ and the exponent of } G_p \text{ is } p^m,\\ \mathbb{Z}_2 \times \mathbb{Z}(2), & \text{if } p = 2 \text{ and the exponent of } G_2 \text{ is infinite},\\ \mathbb{Z}(2^{m-2}) \times \mathbb{Z}(2), & \text{if } p = 2 \text{ and the exponent of } G_2 \text{ is } 2^m > 2,\\ \{0\}, & \text{if } p = 2 \text{ and the exponent of } G_2 \text{ is } 2. \end{cases}$$

(d) *An $\alpha \in \mathrm{Aut}(G)$ is in $\mathrm{SAut}(G)$ iff there is a unit $z \in \widetilde{\mathbb{Z}}^{\times}$ such that*

$$(\forall g \in G)\, \alpha(g) = z{\cdot}g = \prod_p z_p{\cdot}g_p \text{ for } z = \prod_p z_p, g = \prod_p g_p.$$

9. The prime graph of a periodic locally compact abelian group

Now let A be a periodic locally compact abelian group; the Sylow structure of $\mathrm{SAut}(A)$ is now easily discussed: The quotient morphism $\zeta\colon \widetilde{\mathbb{Z}}^{\times} \to \mathrm{SAut}(A)$ of Proposition 36, preserving the Sylow structures, and the structure of $\mathrm{SAut}(A)$ described so far in Theorem 38 allow a precise description of the Sylow structure of $\mathrm{SAut}(A)$.

We associate with A the bipartite graph $\mathcal{G}(A) = (U, V, \mathcal{E}(A), \lambda)$ with U and V as in the mastergraph and with

$$\mathcal{E}(A) = \{e \in E : e = ((p,1),(q,0)) \text{ such that } \operatorname{SAut}(A_q)_p \neq \{\mathrm{id}_A\}\},$$

and for fixed p we let the *cone* $\mathcal{E}_p$ peaking at p be the set of edges in $\mathcal{E}$ emanating from $(p,1)$ and the *funnel* at q be the set of those edges terminating at $(q,0)$.

Let us define

$$\mathbb{S}_e(A) := \operatorname{SAut}(A_q)_p. \tag{35}$$

Finally, for $e \in \mathcal{E}(A)$ from p to q the label is

$$\lambda(e) = \begin{cases} 0, & \text{if } p = q, \\ \nu_p(q-1), & \text{if } p|(q-1). \end{cases} \tag{36}$$

Now let A be a periodic locally compact abelian group; the Sylow structure of $\operatorname{SAut}(A)$ is then easily discussed: The quotient morphism $\zeta\colon \widetilde{\mathbb{Z}}^{\times} \to \operatorname{SAut}(A)$ of Proposition 36, preserving the Sylow structures, and the structure of $\operatorname{SAut}(A)$ described so far in Theorem 38 allow a precise description of the Sylow structure of $\operatorname{SAut}(A)$.

Theorem 39 *(The Sylow Structure of* $\operatorname{SAut}(A)$*). Let A be a periodic locally compact abelian group and* $\operatorname{SAut}(A) = \prod_{p\in\pi} \operatorname{SAut}(A)_p$ *the p-primary decomposition of the profinite group* $\operatorname{SAut}(A) = \prod_{e\in\mathcal{E}(A)} \mathbb{S}_e$*. Then*

(i) *The p-primary decomposition of* $\operatorname{SAut}(A_q)$ *is (additive notation assumed)*

$$\prod_{e\in\mathcal{F}_q} \operatorname{SAut}(A_q)_{p_e} = \prod_{e\in\mathcal{F}_q} \mathbb{S}_e(A),$$

(see Eq. 35) and this group is isomorphic, in case $p = 2$, to

$$\begin{cases} \{0\}, & \textit{if } A_2 \textit{ has exponent} \leq 2, \\ \mathbb{Z}(2^{r-2}) \oplus \mathbb{Z}(2), & \textit{if } A_2 \textit{ has finite exponent } 2^r > 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}(2), & \textit{if } A_2 \textit{ has infinite exponent,} \end{cases}$$

and in case $p > 2$, to

$$\begin{cases} \mathbb{Z}(q^{r-1}) \oplus \bigoplus_{e\in\mathcal{F}_q} \mathbb{Z}(p_e^{\lambda(e)}), & \textit{if } A_q \textit{ has finite exponent } q^r, \\ \mathbb{Z}_q \oplus \bigoplus_{e\in\mathcal{F}_q} \mathbb{Z}(p_e^{\lambda(e)}), & \textit{if } A_q \textit{ has infinite exponent.} \end{cases}$$

(ii) *The structure of the p-primary component* $\operatorname{SAut}(A)_p$ *of* $\operatorname{SAut}(A)$ *(in additive notation) is*

$$\prod_{e\in\mathcal{E}_p} (\operatorname{SAut}(A_{q_e}))_p = \prod_{e\in\mathcal{E}_p} \mathbb{S}_e(A) \cong$$

$$\begin{cases} \prod_{e\in\mathcal{E}_p} \mathbb{Z}(2^{\lambda(e)}), & \textit{if } p = 2 \textit{ and } A_2 \textit{ has exponent} \leq 2, \\ \mathbb{Z}(2^{r-2}) \oplus \mathbb{Z}(2) \oplus \prod_{e\in\mathcal{E}_p} \mathbb{Z}(2^{\lambda(e)}), & \textit{if } p = 2 \textit{ and } A_2 \textit{ has fin. exp. } 2^r{>}2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}(2) \oplus \prod_{e\in\mathcal{E}_p} \mathbb{Z}(2^{\lambda(e)}), & \textit{if } p = 2 \textit{ and } A_2 \textit{ has inf. exponent,} \\ \mathbb{Z}(p^{r-1}) \oplus \prod_{e\in\mathcal{E}_p} \mathbb{Z}(p_e^{\lambda(e)}), & \textit{if } 2 < p \textit{ and } A_p \textit{ has exponent } p^r, \\ \mathbb{Z}_p \oplus \prod_{e\in\mathcal{E}_p} \mathbb{Z}(p_e^{\lambda(e)}), & \textit{if } 2 < p \textit{ and } A_p \textit{ has infinite index.} \end{cases}$$

10. An application

For easy reference we repeat the following definition from the introduction

Definition 40. If (G, A) is a pair consisting of a topological group G and a closed normal subgroup A, then we call it a *special extension of* A if G is a locally compact group and the equivalent conditions of Proposition 1 are satisfied.

We now prove the following result as an example of the methods we are proposing.

Theorem 41. *Let (G, A) be a special extension of a periodic locally compact abelian group. Then for each sloping edge $e \in \mathcal{E}(G, A)$ from some p to some q, all of A_q consists of commutators. In particular, $A_q \subseteq G'$.*

Proof. By definition the existence of a sloping edge e from $(p, 1)$ to $(q, 0)$ in $\mathcal{G}(A)$ we have $p < q$ and there is a p-element $1 \neq g \in G_p$ acting nontrivially on A. Hence $1 \neq r = \rho(g) \in \left(\mathcal{R}(A_q)^\times\right)_p$. By Theorem 38 we know that $\left(\mathcal{R}(A_q)^\times\right)_p$ is a cyclic group of order $p^{\lambda(e)} = p^{\nu_p(q-1)}$.

We claim that $1 - r$ is a unit in the ring $\mathcal{R}(A_q)$ of scalars which is isomorphic to $\mathbb{Z}_q$ or quotient ring thereof depending as A_q has infinite or finite exponent. By way of contradiction suppose that $r - 1$ is not a unit. Since $\mathbb{Z}_q^\times = \mathbb{Z}_q \backslash q\mathbb{Z}_q$, there is an element $u \in \mathcal{R}(A_q)$ such that $1-r = qu$. Then $r = 1-qu \in 1+q\mathcal{R}(A_q)$ which, according to the structure of $\mathbb{Z}_q^\times$ in (4), respectively, of $\mathbb{Z}(q^m)^\times$ in (25), is the q-Sylow subgroup of $\mathcal{R}(A_q)^\times$. But r is a p-element with $p < q$ and this is a contradiction.

Now let $a \in A_q$. For the purpose of this proof we write g additively. Then the commutator of g and a is $[g, a] = \rho(g)(a) - a = r{\cdot}a - a = (r - 1){\cdot}a$. Since $1 - r$ is invertible, we set $b = (r - 1)^{-1}{\cdot}a \in A_q$ and obtain $a = \rho(g)(b) - b = gbg^{-1} - b = [g, b]$.

This shows that every element of A_q is a commutator and thus proves the theorem. □

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References

[1] Ethan D. Bolker, Elementary Number Theory. An Algebraic Approach, W. A. Benjamin, Inc., New York, 1970, xi+180 pp.
[2] W. Herfort, K.H. Hofmann, F.G. Russo, Locally Compact Periodic Groups, de Gruyter Studies in Mathematics, de Gruyter, Berlin, 2018, xii+321pp.
[3] K.H. Hofmann, S.A. Morris, The Structure of Compact Groups, third edition, de Gruyter Studies in Mathematics, vol. 25, de Gruyter, Berlin, 2013, xxii, 924pp.
[4] Edwin Hewitt, Kenneth A. Ross, Abstract Harmonic Analysis. Vol. I: Structure of Topological Groups. Integration Theory, Group Representations, Die Grundlehren der mathematischen Wissenschaften, vol. Bd. 115, Academic Press, Inc., Publishers, New York, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
[5] K.H. Hofmann, F.G. Russo, Near Abelian profinite groups, Forum Math. 27 (2015) 647–698.
[6] Ju.N. Muhin, Automorphisms that fix the closed subgroups of a topological group, Sib. Mat. Zh. 6 (1975) 1231–1239.
[7] L. Ribes, P.A. Zalesskii, Profinite Groups, 40, second edition, Springer Ergebnisse, vol. 40, Springer, 2010, xvi, 404pp.