Selected topics in representation theory

– General representations of quivers and canonical decompositions II –

SS 2006

1 Preliminaries

Kac proved in [3] the following theorem:

Theorem. *Let* $\mathbf{d} \in \mathbb{N}^{Q_0}$*.* $\mathbf{d} = \sum_{a\in I} \mathbf{d_a}$ *is the canonical decomposition of* $\mathbf{d}$ *if and only if all* $\mathbf{d_a}$*,* $a \in I$*, are Schur roots and there exist representations* $U_a \in \mathrm{rep}^0(Q, \mathbf{d_a})$ *such that* $\mathrm{Ext}(U_a, U_b) = 0$ *for* $a \neq b$*. Moreover,* $\bigoplus_{a\in I} U_a \in \mathrm{rep}^0(Q, \mathbf{d}) \cap \mathrm{rep}_{\mathrm{can}}(Q, \mathbf{d})$*.*

If $\mathbf{d} = \sum_{a\in I} \mathbf{d_a}$ *is the canonical decomposition of* $\mathbf{d}$*, then* $\langle \mathbf{d_a}, \mathbf{d_b}\rangle \geq 0$ *for all* $a \neq b$*.*

In the previous lecture (see [2]), a proof of a theorem due to Schofield was given. The theorem contained necessary (almost combinatorial) conditions for a decomposition of a dimension vector to be the canonical one, (which are stricter than those in the second part of Kac's theorem). The aim of this talk is to give also sufficient conditions.

Here is a theorem due to Schofield (see [4]) which will be needed for this:

Theorem. *The following assertions are equivalent:*

1. *Every representation of dimension vector* $\mathbf{d_1} + \mathbf{d_2}$ *has a subrepresentation of dimension vector* $\mathbf{d_1}$*.*
2. *A general representation of dimension vector* $\mathbf{d_1} + \mathbf{d_2}$ *has a subrepresentation of dimension vector* $\mathbf{d_1}$*.*
3. $\mathrm{ext}(\mathbf{d_1}, \mathbf{d_2}) = 0$.

2 Characterisation of canonical decompositions

In order to prove the theorem we will first consider a special case for two dimension vectors.

Proposition. *Let* $\mathbf{d_1}$ *and* $\mathbf{d_2}$ *be Schur roots, and assume that* $\mathrm{ext}(\mathbf{d_1}, \mathbf{d_2}) = 0$*. Then* $\hom(\mathbf{d_2}, \mathbf{d_1}) = 0$ *or* $\mathrm{ext}(\mathbf{d_2}, \mathbf{d_1}) = 0$*. If both* $\mathbf{d_1}$ *and* $\mathbf{d_2}$ *are imaginary, then* $\hom(\mathbf{d_2}, \mathbf{d_1}) = 0$*.*

Proof. If $\mathbf{d_1} = \mathbf{d_2}$, then (by assumption) $\mathrm{ext}(\mathbf{d_2}, \mathbf{d_1}) = 0$.

Since $\dim \mathrm{Ext}(-,-)$ is upper semicontinuous and $\mathrm{ext}(\mathbf{d_1}, \mathbf{d_2}) = 0$, for any two Schur representations R and S of dimension vectors $\mathbf{d_1}$ and $\mathbf{d_2}$, resp., we get $\mathrm{Ext}(R, S) = 0$. If $\hom(\mathbf{d_1}, \mathbf{d_2}) \neq 0$, then $\mathrm{Hom}(R, S) \neq 0$. By the Lemma due to D. Happel and C.M. Ringel ([1], see also previous talk), any non zero homomorphism $f : R \to S$ is injective or surjective, which implies that $\mathbf{d_1} < \mathbf{d_2}$ or $\mathbf{d_2} < \mathbf{d_1}$. (Here, $<$ is to be read componentwise.)

Case 1. $\mathbf{d_2} < \mathbf{d_1}$.

If $\mathbf{d_2}$ is a real Schur root, then $\mathrm{Ext}(S, S) = 0$ for each Schur representation S of dimension vector $\mathbf{d_2}$, in particular $\mathrm{ext}(\mathbf{d_2}, \mathbf{d_2}) = 0$.

If $\hom(\mathbf{d_2}, \mathbf{d_1}) = 0$, we are done. So suppose that $\hom(\mathbf{d_2}, \mathbf{d_1}) \neq 0$. Then we get for each Schur representation R of dimension vector $\mathbf{d_1}$ a Schur subrepresentation S of dimension vector $\mathbf{d_2}$. Therefore, $\mathrm{ext}(\mathbf{d_2}, \mathbf{d_1} - \mathbf{d_2}) = 0$ (by Schofield's theorem in the first part of this lecture). But then $\mathrm{ext}(\mathbf{d_2}, \mathbf{d_1}) = 0$, since $\mathrm{Ext}(S, R/S \oplus S) = 0$.

If $\mathbf{d_2}$ is not a real root, then $\mathrm{Ext}(R,S) = 0$ for all Schur representations R and S of dimension vectors $\mathbf{d_1}$ and $\mathbf{d_2}$, resp. Assume that $\hom(\mathbf{d_2},\mathbf{d_1}) \neq 0$. Then we get for each Schur representation R of dimension vector $\mathbf{d_1}$ a Schur subrepresentation S of dimension vector $\mathbf{d_2}$. Therefore, there is an injective homomorphism $S \to R$. But $\mathrm{Ext}(S,S) \neq 0$, because $\mathbf{d_2}$ is imaginary, and so $\mathrm{Ext}(R,S) \neq 0$. And this means (by the upper semicontinuity of $\dim \mathrm{Ext}(-,-)$) that $\mathrm{ext}(\mathbf{d_1},\mathbf{d_2}) \neq 0$, a contradiction. So $\hom(\mathbf{d_2},\mathbf{d_1}) = 0$.

Case 2. $\mathbf{d_1} < \mathbf{d_2}$.

If $\mathbf{d_1}$ is a real Schur root, then $\mathrm{Ext}(R,R) = 0$ for each Schur representation R of dimension vector $\mathbf{d_1}$, in particular $\mathrm{ext}(\mathbf{d_1},\mathbf{d_1}) = 0$.

If $\hom(\mathbf{d_2},\mathbf{d_1}) = 0$, we are done. So suppose that $\hom(\mathbf{d_2},\mathbf{d_1}) \neq 0$. Then we get for each Schur representation S of dimension vector $\mathbf{d_2}$ a Schur factor representation R of dimension vector $\mathbf{d_1}$. Taking the kernel of the corresponding map $f : S \to R$, we obtain a representation $R' = \ker f$ with dimension vector $\mathbf{d_2} - \mathbf{d_1}$. Therefore, $\mathrm{ext}(\mathbf{d_2} - \mathbf{d_1},\mathbf{d_1}) = 0$ (by Schofield's theorem in the first part of this lecture). But then $\mathrm{ext}(\mathbf{d_2},\mathbf{d_1}) = 0$, since $\mathrm{Ext}(R' \oplus S/R', S/R') = 0$.

If $\mathbf{d_1}$ is not a real root, then $\mathrm{Ext}(R,S) = 0$ for all Schur representations R and S of dimension vectors $\mathbf{d_1}$ and $\mathbf{d_2}$, resp. Assume that $\hom(\mathbf{d_2},\mathbf{d_1}) \neq 0$. Therefore, there is a surjective homomorphism $S \to R$. But $\mathrm{Ext}(R,R) \neq 0$, because $\mathbf{d_1}$ is imaginary, and so $\mathrm{Ext}(R,S) \neq 0$. And this means (by the upper semicontinuity of $\dim \mathrm{Ext}(-,-)$) that $\mathrm{ext}(\mathbf{d_1},\mathbf{d_2}) \neq 0$, a contradiction. So $\hom(\mathbf{d_2},\mathbf{d_1}) = 0$.

Combining the two results for $\mathbf{d_1}$ and $\mathbf{d_2}$ imaginary, we obtain that in this case $\hom(\mathbf{d_2},\mathbf{d_1}) = 0$. □

Now we can prove the main theorem.

Theorem. *Let Q be a quiver, and $\mathbf{d} = \sum \mathbf{d_a}$ be a decomposition of $\mathbf{d}$ into Schur roots. This is the canonical decomposition if and only if $\langle \mathbf{d_a},\mathbf{d_b}\rangle \geq 0$ and $\langle \mathbf{d_a},\mathbf{d_b}\rangle\langle \mathbf{d_b},\mathbf{d_a}\rangle = 0$ and in addition $\mathrm{ext}(\mathbf{d_a},\mathbf{d_b}) = 0$ when $\langle \mathbf{d_a},\mathbf{d_b}\rangle = 0$.*

Proof. The necessity of the conditions was proved in the previous lecture (see [2]).

On the other hand, if $\langle \mathbf{d_a},\mathbf{d_b}\rangle = 0$, then $\mathrm{ext}(\mathbf{d_a},\mathbf{d_b}) = 0$. So let us assume that $\langle \mathbf{d_a},\mathbf{d_b}\rangle > 0$. Then $\langle \mathbf{d_b},\mathbf{d_a}\rangle = 0$, which implies that $\mathrm{ext}(\mathbf{d_b},\mathbf{d_a}) = 0$. Applying the proposition, we obtain that $\hom(\mathbf{d_a},\mathbf{d_b}) = 0$ or $\mathrm{ext}(\mathbf{d_a},\mathbf{d_b}) = 0$. The first case ($\hom(\mathbf{d_a},\mathbf{d_b}) = 0$) cannot happen, because $\langle \mathbf{d_a},\mathbf{d_b}\rangle > 0$. So $\mathrm{ext}(\mathbf{d_a},\mathbf{d_b}) = 0$ for all $a \neq b$, and Kac's theorem gives us that $\mathbf{d} = \sum \mathbf{d_a}$ is the canonical decomposition. □

References

[1] D. Happel and C.M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. **274** (1982), 399–443.

[2] A. Holtmann, *General representations of quivers and canonical decompositions I*, Lecture notes. Available at `http://www.math.uni-bielefeld.de/birep/selected.html`.

[3] V.G. Kac, *Infinite root systems, representations of graphs and invariant theory. II*, J. Algebra **78** (1982), no. 1, 141–162.

[4] A. Schofield, *General representations of quivers*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 46–64.